Inconsistent countable set in second order ZFC and not existence of the strongly inaccessible cardinals.

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Abstract: In this article we derived an importent example of the inconsistent countable set in second order ZFC (ZFC_2) with the full second-order semantic. Main results is:(i) $\neg Con(ZFC_2)$, (ii) let k be an inaccessible cardinal and H_k is a set of all sets having hereditary size less then k, then $\neg Con(ZFC + (V = H_k))$.
Keywords: Gödel encoding, Completion of ZFC_2, Russell's paradox, ω-model, Henkin semantics, full second-order semantic

1.Introduction.

Let's remind that accordingly to naive set theory, any definable collection is a set. Let R be the set of all sets that are not members of themselves. If R qualifies as a member of itself, it would contradict its own definition as a set containing all sets that are not members of themselves. On the other hand, if such a set is not a member of itself, it would qualify as a member of itself by the same definition. This contradiction is Russell's paradox. In 1908, two ways of avoiding the paradox were proposed, Russell's type theory and the Zermelo set theory, the first constructed axiomatic set theory. Zermelo's axioms went well beyond Frege's axioms of extensionality and unlimited set abstraction, and evolved into the now-canonical Zermelo–Fraenkel set theory *ZFC*. *"But how do we know that ZFC is a consistent theory, free of contradictions? The short answer is that we don't; it is a matter of faith (or of skepticism)"*— E.Nelson wrote in his not published paper [1]. However, it is deemed unlikely that even ZFC_2 which is a very stronger than ZFC harbors an unsuspected contradiction; it is widely believed that if ZFC_2 were inconsistent, that fact would have been uncovered by now. This much is certain — ZFC_2 is immune to the classic paradoxes of naive set theory: Russell's paradox, the Burali-Forti paradox, and Cantor's paradox.

Remark 1.1. Note that in this paper we view the second order set theory ZFC_2 under the Henkin semantics [2],[3] and under the full second-order semantics

[4],[5].Thus we interpret the wff's of ZFC_2 language with the full second-order semantics as required in [4],[5].

Designation 1.1. We will be denote by ZFC_2^{Hs} set theory ZFC_2 with the Henkin semantics and we will be denote by ZFC_2^{fss} set theory ZFC_2 with the full second-order semantics.

Remark 1.2. There is no completeness theorem for second-order logic with the full second-order semantics. Nor do the axioms of ZFC_2^{fss} imply a reflection principle which ensures that if a sentence Z of second-order set theory is true, then it is true in some (standard or nonstandard) model $M^{ZFC_2^{fss}}$ of ZFC_2^{fss} [2]. Let Z be the conjunction of all the axioms of ZFC_2^{fss}. We assume now that: Z is true,i.e. $Con\left(ZFC_2^{fss}\right)$. It is known that the existence of a model for Z requires the existence of strongly inaccessible cardinals, i.e. under ZFC it can be shown that κ is a strongly inaccessible if and only if $(H_\kappa, \in)$ is a model of ZFC_2^{fss}. Thus $\neg Con(ZFC_2^{fss} + \exists M^{ZFC_2^{fss}}) \Rightarrow \neg Con(ZFC + (V = H_k))$. In this paper we prove that ZFC_2^{fss} is inconsistent. We will start from a simple naive consideration.Let $\Im$ be the countable collection of all sets X such that $ZFC_2^{fss} \vdash \exists! X\Psi(X)$, where $\Psi(X)$ is any 1-place open wff i.e.,

$$\forall Y\{Y \in \Im \leftrightarrow \exists \Psi(\cdot)\exists! X[\Psi(X) \wedge Y = X]\}. \tag{1.1}$$

Let $X \notin_{\vdash_{ZFC_2^{fss}}} Y$ be a predicate such that $X \notin_{\vdash_{ZFC_2^{fss}}} Y \leftrightarrow ZFC_2^{fss} \vdash X \notin Y$. Let $\Re$ be the countable collection of all sets such that

$$\forall X\left[X \in \Re \leftrightarrow X \notin_{\vdash_{ZFC_2^{fss}}} X\right]. \tag{1.2}$$

From (1.2) one obtain

$$\Re \in \Re \leftrightarrow \Re \notin_{\vdash_{ZFC_2^{fss}}} \Re. \tag{1.3}$$

But obviously this is a contradiction. However contradiction (1.3) it is not a contradiction inside ZFC_2^{fss} for the reason that predicate $X \notin_{\vdash_{ZFC_2^{fss}}} Y$ not is a predicate of ZFC_2^{fss} and therefore countable collections $\Im$ and $\Re$ not is a sets of ZFC_2^{fss}. Nevertheless by using Gödel encoding the above stated contradiction can be shipped in special consistent completion of ZFC_2^{fss}.

Remark 1.3. We note that in order to deduce $\sim Con(ZFC_2^{Hs})$ from $Con(ZFC_2^{Hs})$ by using Gödel encoding, one needs something more than the consistency of ZFC_2^{Hs}, e.g., that ZFC_2^{Hs} has an omega-model $M_{\omega}^{ZFC_2^{Hs}}$ or an standard model $M_{\mathbf{st}}^{ZFC_2^{Hs}}$ i.e., a model in which the *integers are the standard integers* [6].To put it another way, why should we believe a statement just because there's a ZFC_2^{Hs}-proof of it? It's clear that if ZFC_2^{Hs} is inconsistent, then we won't believe ZFC_2^{Hs}-proofs. What's slightly more subtle is that the mere consistency of ZFC_2 isn't quite enough to get us to believe arithmetical theorems of ZFC_2^{Hs}; we must also believe that these arithmetical theorems are asserting something about the standard naturals. It is "conceivable" that ZFC_2^{Hs} might be consistent but that the only nonstandard models $M_{\mathbf{Nst}}^{ZFC_2^{Hs}}$ it has are those in which the integers are nonstandard, in which case we might not "believe" an arithmetical statement such as "ZFC_2^{Hs} is inconsistent" even if there is a ZFC_2^{Hs}-proof of it.

Remark 1.4. However assumption $\exists M_{\mathbf{st}}^{ZFC_2^{Hs}}$ is not necessary. Note thatin any nonstandard model $M_{\mathbf{Nst}}^{Z_2^{Hs}}$ of the second-order arithmetic Z_2^{Hs} the terms $\bar{0}$, $\mathbf{S}\bar{0} = \bar{1}, \mathbf{SS}\bar{0} = \bar{2}, \ldots$ comprise the initial segment isomorphic to $M_{\mathbf{st}}^{Z_2^{Hs}} \subset M_{\mathbf{Nst}}^{Z_2^{Hs}}$. This initial segment is called the standard cut of the $M_{\mathbf{Nst}}^{Z_2^{Hs}}$. The order type of any nonstandard model of $M_{\mathbf{Nst}}^{Z_2^{Hs}}$ is equal to $\mathbb{N} + A \times \mathbb{Z}$ for some linear order A [6],[7]. Thus one can to choose Gödel encoding inside $M_{\mathbf{st}}^{Z_2^{Hs}}$.

