The Gravitational Energy of the Universe

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Abstract

The gravitational energy, momentum, and stress are calculated for the Robertson-Walker metric. The principle of energy conservation is applied, in conjunction with the Friedmann equations. Together, they show that the cosmological constant Λ is non-zero, the curvature index $k = 0$, and the acceleration $\ddot{R}$ is positive. The total energy density is evaluated and found to be always positive.

1. Introduction.

Currently, the energy density of the gravitational field plays no role in standard cosmology. Perhaps, it was thought to be insignificant. Here, it is calculated and shown to dominate the energy of the present Universe. With the assumption of zero material pressure, the Friedmann equations and the energy conservation law suffice to determine the free parameters of the theory. The curvature index k is found to be zero, while the cosmological constant Λ is non-zero. The latter contributes energy, such that the total energy density is positive.

2. Gravitational Energy, Momentum, and Stress.

The gravitational energy tensor is given by [1, 2]

$$T^{(g)}_{\mu\nu} = \frac{c^4}{8\pi G}\left\{Q^{\rho}_{[\lambda\mu]}Q^{\lambda}_{[\rho\nu]} + Q_\mu Q_\nu - \frac{1}{2}g_{\mu\nu}g^{\eta\tau}(Q^{\rho}_{[\lambda\eta]}Q^{\lambda}_{[\rho\tau]} + Q_\eta Q_\tau)\right\} \tag{1}$$

where $Q^{\mu}_{[\nu\lambda]}$ is the gravitational field strength tensor

$$Q^{\mu}_{[\nu\lambda]} = Q^{\mu}_{\nu\lambda} - Q^{\mu}_{\lambda\nu} \tag{2}$$

There are nine independent components $Q^{\mu}_{[\nu\lambda]}$, namely,

$$Q^{0}_{[0i]} = Q^{0}_{0i} = \frac{1}{2}g^{00}\partial_i g_{00} \qquad\qquad Q^{i}_{[j0]} = Q^{i}_{j0} = \frac{1}{2}g^{in}\partial_0 g_{nj} \tag{3}$$

Q_μ is defined by $Q_\mu \equiv Q^{\lambda}_{[\lambda\mu]}$, with components

$$Q_0 = Q^{n}_{n0} = \frac{1}{2}g^{mn}\partial_0 g_{mn} \qquad\qquad Q_i = Q^{0}_{0i} = \frac{1}{2}g^{00}\partial_i g_{00} \tag{4}$$

The FLRW[1] metric is [3]

$$ds^2 = (dx^0)^2 - \frac{R^2(t)}{r_0^2(1+kr^2/4r_0^2)^2}\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right) \tag{5}$$

where $k = -1, 0,$ or $+1$. Since $g_{00} = 1$, all $Q^{0}_{0i} = 0$. This leaves only energy and stress components

[1]Friedmann, LeMaitre, Robertson, Walker

$$
\begin{aligned}
T_{00}^{(g)} &= \frac{c^4}{16\pi G}\left(Q^m_{n0}Q^n_{m0} + Q_0Q_0\right) &(6)\\
T_{0i}^{(g)} &= 0 &(7)\\
T_{ij}^{(g)} &= -\frac{c^4}{16\pi G}\, g_{ij}\left(Q^m_{n0}Q^n_{m0} + Q_0Q_0\right) = -g_{ij}T_{00}^{(g)} &(8)
\end{aligned}
$$

A straightforward calculation yields

$$
Q^m_{n0}Q^n_{m0} = 3\,\frac{\dot{R}^2}{R^2} \qquad\qquad Q_0Q_0 = 9\,\frac{\dot{R}^2}{R^2} \tag{9}
$$

where $\dot{R}$ is the derivative with respect to $x^0 = ct$. Therefore, the non-zero components of the mixed energy tensor are

$$
T_0^{(g)0} = \frac{3c^4}{4\pi G}\frac{\dot{R}^2}{R^2} \qquad\qquad T_i^{(g)j} = -\delta_i^{\ j}\frac{3c^4}{4\pi G}\frac{\dot{R}^2}{R^2} \tag{10}
$$

The stresses are compressive and correspond to an equation of state

$$
p_g = \rho_g c^2 \tag{11}
$$

The gravitational energy and pressure are functions of time alone. They are independent of the curvature index k.

