Christoffel symbols for asymmetric metric tensors.

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Abstract

At first we take two definitions: for the asymmetric metric tensor and for the Christoffel symbols. In order to find the connection of Christoffel symbols with the asymmetric metric tensor we take the derivative from metric tensor with respect to coordinate. In the resulting equation we make the cyclic rearrangement of indexes two times and get two more equations. Then we rearrange the indexes of metric tensor in these three equations and get three more equations.
After this we represent the metric tensor as the sum of symmetric and antisymmetric tensors. Also we represent the Christoffel symbols as the sum of symmetric and antisymmetric symbols.
Resolving our 6 equations we get the formula for Christoffel symbols expressed through the symmetric and antisymmetric parts of metric tensor.
In the item 4) there is the description of Parts : Part VI, Part VII, Part VIII, Part IX on the site www.telnin.narod.ru .

Content

1). Asymmetric metric tensors.

If $\vec{e}_\mu$ is the basis of the curved vector space W, then metric tensor in W defines so :

$$g_{\mu\nu} = (\vec{e}_\mu, \vec{e}_\nu) \qquad (1.1)$$

Then Christoffel symbols $\Gamma^\lambda{}_{\mu\sigma}$ defines so :

$$\frac{\partial}{\partial x^\sigma}\vec{e}_\mu = \partial_\sigma \vec{e}_\mu = \vec{e}_{\mu,\sigma} = \vec{e}_\lambda \cdot \Gamma^\lambda{}_{\mu\sigma}$$

In order to find the connection of Christoffel symbols with metric tensor we take the derivative from (1.1) with respect to x^σ :

$$g_{\mu\nu,\sigma} = \partial_\sigma g_{\mu\nu} = (\partial_\sigma \vec{e}_\mu, \vec{e}_\nu) + (\vec{e}_\mu, \partial_\sigma \vec{e}_\nu) =$$
$$= (\vec{e}_\lambda \cdot \Gamma^\lambda{}_{\mu\sigma}, \vec{e}_\nu) + (\vec{e}_\mu, \vec{e}_\lambda \cdot \Gamma^\lambda{}_{\nu\sigma}) = g_{\lambda\nu} \cdot \Gamma^\lambda{}_{\mu\sigma} + g_{\mu\lambda} \cdot \Gamma^\lambda{}_{\nu\sigma} \qquad (1.2)$$

Let us make in this equation the cyclic rearrangement of indexes two times and we get two more equations :

$$g_{\nu\sigma,\mu} = g_{\lambda\sigma} \cdot \Gamma^\lambda{}_{\nu\mu} + g_{\nu\lambda} \cdot \Gamma^\lambda{}_{\sigma\mu} \qquad (1.3)$$

$$g_{\sigma\mu,\nu} = g_{\lambda\mu} \cdot \Gamma^\lambda{}_{\sigma\nu} + g_{\sigma\lambda} \cdot \Gamma^\lambda{}_{\mu\nu} \qquad (1.4)$$

Now we rearrange the indexes of metric tensor in these three equations – (1.2), (1.3), (1.4) and get three more equations :

$$g_{\nu\mu,\sigma} = g_{\lambda\mu} \cdot \Gamma^\lambda{}_{\nu\sigma} + g_{\nu\lambda} \cdot \Gamma^\lambda{}_{\mu\sigma} \qquad (1.5)$$

$$g_{\sigma\nu,\mu} = g_{\lambda\nu} \cdot \Gamma^\lambda{}_{\sigma\mu} + g_{\sigma\lambda} \cdot \Gamma^\lambda{}_{\nu\mu} \qquad (1.6)$$

$$g_{\mu\sigma,\nu} = g_{\lambda\sigma} \cdot \Gamma^\lambda{}_{\mu\nu} + g_{\mu\lambda} \cdot \Gamma^\lambda{}_{\sigma\nu} \qquad (1.7)$$