Remark 1.5. However there is no any problem as mentioned above in second order set theory ZFC_2 with the full second-order semantics becouse corresponding second order arithmetic Z_2^{fss} is categorical.

Remark 1.6. Note if we view second-order arithmetic Z_2 as a theory in first-order predicate calculus. Thus a model M^{Z_2} of the language of second-order arithmetic Z_2 consists of a set M (which forms the range of individual variables) together with a constant 0 (an element of M), a function S from M to M, two binary operations $+$ and $\times$ on M, a binary relation $<$ on M, and a collection D of subsets of M, which is the range of the set variables. When D is the full powerset of M, the model M^{Z_2} is called a full model. The use of full second-order semantics is equivalent to limiting the models of second-order arithmetic to the full models. In fact, the axioms of second-order arithmetic have only one full model. This follows from the fact that the axioms of Peano arithmetic with the second-order induction axiom have only one model under second-order semantics, i.e. Z_2, with the full semantics, is categorical by Dedekind's argument, so has only one model up to isomorphism. When M is the usual set of natural numbers with its usual operations, M^{Z_2} is called an ω-model. In this case we may identify the model with D, its collection of sets of naturals, because this set is enough to completely determine an ω-model. The unique full omega-model $M_{\omega}^{Z_2^{fss}}$, which is the usual set of natural numbers with its usual structure and all its subsets, is called the intended or standard model of second-order arithmetic.

We assume that: (i) $Con(ZFC_2^{fss})$,(ii) $Con(ZFC_2^{Hs} + \exists(\omega\text{-model of } ZFC_2^{Hs}))$.

Main result is: $\sim Con(ZFC_2^{Hs} + \exists(\omega\text{-model of } ZFC_2^{Hs}))$, $\sim Con(ZFC_2^{fss})$.

2.Derivation inconsistent countable set in

$ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$.

Let **Th** be some fixed, but unspecified, consistent formal theory. For later convenience, we assume that the encoding is done in some fixed formal second order theory **S** and that **Th** contains **S**.The sense in which **S** is contained in **Th** is better exemplified than explained: if **S** is a formal system of a second order arithmetic Z_2^{Hs} and **Th** is, say, ZFC_2^{Hs}, then **Th** contains **S** in the sense that there is a well-known embedding, or interpretation, of **S** in **Th**. Since encoding is to take place in **S**, it will have to have a large supply of constants and closed terms to be used as codes. (e.g. in formal arithmetic, one has $\bar{0}, \bar{1}, \ldots$.) **S** will also have certain function symbols to be described shortly.To each formula, Φ, of the language of **Th** is assigned a closed term, $[\Phi]^c$, called the code of Φ. We note that if $\Phi(x)$ is a formula with free variable x, then $[\Phi(x)]^c$ is a closed term encoding the formula $\Phi(x)$ with x viewed as a syntactic object and not as a parameter. Corresponding to the logical connectives and quantifiers are function symbols, $neg(\bullet)$, $imp(\bullet)$, etc., such that, for all formulae Φ, Ψ : $\mathbf{S} \vdash neg([\Phi]^c) = [\neg\Phi]^c$, $\mathbf{S} \vdash imp([\Phi]^c, [\Psi]^c) = [\Phi \to \Psi]^c$ etc. Of particular importance is the substitution operator, represented by the function symbol $sub(\bullet, \bullet)$. For formulae $\Phi(x)$, terms t with codes $[t]^c$:

$$\mathbf{S} \vdash sub([\Phi(x)]^c, [t]^c) = [\Phi(t)]^c. \tag{2.1}$$

It well known [8] that one can also encode derivations and have a binary relation $\mathbf{Prov_{Th}}(x,y)$ (read "x proves y " or "x is a proof of y") such that for closed t_1, t_2 : $\mathbf{S} \vdash \mathbf{Prov_{Th}}(t_1, t_2)$ iff t_1 is the code of a derivation in **Th** of the formula with code t_2 . It follows that

$$\mathbf{Th} \vdash \Phi \text{ iff } \mathbf{S} \vdash \mathbf{Prov_{Th}}(t, [\Phi]^c) \tag{2.2}$$

for some closed term t. Thus one can define

$$\mathbf{Pr_{Th}}(y) \leftrightarrow \exists x \mathbf{Prov_{Th}}(x,y), \tag{2.3}$$

and therefore one obtain a predicate asserting provability. We note that is not always the case that [8]:

$$\mathbf{Th} \vdash \Phi \text{ iff } \mathbf{S} \vdash \mathbf{Pr_{Th}}([\Phi]^c). \tag{2.4}$$

It well known [8] that the above encoding can be carried out in such a way that the following important conditions $\mathbf{D}1, \mathbf{D}2$ and $\mathbf{D}3$ are meet for all sentences [8]:

$$\mathbf{D}1.\,\mathbf{Th} \vdash \Phi \text{ implies } \mathbf{S} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c),$$

$$\mathbf{D}2.\,\mathbf{S} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c) \to \mathbf{Pr}_{\mathbf{Th}}([\mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)]^c), \tag{2.5}$$

$$\mathbf{D}3.\,\mathbf{S} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c) \wedge \mathbf{Pr}_{\mathbf{Th}}([\Phi \to \Psi]^c) \to \mathbf{Pr}_{\mathbf{Th}}([\Psi]^c).$$

Conditions $\mathbf{D}1, \mathbf{D}2$ and $\mathbf{D}3$ are called the Derivability Conditions.

Lemma 2.1. Assume that: (i) $Con(\mathbf{Th})$ and (ii) $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$, where Φ is a closed formula.Then $\mathbf{Th} \nvdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi]^c)$.

Proof. Let $Con_{\mathbf{Th}}(\Phi)$ be a formula

$$Con_{\mathbf{Th}}(\Phi) \triangleq \forall t_1 \forall t_2 \neg[\mathbf{Prov}_{\mathbf{Th}}(t_1, [\Phi]^c) \wedge \mathbf{Prov}_{\mathbf{Th}}(t_2, neg([\Phi]^c))] \leftrightarrow$$
$$\neg\exists t_1 \neg\exists t_2 [\mathbf{Prov}_{\mathbf{Th}}(t_1, [\Phi]^c) \wedge \mathbf{Prov}_{\mathbf{Th}}(t_2, neg([\Phi]^c))]. \tag{2.6}$$

where t_1, t_2 is a closed term. We note that $\mathbf{Th} + Con(\mathbf{Th}) \vdash Con_{\mathbf{Th}}(\Phi)$ for any closed Φ. Suppose that $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi]^c)$, then (ii) gives

$$\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c) \wedge \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi]^c). \tag{2.7}$$

From (2.3) and (2.7) we obtain

$$\exists t_1 \exists t_2 [\mathbf{Prov}_{\mathbf{Th}}(t_1, [\Phi]^c) \wedge \mathbf{Prov}_{\mathbf{Th}}(t_2, neg([\Phi]^c))]. \tag{2.8}$$

But the formula (2.6) contradicts the formula (2.8). Therefore $\mathbf{Th} \nvdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi]^c)$.