3. Conservation of Energy.

The gravitational field equations [3]

$$
R_\mu^{\ \nu} - \frac{1}{2}\delta_\mu^{\ \nu}R + \Lambda\delta_\mu^{\ \nu} = -\frac{8\pi G}{c^4}T_\mu^{(m)\nu} \tag{12}
$$

with the material energy tensor

$$
T_\mu^{(m)\nu} = \begin{pmatrix} \rho_m c^2 & & & \\ & -p_m & & 0 \\ 0 & & -p_m & \\ & & & -p_m \end{pmatrix} \tag{13}
$$

yield the Friedmann equations

$$
\begin{aligned}
\frac{3\dot{R}^2}{R^2} + \frac{3k}{R^2} - \Lambda &= \frac{8\pi G}{c^4}\,\rho_m c^2 &(14)\\
\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} - \Lambda &= -\frac{8\pi G}{c^4}\,p_m &(15)
\end{aligned}
$$

At the present time, the material pressure $p_m \ll \rho_m c^2$. It will be ignored for the remainder of the paper ($p_m = 0$). Equation (15) then becomes

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} - \Lambda = 0 \tag{16}$$

Eliminate k and $\dot{R}^2$ to find

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G}{3c^2}\,\rho_m \tag{17}$$

It follows that a positive cosmological constant is required, if $\ddot{R} > 0$.

The differential law of energy and momentum conservation is (appendix)

$$\mathrm{div}\, T_\mu{}^\nu = T_\mu{}^\nu{}_{;\nu} + Q^\beta_{[\alpha\mu]} T_\beta{}^\alpha = 0 \tag{18}$$

where $T_\mu{}^\nu{}_{;\nu}$ is the covariant derivative. The total density of energy, momentum, and stress is given by

$$T_\mu{}^\nu = T_\mu^{(g)\nu} + T_\mu^{(m)\nu} + T_\mu^{(\Lambda)\nu} \tag{19}$$

The final term is implied by the cosmological constant in the field equations. It is yet to be determined, but it must have the form $T_\mu^{(\Lambda)\nu} = C\delta_\mu{}^\nu$. The material equations of motion give

$$T^{(m)\nu}_{\mu\,;\nu} = 0 \tag{20}$$

so that energy conservation is expressed by

$$\begin{aligned} \mathrm{div}\, T_0{}^\nu &= T^{(g)\nu}_{0\,;\nu} + T^{(\Lambda)\nu}_{0\,;\nu} + Q^j_{[i0]}(T^{(g)} + T^{(m)} + T^{(\Lambda)})^i_j \\ &= \partial_0 T_0^{(g)0} + \Gamma^n_{n0} T_0^{(g)0} + \Gamma^n_{n0} T_0^{(\Lambda)0} = 0 \end{aligned} \tag{21}$$

Substitute (10) to find

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{4\pi G}{c^4} T_0^{(\Lambda)0} = 0 \tag{22}$$

Comparison with (16) shows that $k = 0$ and

$$T_\mu^{(\Lambda)\nu} = -\frac{c^4}{4\pi G}\Lambda\delta_\mu{}^\nu \tag{23}$$

This energy tensor corresponds to an equation of state

$$p_\Lambda = -\rho_\Lambda c^2 \tag{24}$$

In equation (14), set $k = 0$ and rearrange to find

$$\Lambda = \frac{3\dot{R}^2}{R^2} - \frac{8\pi G}{c^2}\,\rho_m \tag{25}$$

Substitution into (19) gives

$$\begin{aligned} T_0{}^0 &= \frac{3c^4}{4\pi G}\frac{\dot{R}^2}{R^2} + \rho_m c^2 - \frac{c^4}{4\pi G}\,\Lambda \\ &= 3\rho_m c^2 \end{aligned} \tag{26}$$

Therefore, the energy density of the Universe is always positive.

4. Concluding Remarks.

Formula (25) makes possible an evaluation of the constant Λ, in terms of the mass density and the Hubble ratio

$$\frac{H}{c} = \frac{\dot{R}}{R} \tag{27}$$

The experimental value of the Hubble constant is stated to be

$$H_0 = 71\,\frac{\text{km-s}^{-1}}{\text{Mpc}} = 2.3 \times 10^{-18}\ \text{s}^{-1} \tag{28}$$

For historical reasons, the mass density is expressed in terms of a "critical density"

$$\rho_{cr} = \frac{3H_0^2}{8\pi G} = 9.5 \times 10^{-30}\,\text{g-cm}^{-3} \tag{29}$$

The mass density, including the missing mass, is estimated to be

$$\rho_0 = 0.27\,\rho_{cr} = 2.6 \times 10^{-30}\,\text{g-cm}^{-3} \tag{30}$$

Substitution into (25) yields a positive cosmological constant

$$\begin{aligned} \Lambda &= \frac{8\pi G}{c^2}\left(\frac{3H_0^2}{8\pi G} - \rho_0\right) \\ &= \frac{8\pi G}{c^2}(0.73\,\rho_{cr}) = 1.3 \times 10^{-56}\,\text{cm}^{-2} \end{aligned} \tag{31}$$

According to formula (17), the acceleration will be positive, if

$$\rho_m < \frac{c^2}{4\pi G}\,\Lambda = 1.46\,\rho_{cr} = 1.4 \times 10^{-29}\,\text{g-cm}^{-3} \tag{32}$$

Therefore, the acceleration $\ddot{R}$ is now positive and will remain so in the future. It was apparently negative at times in the distant past. The total pressure is

$$p = p_g + p_\Lambda = \frac{3H^2c^2}{4\pi G} + \frac{c^4}{4\pi G}\,\Lambda \tag{33}$$

which is positive.