Let us represent $g_{\mu\nu}$ as the sum of symmetric and antisymmetric tensors :

$$g_{\mu\nu} = a_{\mu\nu} + b_{\mu\nu} \qquad a_{\mu\nu} = a_{\nu\mu} \qquad b_{\mu\nu} = -b_{\nu\mu} \qquad (1.8)$$

Then go to the new view of equations :

$$(1.2)+(1.5) \qquad a_{\mu\nu,\sigma} = a_{\nu\lambda} \cdot \Gamma^\lambda{}_{\mu\sigma} + a_{\mu\lambda} \cdot \Gamma^\lambda{}_{\nu\sigma} \qquad (1.9)$$

$$(1.3)+(1.6) \qquad a_{\nu\sigma,\mu} = a_{\sigma\lambda} \cdot \Gamma^\lambda{}_{\nu\mu} + a_{\nu\lambda} \cdot \Gamma^\lambda{}_{\sigma\mu} \qquad (1.10)$$

$$(1.4)+(1.7) \qquad a_{\sigma\mu,\nu} = a_{\mu\lambda} \cdot \Gamma^\lambda{}_{\sigma\nu} + a_{\sigma\lambda} \cdot \Gamma^\lambda{}_{\mu\nu} \qquad (1.11)$$

$$(1.2)-(1.5) \qquad b_{\mu\nu,\sigma} = -b_{\nu\lambda} \cdot \Gamma^\lambda{}_{\mu\sigma} + b_{\mu\lambda} \cdot \Gamma^\lambda{}_{\nu\sigma} \qquad (1.12)$$

$$(1.3)-(1.6) \qquad b_{\nu\sigma,\mu} = -b_{\sigma\lambda} \cdot \Gamma^\lambda{}_{\nu\mu} + b_{\nu\lambda} \cdot \Gamma^\lambda{}_{\sigma\mu} \qquad (1.13)$$

$$(1.4)-(1.7) \qquad b_{\sigma\mu,\nu} = -b_{\mu\lambda} \cdot \Gamma^\lambda{}_{\sigma\nu} + b_{\sigma\lambda} \cdot \Gamma^\lambda{}_{\mu\nu} \qquad (1.14)$$

After that we shall introduce new values :

$$\Gamma^\lambda{}_{\mu\sigma} + \Gamma^\lambda{}_{\sigma\mu} = F^\lambda{}_{\mu\sigma} \qquad (1.15)$$

$$\Gamma^\lambda{}_{\mu\sigma} - \Gamma^\lambda{}_{\sigma\mu} = G^\lambda{}_{\mu\sigma} \qquad (1.16)$$

$$\Gamma^\lambda{}_{\mu\sigma} = \tfrac{1}{2} \cdot (F^\lambda{}_{\mu\sigma} + G^\lambda{}_{\mu\sigma}) \qquad (1.17)$$

Now let us form two equations for $F^\lambda{}_{\mu\sigma}$ and $G^\lambda{}_{\mu\sigma}$:

(1.9) + (1.10) – (1.11) :

$$a_{\mu\nu,\sigma} + a_{\nu\sigma,\mu} - a_{\sigma\mu,\nu} = a_{\nu\lambda} \cdot F^\lambda{}_{\mu\sigma} + a_{\mu\lambda} \cdot G^\lambda{}_{\nu\sigma} + a_{\sigma\lambda} \cdot G^\lambda{}_{\nu\mu} \qquad (1.18)$$

(1.12) – (1.13) + (1.14) :

$$b_{\mu\nu,\sigma} - b_{\nu\sigma,\mu} + b_{\sigma\mu,\nu} = -b_{\nu\lambda} \cdot F^{\lambda}{}_{\mu\sigma} + b_{\mu\lambda} \cdot G^{\lambda}{}_{\nu\sigma} + b_{\sigma\lambda} \cdot F^{\lambda}{}_{\nu\mu} \qquad (1.19)$$