Lemma 2.2. Assume that: (i) $Con(\mathbf{Th})$ and (ii) $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi]^c)$, where Φ is a closed formula.Then $\mathbf{Th} \nvdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$.

Assumption 2.1. Let $\mathbf{Th}$ be an second order theory with the Henkin semantics. We assume now that:

(i) the language of $\mathbf{Th}$ consists of:

numerals $\overline{0}, \overline{1}$,...

countable set of the numerical variables: $\{v_0, v_1, \ldots\}$

countable set $\mathcal{F}$ of the set variables: $\mathcal{F} = \{x, y, z, X, Y, Z, \Re, \ldots\}$

countable set of the n-ary function symbols: $f_0^n, f_1^n, \ldots$

countable set of the n-ary relation symbols: $R_0^n, R_1^n, \ldots$
connectives: $\neg, \rightarrow$
quantifier:$\forall$.
(ii) $\mathbf{Th}$ contains ZFC_2,
(iii) $\mathbf{Th}$ has an an ω-model $M_\omega^{\mathbf{Th}}$ or
(iv) $\mathbf{Th}$ has an nonstandard model $M_{Nst.}^{\mathbf{Th}}$.

Definition 2.1. An $\mathbf{Th}$-wff Φ (well-formed formula Φ) is closed - i.e. Φ is a sentence - if it has no free variables; a wff is open if it has free variables.We'll use the slang 'k-place open wff ' to mean a wff with k distinct free variables.

Definition 2.2.We said that,$\mathbf{Th}^\#$ is a nice theory or a nice extension of the $\mathbf{Th}$ iff

(i) $\mathbf{Th}^\#$ contains $\mathbf{Th}$;(ii) Let Φ be any closed formula, then $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$ implies $\mathbf{Th}^\# \vdash \Phi$.

Definition 2.3.We said that,$\mathbf{Th}^\#$ is a maximally nice theory or a maximally nice extension of the $\mathbf{Th}$ iff $\mathbf{Th}^\#$ is consistent and for any consistent nice extension $\mathbf{Th}'$ of the $\mathbf{Th}$: $\mathbf{Ded}(\mathbf{Th}^\#) \subseteq \mathbf{Ded}(\mathbf{Th}')$ implies $\mathbf{Ded}(\mathbf{Th}^\#) = \mathbf{Ded}(\mathbf{Th}')$.

Remark 2.1. We note that a theory $\mathbf{Th}^\#$ depend on model $M_\omega^{\mathbf{Th}}$ or $M_{Nst.}^{\mathbf{Th}}$, i.e. $\mathbf{Th}^\# = \mathbf{Th}^\#[M_\omega^{\mathbf{Th}}]$ or $\mathbf{Th}^\# = \mathbf{Th}^\#[M_{\mathbf{Nst}}^{\mathbf{Th}}]$ correspondingly. We will consider the case $\mathbf{Th}^\# \triangleq \mathbf{Th}^\#[M_\omega^{\mathbf{Th}}]$ without loss of generality.

Proposition 2.1. Assume that (i) $Con(\mathbf{Th})$ and (ii) $\mathbf{Th}$ has an ω-model $M_\omega^{\mathbf{Th}}$.Then theory $\mathbf{Th}$ can be extended to a maximally consistent nice theory $\mathbf{Th}^\# \triangleq \mathbf{Th}^\#[M_\omega^{\mathbf{Th}}]$.

Proof. Let $\Phi_1 \ldots \Phi_i \ldots$ be an enumeration of all wff's of the theory $\mathbf{Th}$ (this can be achieved if the set of propositional variables can be enumerated). Define a chain $\wp = \{\mathbf{Th}_i | i \in \mathbb{N}\}, \mathbf{Th}_1 = \mathbf{Th}$ of consistent theories inductively as follows: assume that theory $\mathbf{Th}_i$ is defined. (i) Suppose that a statement (2.9) is satisfied

$$\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \text{ and } [\mathbf{Th}_i \nvdash \Phi_i] \wedge [M_\omega^{\mathbf{Th}} \models \Phi_i]. \tag{2.9}$$

Then we define a theory $\mathbf{Th}_{i+1}$ as follows $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\Phi_i\}$.Using Lemma 2.1 we will rewrite the condition (2.9) symbolically as follows

$$\begin{gathered}\mathbf{Th}_i \vdash \mathbf{Pr}^\#_{\mathbf{Th}_i}([\Phi_i]^c),\\ \mathbf{Pr}^\#_{\mathbf{Th}_i}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \wedge [M_\omega^{\mathbf{Th}} \models \Phi_i].\end{gathered} \tag{2.10}$$

(ii) Suppose that a statement (2.11) is satisfied

$$\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \text{ and } [\mathbf{Th}_i \nvdash \neg\Phi_i] \wedge [M_\omega^{\mathbf{Th}} \models \neg\Phi_i]. \tag{2.11}$$

Then we define theory $\mathbf{Th}_{i+1}$ as follows: $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\neg\Phi_i\}$. Using Lemma 2.2 we will rewrite the condition (2.11) symbolically as follows

$$\mathbf{Th}_i \vdash \mathbf{Pr}^{\#}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \,,$$

$$\mathbf{Pr}^{\#}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \wedge [M^{\mathbf{Th}}_{\omega} \models \neg\Phi_i]. \tag{2.12}$$

(iii) Suppose that a statement (2.13) is satisfied

$$\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \text{ and } \mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \Rightarrow \Phi_i. \tag{2.13}$$

We will rewrite the condition (2.13) symbolically as follows

$$\mathbf{Th}_i \vdash \mathbf{Pr}^{*}_{\mathbf{Th}_i}([\Phi_i]^c)$$

$$\mathbf{Pr}^{*}_{\mathbf{Th}_i}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \Rightarrow \Phi_i] \tag{2.14}$$

Then we define a theory $\mathbf{Th}_{i+1}$ as follows: $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i$.