The gravitational field equations (12) are derived by variation of the action

$$\delta \int \Big\{ \frac{c^4}{16\pi G}\Big(g^{\mu\nu}R_{\mu\nu} - 2\Lambda\Big) + L^{(m)} \Big\} \sqrt{-g}\, d^4x = 0 \tag{34}$$

The final term defines the material energy tensor

$$\delta \int L^{(m)} \sqrt{-g}\, d^4x = \frac{1}{2}\int T^{(m)}_{\mu\nu}\,\delta g^{\mu\nu} \sqrt{-g}\, d^4x \tag{35}$$

This statement is true of the perfect fluid, the Maxwell field, the Dirac field, etc. In contrast, the gravitational energy tensor (1) has been constructed from the field strength tensor $Q^{\mu}_{[\nu\lambda]}$. The coefficient in $T^{(g)}_{\mu\nu}$ is such that it reduces to the Newtonian stress-energy tensor

$$T^{(g)}_{00} = \frac{1}{8\pi G}(\nabla\psi)^2 \tag{36}$$

$$T^{(g)}_{0i} = 0 \tag{37}$$

$$T^{(g)}_{ij} = \frac{1}{4\pi G}\Big\{\partial_i\psi\,\partial_j\psi - \frac{1}{2}\delta_{ij}(\nabla\psi)^2\Big\} \tag{38}$$

when

$$g_{00} = 1 + \frac{2}{c^2}\psi \tag{39}$$

The cosmological tensor $T^{(\Lambda)}_{\mu\nu}$ has been found by appealing to the conservation law. The suggestion is that $T^{(\Lambda)}_{\mu\nu}$ should not be regarded as a separate entity, but as part of the gravitational energy tensor, just as the cosmological action is part of the gravitational action.

Appendix: Conservation of Energy and Momentum.

The differential law of conservation is derived by summing the invariant expression $e_\mu T^{\mu\nu}\, dV_\nu$

$$\sum_{\delta R} e_\mu T^{\mu\nu}\, dV_\nu = \Big\{ e_\mu\, \partial_\nu(\sqrt{-g}\, T^{\mu\nu}) + (\nabla_\nu e_\mu)\sqrt{-g}\, T^{\mu\nu} \Big\} d^4x \tag{40}$$

The region δR is closed and infinitesimal, while dV_ν is the vector

$$dV_\nu = \sqrt{-g}\, (dx^1 dx^2 dx^3,\, dx^0 dx^2 dx^3,\, \ldots) \tag{41}$$

By definition, $\nabla_\nu e_\mu = e_\lambda Q^\lambda_{\mu\nu}$, so that

$$\sum_{\delta R} e_\mu T^{\mu\nu}\, dV_\nu = e_\mu \Big\{ \frac{1}{\sqrt{-g}} \partial_\nu(\sqrt{-g}\, T^{\mu\nu}) + Q^\mu_{\lambda\nu} T^{\lambda\nu} \Big\} \sqrt{-g}\, d^4x \tag{42}$$

Energy and momentum are conserved, if

$$\operatorname{div} T^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu(\sqrt{-g}\, T^{\mu\nu}) + Q^\mu_{\lambda\nu} T^{\lambda\nu} = 0 \tag{43}$$

The $Q^\mu_{\lambda\nu}$ are related to the Christofel coefficients $\Gamma^\mu_{\lambda\nu}$ by the formula

$$Q^\mu_{\lambda\nu} = \Gamma^\mu_{\lambda\nu} + g^{\mu\alpha} g_{\nu\beta} Q^\beta_{[\lambda\alpha]} \tag{44}$$

Therefore, the divergence may be written in the form

$$\operatorname{div} T^{\mu\nu} = T^{\mu\nu}_{\ \ ;\nu} + g^{\mu\alpha} Q^\beta_{[\lambda\alpha]} T_\beta{}^\lambda \tag{45}$$

where $T^{\mu\nu}_{\ \ ;\nu}$ is the (contracted) covariant derivative. Similarly, the divergence of the mixed energy tensor is

$$\operatorname{div} T_\mu{}^\nu = T_\mu{}^\nu{}_{;\nu} + Q^\beta_{[\alpha\mu]} T_\beta{}^\alpha \tag{46}$$

References.

1. K. Dalton, "Gravity and the Fermion Mass", *Nuovo Cim.* 124B(2009)1175.

2. K. Dalton, "Gravitational Waves", to be published in the *Hadronic Journal.*

3. R. Adler, M. Bazin, M. Schiffer, *Introduction to General Relativity*, (McGraw-Hill, New York, 2nd ed., 1975) chap. 12, 13.