From (1.18) we get

$$F^{\rho}{}_{\mu\sigma} = a^{\rho\nu} \cdot (a_{\mu\nu,\sigma} + a_{\nu\sigma,\mu} - a_{\sigma\mu,\nu} - a_{\mu\lambda} \cdot G^{\lambda}{}_{\nu\sigma} - a_{\sigma\lambda} \cdot G^{\lambda}{}_{\nu\mu}) \qquad (1.20)$$

and substitute it into (1.19) :

$$D_{\lambda}{}^{sq}{}_{\mu\nu\sigma} \cdot G^{\lambda}{}_{sq} = B_{\mu\nu\sigma} \qquad (1.21)$$

where

$$D_{\lambda}{}^{sq}{}_{\mu\nu\sigma} = b_{\mu\lambda} \cdot \delta^{s}{}_{\nu} \cdot \delta^{q}{}_{\sigma} + b_{\nu\rho} \cdot a^{\rho s} \cdot (a_{\mu\lambda} \cdot \delta^{q}{}_{\sigma} + a_{\sigma\lambda} \cdot \delta^{q}{}_{\mu}) -$$

$$- b_{\sigma\rho} \cdot a^{\rho s} \cdot (a_{\nu\lambda} \cdot \delta^{q}{}_{\mu} + a_{\mu\lambda} \cdot \delta^{q}{}_{\nu}) \qquad (1.22)$$

$$B_{\mu\nu\sigma} = (b_{\mu\nu,\sigma} - b_{\nu\sigma,\mu} + b_{\sigma\mu,\nu}) + b_{\nu\rho} \cdot a^{\rho r} \cdot (a_{\mu r,\sigma} + a_{r\sigma,\mu} - a_{\sigma\mu,r}) -$$

$$- b_{\sigma\rho} \cdot a^{\rho\omega} \cdot (a_{\nu\omega,\mu} + a_{\omega\mu,\nu} - a_{\mu\nu,\omega}) \qquad (1.23)$$

If we substitute the formule

$$a_{\mu\nu} = 1/2 \cdot (g_{\mu\nu} + g_{\nu\mu}) \qquad (1.24)$$

$$b_{\mu\nu} = 1/2 \cdot (g_{\mu\nu} - g_{\nu\mu}) \qquad (1.25)$$

into (1.23), (1.22), (1.20) then we express $F^{\rho}{}_{\mu\sigma}$ and $G^{\lambda}{}_{sq}$ by metric tensor. And if we substitute these $F^{\lambda}{}_{\mu\sigma}$ and $G^{\lambda}{}_{\mu\sigma}$ into (1.17) then we express $\Gamma^{\lambda}{}_{\mu\sigma}$ by metric tensor.

Using :

$$g^{\mu\lambda} \cdot g_{\nu\lambda} = g_{\nu\lambda} \cdot g^{\mu\lambda} = \delta^{\mu}{}_{\nu} \qquad (1.26)$$

$$g^{\lambda\mu} \cdot g_{\lambda\nu} = (g_{\nu\lambda} \cdot g^{\mu\lambda})^{T} = (\delta^{\mu}{}_{\nu})^{T} = \delta_{\nu}{}^{\mu} = \delta^{\mu}{}_{\nu} \qquad (1.27)$$

$$a^{\mu\nu} = 1/2 \cdot (g^{\mu\nu} + g^{\nu\mu}) \quad (1.28) \qquad a_{\nu\lambda} \cdot a^{\lambda\rho} = \delta_{\nu}{}^{\rho} \quad (1.29)$$

$$c^{\sigma\mu} \cdot b_{\mu\lambda} = \delta^{\sigma}{}_{\lambda} \quad (1.30) \qquad c^{\sigma\mu} = -c^{\mu\sigma} \quad (1.31)$$

$$b_{\nu\rho} \cdot a^{\rho s} = 0 \qquad (1.32)$$

we obtain:

(1.23) + (1.32) + (1.30) = (1.33)

$$G^{\rho}{}_{\mu\sigma} = c^{\rho\omega} \cdot (b_{\omega\mu,\sigma} - b_{\mu\sigma,\omega} + b_{\sigma\omega,\mu}) \qquad (1.33)$$