(iv) Suppose that a statement (2.15) is satisfied

$$\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \text{ and } \mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \Rightarrow \neg\Phi_i. \tag{2.15}$$

We will rewrite the condition (2.15) symbolically as follows

$$\mathbf{Th}_i \vdash \mathbf{Pr}^{*}_{\mathbf{Th}_i}([\Phi_i]^c)$$

$$\mathbf{Pr}^{*}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \Rightarrow \neg\Phi_i] \tag{2.16}$$

Then we define a theory $\mathbf{Th}_{i+1}$ as follows: $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i$. We define now a theory $\mathbf{Th}^{\#}$ as follows:

$$\mathbf{Th}^{\#} \triangleq \bigcup_{i \in \mathbb{N}} \mathbf{Th}_i. \tag{2.17}$$

First, notice that each $\mathbf{Th}_i$ is consistent. This is done by induction on i and by Lemmas 2.1-2.2. By assumption, the case is true when $i = 1$. Now, suppose $\mathbf{Th}_i$ is consistent. Then its deductive closure $\mathbf{Ded}(\mathbf{Th}_i)$ is also consistent. If a statement (2.14) is satisfied,i.e. $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi_i]^c)$ and $\mathbf{Th} \vdash \Phi_i$, then clearly $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\Phi_i\}$ is consistent since it is a subset of closure $\mathbf{Ded}(\mathbf{Th}_i)$. If a statement (2.15) is satisfied,i.e. $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi_i]^c)$ and $\mathbf{Th} \vdash \neg\Phi_i$, then clearly $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\neg\Phi_i\}$ is consistent since it is a subset of closure $\mathbf{Ded}(\mathbf{Th}_i)$. Otherwise:(i) if a statement (2.9) is satisfied,i.e. $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi_i]^c)$ and $[\mathbf{Th}_i \nvdash \Phi_i] \wedge [M_\omega^{\mathbf{Th}} \models \Phi_i]$ then clearly $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\Phi_i\}$ is consistent by Lemma 2.1 and by one of the standard properties of consistency: $\Delta \cup \{A\}$ is consistent iff $\Delta \nvdash \neg A$; (ii) if a statement (2.11) is satisfied,i.e. $\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi_i]^c)$ and $[\mathbf{Th}_i \nvdash \neg\Phi_i] \wedge [M_\omega^{\mathbf{Th}} \models \neg\Phi_i]$ then clearly $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\neg\Phi_i\}$ is consistent by Lemma 2.2 and by one of the standard properties of consistency: $\Delta \cup \{\neg A\}$ is consistent iff $\Delta \nvdash A$. Next, notice $\mathbf{Ded}(\mathbf{Th}^{\#})$ is maximally consistent nice extension of the $\mathbf{Ded}(\mathbf{Th})$. $\mathbf{Ded}(\mathbf{Th}^{\#})$ is consistent because, by the standard Lemma 2.3 belov, it is the union of a chain of consistent sets. To see that $\mathbf{Ded}(\mathbf{Th}^{\#})$ is maximal, pick any wff Φ. Then Φ is some Φ_i in the enumerated list of all wff's. Therefore for any Φ such that $\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\Phi]^c)$ or $\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi]^c)$, either $\Phi \in \mathbf{Th}^{\#}$ or $\neg\Phi \in \mathbf{Th}^{\#}$. Since $\mathbf{Ded}(\mathbf{Th}_{i+1}) \subseteqq \mathbf{Ded}(\mathbf{Th}^{\#})$, we have $\Phi \in \mathbf{Ded}(\mathbf{Th}^{\#})$ or $\neg\Phi \in \mathbf{Ded}(\mathbf{Th}^{\#})$,which implies that $\mathbf{Ded}(\mathbf{Th}^{\#})$ is maximally consistent nice extension of the $\mathbf{Ded}(\mathbf{Th})$.

Lemma 2.3. The union of a chain $\wp = \{\Gamma_i | i \in \mathbb{N}\}$ of consistent sets Γ_i, ordered by $\subseteqq$, is consistent.

Definition 2.4. We define now predicate $\mathbf{Pr}_{\mathbf{Th}^{\#}}([\Phi_i]^c)$ asserting provability in $\mathbf{Th}^{\#}$:

$$\mathbf{Pr}_{\mathbf{Th}^{\#}}([\Phi_i]^c) \Leftrightarrow [\mathbf{Pr}^{\#}_{\mathbf{Th}_i}([\Phi_i]^c)] \vee [\mathbf{Pr}^{*}_{\mathbf{Th}_i}([\Phi_i]^c)],$$

$$\mathbf{Pr}_{\mathbf{Th}^{\#}}([\neg\Phi_i]^c) \Leftrightarrow [\mathbf{Pr}^{\#}_{\mathbf{Th}_i}([\neg\Phi_i]^c)] \vee [\mathbf{Pr}^{*}_{\mathbf{Th}_i}([\neg\Phi_i]^c)]. \tag{2.18}$$

Definition 2.5. Let $\Psi = \Psi(x)$ be one-place open wff such that the conditions:

$(*)$ $\mathbf{Th} \vdash \exists! x_\Psi [\Psi(x_\Psi)]$ or

$(**)$ $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\exists! x_\Psi[\Psi(x_\Psi)]]^c)$ and $M_\omega^{\mathbf{Th}} \models \exists! x_\Psi[\Psi(x_\Psi)]$ is satisfied.

Then we said that, a set y is a $\mathbf{Th}^{\#}$-set iff there is exist one-place open wff $\Psi(x)$ such that $y = x_\Psi$. We write $y[\mathbf{Th}^{\#}]$ iff y is a $\mathbf{Th}^{\#}$-set.

Remark 2.2. Note that $[(*) \vee (**)] \Rightarrow \mathbf{Th}^{\#} \vdash \exists! x_\Psi[\Psi(x_\Psi)]$.

Remark 2.3. Note that $y[\mathbf{Th}^{\#}] \Leftrightarrow \exists\Psi[(y = x_{\Psi}) \wedge \mathbf{Pr}_{\mathbf{Th}^{\#}}([\exists!x_{\Psi}[\Psi(x_{\Psi})]]^{c})]$

Definition 2.6. Let $\Im$ be a collection such that : $\forall x\left[x \in \Im \leftrightarrow x \text{ is a } \mathbf{Th}^{\#}\text{-set}\right]$.

Proposition 2.2. Collection $\Im$ is a $\mathbf{Th}^{\#}$-set.

Proof. Let us consider an one-place open wff $\Psi(x)$ such that conditions $(*)$ or $(**)$ is satisfied, i.e. $\mathbf{Th}^{\#} \vdash \exists!x_{\Psi}[\Psi(x_{\Psi})]$. We note that there exists countable collection $\mathcal{F}_{\Psi}$ of the one-place open wff's $\mathcal{F}_{\Psi} = \{\Psi_n(x)\}_{n\in\mathbb{N}}$ such that: (i) $\Psi(x) \in \mathcal{F}_{\Psi}$ and (ii)

$$\mathbf{Th} \vdash \exists!x_{\Psi}[[\Psi(x_{\Psi})] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi})]\}]$$

or

$$\mathbf{Th} \vdash \exists!x_{\Psi}[\mathbf{Pr}_{\mathbf{Th}}([\Psi(x_{\Psi})]^{c}) \wedge \{\forall n(n \in \mathbb{N})\mathbf{Pr}_{\mathbf{Th}}([\Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi})]^{c})\}] \tag{2.19}$$

and

$$M_{\omega}^{\mathbf{Th}} \models \exists!x_{\Psi}[[\Psi(x_{\Psi})] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi})]\}]$$

or of the equivalent form

$$\mathbf{Th} \vdash \exists!x_1[[\Psi_1(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]\}]$$

or

$$\mathbf{Th} \vdash \exists!x_{\Psi}[\mathbf{Pr}_{\mathbf{Th}}([\Psi(x_1)]^{c}) \wedge \{\forall n(n \in \mathbb{N})\mathbf{Pr}_{\mathbf{Th}}([\Psi(x_1) \leftrightarrow \Psi_n(x_1)]^{c})\}] \tag{2.20}$$

and

$$M_{\omega}^{\mathbf{Th}} \models \exists!x_{\Psi}[[\Psi(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]\}]$$

where we set $\Psi(x) = \Psi_1(x_1), \Psi_n(x_1) = \Psi_{n,1}(x_1)$ and $x_{\Psi} = x_1$. We note that any collection $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n\in\mathbb{N}}, k = 1,2,\ldots$ such above defines an unique set x_{Ψ_k}, i.e. $\mathcal{F}_{\Psi_{k_1}} \bigcap \mathcal{F}_{\Psi_{k_2}} = \varnothing$ iff $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$. We note that collections $\mathcal{F}_{\Psi_k}, k = 1,2,..$ is no part of the ZFC_2, i.e. collection $\mathcal{F}_{\Psi_k}$ there is no set in sense of ZFC_2. However that is no problem, because by using Gödel numbering one can to replace any collection $\mathcal{F}_{\Psi_k}, k = 1,2,..$ by collection $\Theta_k = g(\mathcal{F}_{\Psi_k})$ of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n\in\mathbb{N}}, k = 1,2,\ldots\ . \tag{2.21}$$

It is easy to prove that any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}^{\#}$-set.This is done by Gödel encoding [8],[10] of the statament (2.19) by Proposition 2.1 and by axiom schema of separation [9]. Let $g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1,2,..$ be a Gödel number of the wff $\Psi_{n,k}(x_k)$. Therefore $g(\mathcal{F}_k) = \{g_{n,k}\}_{n\in\mathbb{N}}$, where we set $\mathcal{F}_k = \mathcal{F}_{\Psi_k}$, $k = 1,2,..$ and

$$\forall k_1 \forall k_2[\{g_{n,k_1}\}_{n\in\mathbb{N}} \bigcap \{g_{n,k_2}\}_{n\in\mathbb{N}} = \varnothing \leftrightarrow x_{k_1} \neq x_{k_2}]. \quad (2.22)$$

Let $\{\{g_{n,k}\}_{n\in\mathbb{N}}\}_{k\in\mathbb{N}}$ be a family of the all sets $\{g_{n,k}\}_{n\in\mathbb{N}}$. By axiom of choice [9] one obtain unique set $\mathfrak{I}' = \{g_k\}_{k\in\mathbb{N}}$ such that $\forall k[g_k \in \{g_{n,k}\}_{n\in\mathbb{N}}]$. Finally one obtain a set $\mathfrak{I}$ from a set $\mathfrak{I}'$ by axiom schema of replacement [9].Thus one can define a $\mathbf{Th}^{\#}$-set

$\mathfrak{R}_c \subsetneqq \mathfrak{I}$:

$$\forall x[x \in \mathfrak{R}_c \leftrightarrow (x \in \mathfrak{I}) \wedge \mathbf{Pr}_{\mathbf{Th}^{\#}}([x \notin x]^c)]. \quad (2.23)$$

Proposition 2.3. Any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}^{\#}$-set.

Proof. We define $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$. Therefore $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$ (see [10]). Let us define now predicate $\Pi(g_{n,k}, v_k)$

$$\Pi(g_{n,k}, v_k) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}}([\exists! x_k[\Psi_{1,k}(x_1)]]^c) \wedge$$
$$\wedge \exists! x_k(v_k = [x_k]^c)[\forall n(n \in \mathbb{N})[\mathbf{Pr}_{\mathbf{Th}}([[\Psi_{1,k}(x_k)]]^c) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}}(\mathbf{Fr}(g_{n,k}, v_k))]]. \quad (2.24)$$

We define now a set Θ_k such that

$$\Theta_k = \Theta'_k \cup \{g_k\},$$
$$\forall n(n \in \mathbb{N})[g_{n,k} \in \Theta'_k \leftrightarrow \Pi(g_{n,k}, v_k)] \quad (2.25)$$

But obviously definitions (2.19) and (2.25) is equivalent by Proposition 2.1.

Proposition 2.4. (i) $\mathbf{Th}^{\#} \vdash \exists \mathfrak{R}_c$,(ii) $\mathfrak{R}_c$ is a countable $\mathbf{Th}^{\#}$-set.

Proof.(i) Statement $\mathbf{Th}^{\#} \vdash \exists \mathfrak{R}_c$ follows immediately by using statement $\exists \mathfrak{I}$ and axiom schema of separation [4]. (ii) follows immediately from countability of a set $\mathfrak{I}$.

Proposition 2.5. A set $\mathfrak{R}_c$ is inconsistent.

Proof.From formla (2.18) one obtain

$$\mathbf{Th}^{\#} \vdash \mathfrak{R}_c \in \mathfrak{R}_c \leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}}([\mathfrak{R}_c \notin \mathfrak{R}_c]^c). \quad (2.21)$$

From formula (2.21) and Proposition 2.1 one obtain

$$\mathbf{Th}^{\#} \vdash \Re_c \in \Re_c \leftrightarrow \Re_c \notin \Re_c \tag{2.22}$$

and therefore

$$\mathbf{Th}^{\#} \vdash (\Re_c \in \Re_c) \wedge (\Re_c \notin \Re_c). \tag{2.23}$$

But this is a contradiction.

Proposition 2.6.Assume that (i) $Con(\mathbf{Th})$ and (ii) $\mathbf{Th}$ has an ω-model $M_{Nst}^{\mathbf{Th}}$. Then theory $\mathbf{Th}$ can be extended to a maximally consistent nice theory $\mathbf{Th}^{\#} \triangleq \mathbf{Th}^{\#}[M_{\mathbf{Nst}}^{\mathbf{Th}}]$.