(1.18) + (1.33) + (1.29) = (1.34)

$$F^{\rho}{}_{\mu\sigma} = a^{\rho\nu} \cdot (a_{\mu\nu,\sigma} + a_{\nu\sigma,\mu} - a_{\sigma\mu,\nu} -$$
$$- a_{\mu\lambda} \cdot c^{\lambda\omega} \cdot [b_{\omega\nu,\sigma} - b_{\nu\sigma,\omega} + b_{\sigma\omega,\nu}] -$$
$$- a_{\sigma\lambda} \cdot c^{\lambda\omega} \cdot [b_{\omega\nu,\mu} - b_{\nu\mu,\omega} + b_{\mu\omega,\nu}]) \qquad (1.34)$$

(1.17) + (1.34) + (1.33) = (1.35)

$$\Gamma^{\rho}{}_{\mu\sigma} = 1/2 \cdot \{a^{\rho\nu} \cdot (a_{\mu\nu,\sigma} + a_{\nu\sigma,\mu} - a_{\sigma\mu,\nu} -$$
$$- a_{\mu\lambda} \cdot c^{\lambda\omega} \cdot [b_{\omega\nu,\sigma} - b_{\nu\sigma,\omega} + b_{\sigma\omega,\nu}] -$$
$$- a_{\sigma\lambda} \cdot c^{\lambda\omega} \cdot [b_{\omega\nu,\mu} - b_{\nu\mu,\omega} + b_{\mu\omega,\nu}]) +$$
$$+ c^{\rho\omega} \cdot (b_{\omega\mu,\sigma} - b_{\mu\sigma,\omega} + b_{\sigma\omega,\mu})\} \qquad (1.35)$$

2). Symmetric metric tensor.

If we consider symmetric $g_{\mu\nu}$ (т.е. $b_{\mu\nu} = 0$)

then from (1.35) it follows that

$$\Gamma^{\lambda}{}_{\mu\sigma} = 1/2 \cdot a^{\lambda\nu} \cdot (a_{\mu\nu,\sigma} + a_{\nu\sigma,\mu} - a_{\sigma\mu,\nu}) \qquad (2.1)$$

And from symmetry $g_{\mu\nu} = a_{\mu\nu} = a_{\nu\mu} = g_{\nu\mu}$ it follows that

$$\Gamma^{\lambda}{}_{\mu\sigma} = \Gamma^{\lambda}{}_{\sigma\mu}$$

3). Antisymmetric metric tensor.

a If we consider antisymmetric $g_{\mu\nu}$ $(a_{\mu\nu} = 0)$

then from (1.35) it follows that

$$\Gamma^{\lambda}{}_{\nu\sigma} = 1/2 \cdot c^{\lambda\mu} \cdot (b_{\mu\nu,\sigma} - b_{\nu\sigma,\mu} + b_{\sigma\mu,\nu}) \qquad (3.1)$$

And from symmetry $g_{\mu\nu} = b_{\mu\nu} = -b_{\nu\mu} = -g_{\nu\mu}$ it follows that

$$\Gamma^{\lambda}{}_{\nu\sigma} = -\Gamma^{\lambda}{}_{\sigma\nu}$$

4) The generalization of the First Noether theorem on asymmetric metric tensors and some others generalizations.

On site www.telnin.narod.ru the Part VI contains the description of Christoffel symbols for asymmetric metric tensors.

Then – the main part – Part VII – deals with the First Noether theorem. This theorem is generalized on the curved spaces and on the asymmetric metric tensors.

And also it takes into account the second derivatives of the matter fields by the coordinates in Lagrangian (this permits to apply this theorem to gravity).

The Part VIII is the simple example of using by this generalized theorem. It gives the definitions for the energy-momentum vectors for matter and also for the gravitational field.

The Part IX is the application of Part VIII to the Schwarzschild metric. And it gives the energy of the Earth gravitational field.