Proof. Let $\Phi_1 \ldots \Phi_i \ldots$ be an enumeration of all wff's of the theory $\mathbf{Th}$ (this can be achieved if the set of propositional variables can be enumerated). Define a chain $\wp = \{\mathbf{Th}_i | i \in \mathbb{N}\}, \mathbf{Th}_1 = \mathbf{Th}$ of consistent theories inductively as follows: assume that theory $\mathbf{Th}_i$ is defined. (i) Suppose that a statement (2.24) is satisfied

$$\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \text{ and } [\mathbf{Th}_i \nvdash \Phi_i] \wedge [M_{Nst}^{\mathbf{Th}} \models \Phi_i]. \tag{2.24}$$

Then we define a theory $\mathbf{Th}_{i+1}$ as follows $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\Phi_i\}$. Using Lemma 2.1 we will rewrite the condition (2.24) symbolically as follows

$$\begin{gathered} \mathbf{Th}_i \vdash \mathbf{Pr}^{\#}_{\mathbf{Th}_i}([\Phi_i]^c), \\ \mathbf{Pr}^{\#}_{\mathbf{Th}_i}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \wedge [M_{Nst}^{\mathbf{Th}} \models \Phi_i]. \end{gathered} \tag{2.25}$$

(ii) Suppose that a statement (2.26) is satisfied

$$\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \text{ and } [\mathbf{Th}_i \nvdash \neg\Phi_i] \wedge [M_{Nst}^{\mathbf{Th}} \models \neg\Phi_i]. \tag{2.26}$$

Then we define theory $\mathbf{Th}_{i+1}$ as follows: $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\neg\Phi_i\}$. Using Lemma 2.2 we will rewrite the condition (2.26) symbolically as follows

$$\mathbf{Th}_i \vdash \mathbf{Pr}^{\#}_{\mathbf{Th}_i}([\neg\Phi_i]^c)\ ,$$

$$\mathbf{Pr}^{\#}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \wedge [M^{\mathbf{Th}}_{\omega} \models \neg\Phi_i]. \tag{2.27}$$

(iii) Suppose that a statement (2.28) is satisfied

$$\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \text{ and } \mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \Rightarrow \Phi_i. \tag{2.28}$$

We will rewrite the condition (2.28) symbolically as follows

$$\mathbf{Th}_i \vdash \mathbf{Pr}^{*}_{\mathbf{Th}_i}([\Phi_i]^c)$$

$$\mathbf{Pr}^{*}_{\mathbf{Th}_i}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \Rightarrow \Phi_i] \tag{2.29}$$

Then we define a theory $\mathbf{Th}_{i+1}$ as follows: $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i$.

(iv) Suppose that a statement (2.30) is satisfied

$$\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \text{ and } \mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \Rightarrow \neg\Phi_i. \tag{2.30}$$

We will rewrite the condition (2.30) symbolically as follows

$$\mathbf{Th}_i \vdash \mathbf{Pr}^{*}_{\mathbf{Th}_i}([\Phi_i]^c)$$

$$\mathbf{Pr}^{*}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi_i]^c) \Rightarrow \neg\Phi_i] \tag{2.31}$$

Then we define a theory $\mathbf{Th}_{i+1}$ as follows: $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i$. We define now a theory $\mathbf{Th}^{\#}$ as follows:

$$\mathbf{Th}^{\#} \triangleq \bigcup_{i\in\mathbb{N}} \mathbf{Th}_i. \tag{2.32}$$

First, notice that each $\mathbf{Th}_i$ is consistent. This is done by induction on i and by Lemmas 2.1-2.2. By assumption, the case is true when $i = 1$. Now, suppose $\mathbf{Th}_i$ is consistent. Then its deductive closure $\mathbf{Ded}(\mathbf{Th}_i)$ is also consistent. If a statement (2.28) is satisfied,i.e. $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi_i]^c)$ and $\mathbf{Th} \vdash \Phi_i$, then clearly $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\Phi_i\}$ is consistent since it is a subset of closure $\mathbf{Ded}(\mathbf{Th}_i)$. If a statement (2.30) is satisfied,i.e. $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi_i]^c)$ and $\mathbf{Th} \vdash \neg\Phi_i$, then clearly $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\neg\Phi_i\}$ is consistent since it is a subset of closure $\mathbf{Ded}(\mathbf{Th}_i)$. Otherwise:(i) if a statement (2.24) is satisfied,i.e. $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi_i]^c)$ and $[\mathbf{Th}_i \nvdash \Phi_i] \wedge [M_\omega^{\mathbf{Th}} \models \Phi_i]$ then clearly $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\Phi_i\}$ is consistent by Lemma 2.1 and by one of the standard properties of consistency: $\Delta \cup \{A\}$ is consistent iff $\Delta \nvdash \neg A$; (ii) if a statement (2.26) is satisfied,i.e. $\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi_i]^c)$ and $[\mathbf{Th}_i \nvdash \neg\Phi_i] \wedge [M_\omega^{\mathbf{Th}} \models \neg\Phi_i]$ then clearly $\mathbf{Th}_{i+1} \triangleq \mathbf{Th}_i \cup \{\neg\Phi_i\}$ is consistent by Lemma 2.2 and by one of the standard properties of consistency: $\Delta \cup \{\neg A\}$ is consistent iff $\Delta \nvdash A$. Next, notice $\mathbf{Ded}(\mathbf{Th}^\#)$ is maximally consistent nice extension of the $\mathbf{Ded}(\mathbf{Th})$. $\mathbf{Ded}(\mathbf{Th}^\#)$ is consistent because, by the standard Lemma 2.3 belov, it is the union of a chain of consistent sets. To see that $\mathbf{Ded}(\mathbf{Th}^\#)$ is maximal, pick any wff Φ. Then Φ is some Φ_i in the enumerated list of all wff's. Therefore for any Φ such that $\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\Phi]^c)$ or $\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\neg\Phi]^c)$, either $\Phi \in \mathbf{Th}^\#$ or $\neg\Phi \in \mathbf{Th}^\#$. Since $\mathbf{Ded}(\mathbf{Th}_{i+1}) \subseteqq \mathbf{Ded}(\mathbf{Th}^\#)$, we have $\Phi \in \mathbf{Ded}(\mathbf{Th}^\#)$ or $\neg\Phi \in \mathbf{Ded}(\mathbf{Th}^\#)$, which implies that $\mathbf{Ded}(\mathbf{Th}^\#)$ is maximally consistent nice extension of the $\mathbf{Ded}(\mathbf{Th})$.

Definition 2.7. We define now predicate $\mathbf{Pr}_{\mathbf{Th}^\#}([\Phi_i]^c)$ asserting provability in $\mathbf{Th}^\#$:

$$\mathbf{Pr}_{\mathbf{Th}^\#}([\Phi_i]^c) \Leftrightarrow [\mathbf{Pr}^\#_{\mathbf{Th}_i}([\Phi_i]^c)] \vee [\mathbf{Pr}^*_{\mathbf{Th}_i}([\Phi_i]^c)],$$

$$\mathbf{Pr}_{\mathbf{Th}^\#}([\neg\Phi_i]^c) \Leftrightarrow [\mathbf{Pr}^\#_{\mathbf{Th}_i}([\neg\Phi_i]^c)] \vee [\mathbf{Pr}^*_{\mathbf{Th}_i}([\neg\Phi_i]^c)]. \quad (2.33)$$

Definition 2.8. Let $\Psi = \Psi(x)$ be one-place open wff such that the conditions:

$(*)$ $\mathbf{Th} \vdash \exists! x_\Psi[\Psi(x_\Psi)]$ or

$(**)$ $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\exists! x_\Psi[\Psi(x_\Psi)]]^c)$ and $M_{Nst}^{\mathbf{Th}} \models \exists! x_\Psi[\Psi(x_\Psi)]$ is satisfied.

Then we said that, a set y is a $\mathbf{Th}^\#$-set iff there is exist one-place open wff $\Psi(x)$ such that $y = x_\Psi$. We write $y[\mathbf{Th}^\#]$ iff y is a $\mathbf{Th}^\#$-set.

Remark 2.4. Note that $[(*) \vee (**)] \Rightarrow \mathbf{Th}^\# \vdash \exists! x_\Psi[\Psi(x_\Psi)]$.

Remark 2.5. Note that $y[\mathbf{Th}^\#] \Leftrightarrow \exists\Psi[(y = x_\Psi) \wedge \mathbf{Pr}_{\mathbf{Th}^\#}([\exists! x_\Psi[\Psi(x_\Psi)]]^c)]$

Definition 2.9. Let $\Im$ be a collection such that : $\forall x\big[x \in \Im \leftrightarrow x$ is a $\mathbf{Th}^\#$-set$\big]$.

Proposition 2.7. Collection $\Im$ is a $\mathbf{Th}^\#$-set.

Proof. Let us consider an one-place open wff $\Psi(x)$ such that conditions $(*)$ or $(**)$ is satisfied, i.e. $\mathbf{Th}^\# \vdash \exists! x_\Psi[\Psi(x_\Psi)]$. We note that there exists countable collection $\mathcal{F}_\Psi$ of the one-place open wff's $\mathcal{F}_\Psi = \{\Psi_n(x)\}_{n\in\mathbb{N}}$ such that: (i) $\Psi(x) \in \mathcal{F}_\Psi$ and (ii)

$$\mathbf{Th} \vdash \exists! x_{\Psi}\Big[[\Psi(x_{\Psi})] \wedge \Big\{\forall n\Big(n \in M_{\mathbf{st}}^{Z_2^{Hs}}\Big)[\Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi})]\Big\}\Big]$$

or

$$\mathbf{Th} \vdash \exists! x_{\Psi}\Big[\mathbf{Pr_{Th}}([\Psi(x_{\Psi})]^c) \wedge \Big\{\forall n\Big(n \in M_{\mathbf{st}}^{Z_2^{Hs}}\Big)\mathbf{Pr_{Th}}([\Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi})]^c)\Big\}\Big] \quad (2.34)$$

and

$$M_{Nst}^{\mathbf{Th}} \models \exists! x_{\Psi}\Big[[\Psi(x_{\Psi})] \wedge \Big\{\forall n\Big(n \in M_{\mathbf{st}}^{Z_2^{Hs}}\Big)[\Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi})]\Big\}\Big]$$

or of the equivalent form

$$\mathbf{Th} \vdash \exists! x_1\Big[[\Psi_1(x_1)] \wedge \Big\{\forall n\Big(n \in M_{\mathbf{st}}^{Z_2^{Hs}}\Big)[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]\Big\}\Big]$$

or

$$\mathbf{Th} \vdash \exists! x_{\Psi}\Big[\mathbf{Pr_{Th}}([\Psi(x_1)]^c) \wedge \Big\{\forall n\Big(n \in M_{\mathbf{st}}^{Z_2^{Hs}}\Big)\mathbf{Pr_{Th}}([\Psi(x_1) \leftrightarrow \Psi_n(x_1)]^c)\Big\}\Big] \quad (2.35)$$

and

$$M_{Nst}^{\mathbf{Th}} \models \exists! x_{\Psi}\Big[[\Psi(x_1)] \wedge \Big\{\forall n\Big(n \in M_{\mathbf{st}}^{Z_2^{Hs}}\Big)[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]\Big\}\Big]$$

where we set $\Psi(x) = \Psi_1(x_1), \Psi_n(x_1) = \Psi_{n,1}(x_1)$ and $x_{\Psi} = x_1$. We note that any collection $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n\in\mathbb{N}}, k = 1,2,\ldots$ such above defines an unique set x_{Ψ_k}, i.e. $\mathcal{F}_{\Psi_{k_1}} \bigcap \mathcal{F}_{\Psi_{k_2}} = \varnothing$ iff $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$. We note that collections $\mathcal{F}_{\Psi_k}, k = 1,2,..$ is no part of the ZFC_2^{Hs}, i.e. collection $\mathcal{F}_{\Psi_k}$ there is no set in sense of ZFC_2^{Hs}. However that is no problem, because by using Gödel numbering one can to replace any collection $\mathcal{F}_{\Psi_k}, k = 1,2,..$ by collection $\Theta_k = g(\mathcal{F}_{\Psi_k})$ of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n\in\mathbb{N}}, k = 1,2,\ldots\ . \quad (2.36)$$

It is easy to prove that any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}^{\#}$-set.This is done by Gödel encoding [8],[10] of the statament (2.19) by Proposition 2.6 and by axiom schema of separation [4]. Let $g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1,2,..$ be a Gödel number of the wff $\Psi_{n,k}(x_k)$. Therefore $g(\mathcal{F}_k) = \{g_{n,k}\}_{n\in\mathbb{N}}$, where we set $\mathcal{F}_k = \mathcal{F}_{\Psi_k}$, $k = 1,2,..$ and

$$\forall k_1 \forall k_2[\{g_{n,k_1}\}_{n\in\mathbb{N}} \bigcap \{g_{n,k_2}\}_{n\in\mathbb{N}} = \varnothing \leftrightarrow x_{k_1} \neq x_{k_2}]. \quad (2.37)$$

Let $\{\{g_{n,k}\}_{n\in\mathbb{N}}\}_{k\in\mathbb{N}}$ be a family of the all sets $\{g_{n,k}\}_{n\in\mathbb{N}}$. By axiom of choice [9] one obtain unique set $\mathfrak{I}' = \{g_k\}_{k\in\mathbb{N}}$ such that $\forall k[g_k \in \{g_{n,k}\}_{n\in\mathbb{N}}]$. Finally one obtain a set $\mathfrak{I}$ from a set $\mathfrak{I}'$ by axiom schema of replacement [9].Thus one can define a $\mathbf{Th}^{\#}$-set

$\mathfrak{R}_c \subsetneqq \mathfrak{I}$:

$$\forall x[x \in \mathfrak{R}_c \leftrightarrow (x \in \mathfrak{I}) \wedge \mathbf{Pr}_{\mathbf{Th}^{\#}}([x \notin x]^c)]. \tag{2.38}$$

Proposition 2.8. Any collection $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1,2,..$ is a $\mathbf{Th}^{\#}$-set.

Proof. We define $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$. Therefore $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$ (see [10]). Let us define now predicate $\Pi(g_{n,k}, v_k)$

$$\Pi(g_{n,k}, v_k) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}}([\exists! x_k[\Psi_{1,k}(x_1)]]^c) \wedge$$
$$\wedge \exists! x_k(v_k = [x_k]^c)\Big[\forall n\Big(n \in M_{\mathbf{st}}^{Z_2^{Hs}}\Big)[\mathbf{Pr}_{\mathbf{Th}}([[\Psi_{1,k}(x_k)]]^c) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}}(\mathbf{Fr}(g_{n,k}, v_k))]\Big]. \tag{2.39}$$

We define now a set Θ_k such that

$$\Theta_k = \Theta_k' \cup \{g_k\},$$
$$\forall n(n \in \mathbb{N})[g_{n,k} \in \Theta_k' \leftrightarrow \Pi(g_{n,k}, v_k)] \tag{2.40}$$

But obviously definitions (2.39) and (2.40) is equivalent by Proposition 2.6.

Proposition 2.9. (i) $\mathbf{Th}^{\#} \vdash \exists \mathfrak{R}_c$,(ii) $\mathfrak{R}_c$ is a countable $\mathbf{Th}^{\#}$-set.

Proof.(i) Statement $\mathbf{Th}^{\#} \vdash \exists \mathfrak{R}_c$ follows immediately by using statement $\exists \mathfrak{I}$ and axiom schema of separation [9]. (ii) follows immediately from countability of a set $\mathfrak{I}$.

Proposition 2.10. A set $\mathfrak{R}_c$ is inconsistent.

Proof.From formla (2.18) one obtain

$$\mathbf{Th}^{\#} \vdash \mathfrak{R}_c \in \mathfrak{R}_c \leftrightarrow \mathbf{Pr}_{\mathbf{Th}^{\#}}([\mathfrak{R}_c \notin \mathfrak{R}_c]^c). \tag{2.41}$$

From formula (2.41) and Proposition 2.6 one obtain

$$\mathbf{Th}^{\#} \vdash \Re_c \in \Re_c \leftrightarrow \Re_c \notin \Re_c \tag{2.42}$$

and therefore

$$\mathbf{Th}^{\#} \vdash (\Re_c \in \Re_c) \wedge (\Re_c \notin \Re_c). \tag{2.43}$$

But this is a contradiction.

3.Derivation inconsistent countable set in ZFC_2 with the full semantics.

Let $\mathbf{Th}$ be an second order theory with the full second order semantics.We assume now that: (i) $\mathbf{Th}$ contains ZFC_2^{fss}, (ii) $\mathbf{Th}$ has no any model.

Definition 3.1. Using formula (2.3) one can define predicate $\mathbf{Pr}^{\omega}_{\mathbf{Th}}(y)$ really asserting provability in ZFC_2^{fss}

$$\mathbf{Pr}^{\omega}_{\mathbf{Th}}(y) \leftrightarrow \exists x\left(x \in M^{Z_2}_{\omega}\right)\mathbf{Prov}_{\mathbf{Th}}(x,y), \tag{3.1}$$

Let Φ be any closed formula with code $y = [\Phi]^c \in M^{Z_2}_{\omega}$, then $\mathbf{Th} \vdash \mathbf{Pr}^{\omega}_{\mathbf{Th}}([\Phi]^c)$ implies $\mathbf{Th} \vdash \Phi$.

Definition 3.2. Let $\Psi = \Psi(x)$ be one-place open wff such that the conditions:

$(*)$ $\mathbf{Th} \vdash \exists! x_{\Psi}[\Psi(x_{\Psi})]$ or

$(**)$ $\mathbf{Th} \vdash \mathbf{Pr}^{\omega}_{\mathbf{Th}}([\exists! x_{\Psi}[\Psi(x_{\Psi})]]^c)$ is satisfied.

Then we said that, a set y is a $\mathbf{Th}$-set iff there is exist one-place open wff $\Psi(x)$ such that $y = x_{\Psi}$. We write $y[\mathbf{Th}]$ iff y is a $\mathbf{Th}$-set.

Remark 3.1. Note that $[(*) \vee (**)] \Rightarrow \mathbf{Th} \vdash \exists! x_{\Psi}[\Psi(x_{\Psi})]$.

Remark 3.2. Note that $y[\mathbf{Th}] \Leftrightarrow \exists\Psi[(y = x_{\Psi}) \wedge \mathbf{Pr}^{\omega}_{\mathbf{Th}}([\exists! x_{\Psi}[\Psi(x_{\Psi})]]^c)]$

Definition 3.3. Let $\Im$ be a collection such that : $\forall x\left[x \in \Im \leftrightarrow x \text{ is a } \mathbf{Th}\text{-set}\right]$.

Proposition 3.2. Collection $\Im$ is a $\mathbf{Th}$-set.

Definition 3.4. We define now a $\mathbf{Th}$-set $\Re_c \subsetneq \Im$:

$$\forall x[x \in \Re_c \leftrightarrow (x \in \Im) \wedge \mathbf{Pr}^{\omega}_{\mathbf{Th}}([x \notin x]^c)]. \tag{3.2}$$

Proposition 3.3. (i) $\mathbf{Th} \vdash \exists \Re_c$, (ii) $\Re_c$ is a countable $\mathbf{Th}$-set.

Proof.(i) Statement $\mathbf{Th} \vdash \exists \Re_c$ follows immediately by using statement $\exists \Im$ and axiom schema of separation [4]. (ii) follows immediately from countability of a set $\mathfrak{I}$.

Proposition 3.4. A set $\mathfrak{R}_c$ is inconsistent.

Proof.From formla (3.2) one obtain

$$\mathbf{Th} \vdash \mathfrak{R}_c \in \mathfrak{R}_c \leftrightarrow \mathbf{Pr}^{\omega}_{\mathbf{Th}}([\mathfrak{R}_c \notin \mathfrak{R}_c]^c). \tag{3.3}$$

From formula (3.3) and definition 3.1 one obtain

$$\mathbf{Th} \vdash \mathfrak{R}_c \in \mathfrak{R}_c \leftrightarrow \mathfrak{R}_c \notin \mathfrak{R}_c \tag{3.4}$$

and therefore

$$\mathbf{Th} \vdash (\mathfrak{R}_c \in \mathfrak{R}_c) \wedge (\mathfrak{R}_c \notin \mathfrak{R}_c). \tag{3.5}$$

But this is a contradiction.

4.**Conclusion**.

In this paper we have proved that the second order ZFC with the full second-order semantic is a contradictory,i.e. $\neg Con(ZFC_2)$.Main result is: let k be an inaccessible cardinal and H_k is a set of all sets having hereditary size less then k, then $\neg Con(ZFC + (V = H_k))$.

This result was obtained in [7],[13] by using essentially another approach.

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