Resolving the decision version of the directed Hamiltonian path (cycle) problem under two special conditions by method of matrix determinant: An overview.

Okunoye Babatunde O.

Department of Pure and Applied
Biology, Ladoke Akintola University
of Technology, Ogbomoso, Nigeria.
to.csgroups@gmail.com

Abstract: In computational complexity, the Decision version of the Directed Hamiltonian Path Problem is known to be NP-complete (Nondeterministic-Polynomial complete). There are no known efficient algorithms for its resolution in Polynomial time. In three papers, the author shows that this problem can be resolved in Polynomial time under two special conditions relating to the determinant of a matrix: the absence of zero rows (columns) and similar rows (columns). In this paper, the author gives a brief overview of the proposed solution and the P vs NP problem.

Keywords: Directed Hamiltonian Path (Cycle) Problem, P vs NP, Matrix Determinant

I. Introduction

What do these three problems in biology have in common?

- The Multiple sequence alignment problem (Carrillo and Lipman, 1988)
- The Interval constrained colouring problem (Althaus *et al*, 2008)
- The Double digest problem (Goldstein and Waterman, 1987)

The perhaps astonishing answer is that they are NP-complete (Nondeterministic-Polynomial complete) and this shows the relevance of computational complexity outside the core field of computer science and especially in the biological sciences. The P vs NP question has been described as among “the most central open problems in mathematics” (Fortnow and Homer, 2003) and one of “the most important problems in contemporary mathematics and theoretical computer science” (Sipser, 1992). Briefly stated, the P vs NP question asks whether every algorithmic problem with efficiently verifiable solutions also have efficiently computable solutions. That is, the class P refers to the class of problems with efficiently computable solutions while class NP the class of problems with efficiently verifiable solutions.

Cook (1971) and Levin (1973) further introduced the concept of NP-completeness. An NP-complete problem is a problem in NP to which all other problems in NP can be reduced to. Karp (1972) in his seminal paper showed that several computationally hard problems are NP-complete. Today we know that there are thousands of NP-complete problems (see Garey and Johnson, 1979). Some of the more common NP-complete problems include:

(1). The Hamiltonian Cycle: Given a graph, is there a cycle which visits all vertices exactly once?

(2). Clique: Given (G, k) a graph and integer: Are there k nodes in G that are all connected to each other?

(3). Independent Set: Given (G, k) a graph and an integer, are there k nodes in G none of which are connected to each other?

(4). Travelling Salesman Problem: Given a list of cities and pairwise distances between them, is there a tour which visits each city exactly once and has length at most k?

An efficient solution (i.e. in polynomial time) to any of these problems of considerable industrial and technological importance would imply P=NP.

II. RESOLVING THE DECISION VERSION OF THE DIRECTED HAMILTONIAN PATH PROBLEM UNDER TWO SPECIAL CONDITIONS BY METHOD OF MATRIX DETERMINANT

A method of resolving NP complete problems is to find solutions which are fast on average with respect to a natural distribution on inputs. The notion of "average case completeness" is attributable to Levin (1986). The goal is to find solutions to hard optimization problems which are useful in many practical real world applications. Earlier attempts at resolving NP-hard problems included methods such as brute force (Yablonski, 1959; Edmonds, 1965) and reductions to linear programming (Khachiyan, 1979; Yannakakis, 1991).

The author in resolving the decision version of the Directed Hamiltonian cycle problem used the unconventional method of representing the directed graph as an adjacency matrix (Okunoye 2012 a,b,c). The special scalar – the determinant of the matrix was used to ascertain whether any given adjacency matrix encodes a Directed Hamiltonian Path. Crucially, the determinant of a matrix is efficiently (in polynomial time) resolved using techniques including Gaussian elimination (Fang and Havas, 1997; Lipschutz and Lipson, 2009) and as such presents a route in the resolution of the problem under two special conditions relating to the determinant of a matrix: the absence of a zero row (column) and the absence of similar rows (columns). The theorems from (Lipschutz and Lipson, 2009) establishing the conditions are given below with proofs (see Shilov, 1977; Lancaster and Tismenetsky, 1985; Kolman and Hill, 1993; and Bronson and Costa, 2007 for similar proofs).

Theorem 1: For a matrix A, if A has a row (column) of zeros, then |A| is zero.

Proof: Each term in |A| contains a factor from every row (column), and so from the row (column) of zeros. Thus each term of |A| is zero, and so |A| = 0.

To prove theorem 2 (If A has two identical rows (columns), then |A| = 0) it is necessary first to prove a related theorem:

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Theorem 3: If two rows (columns) are interchanged, then |B| = - |A|

Proof: The theorem is proven for the case that the two columns are interchanged. Let τ be the transposition that interchanges the two numbers corresponding to the two columns of A that are interchanged. If A = $|a_{ij}|$ and B = $|b_{ij}|$, then $b_{ij} = a_{i\tau(j)}$.

Hence, for any permutation σ,

$$b_{1\sigma(1)}\, b_{2\sigma(2)} \ldots b_{n\sigma(n)} = a_{1(\tau o\sigma)(1)}\, a_{2(\tau o\sigma)(2)} \ldots a_{n(\tau o\sigma)(n)}$$

$$\text{Thus } |B| = \textstyle\sum_{\sigma \epsilon Sn} b_{1\sigma(1)}\, b_{2\sigma(2)} \ldots b_{n\sigma(n)} = \sum_{\sigma \epsilon Sn} (sgn\ \sigma)\, a_{1(\tau o\sigma)(1)}\, a_{2(\tau o\sigma)(2)} \ldots a_{n(\tau o\sigma)(n)}$$

Because the transposition τ is an odd permutation, $sgn\ (\tau o\sigma) = (sgn\ \tau)(sgn\ \sigma) = -sgn\ \sigma$

It follows then, $sgn\ \sigma = -sgn\ (\tau o\sigma)$ and so

|B| = $-\sum_{\sigma \epsilon Sn}[(sgn\ \tau o\sigma)]\, a_{1(\tau o\sigma)(1)}\, a_{2(\tau o\sigma)(2)} \ldots a_{n(\tau o\sigma)(n)}$

As σ runs through all the elements of S_n, $\tau\ o\ \sigma$ also runs through all the elements of S_n.

Hence,

|B| = -|A|.

Theorem 2: If A has two identical rows (columns), then |A| = 0.

Proof: Suppose 1 + 1 ≠ 0 in K. If we interchange the two identical rows of A, we will still obtain the matrix A.
Hence by Theorem 3, |A| = -|A| and so |A| = 0.

Now suppose 1 + 1 = 0 in K. Then sgn σ = 1 for every σ□S_n. Because A has two identical rows, we can arrange the terms

of A into pairs of equal terms. Because each term is zero, the determinant of A is zero.

These theorems imply that in a matrix where there are no zero rows (columns) and no similar rows (columns), the determinant value is non-zero. It seems clear that since a directed graph can be represented by an adjacency matrix, these theorems and the two conditions they specify could lead to a method for deciding whether an arbitrary adjacency matrix encodes a Hamiltonian path. In a separate paper (Okunoye, 2012c); based on these theorems, the author gave a proof (by deduction) of a proposition as follows:

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Proposition 1: In an adjacency matrix which encodes for a directed Hamiltonian Path, a non-zero determinant value certifies the existence of a directed Hamiltonian Path when no zero rows (columns) and no similar rows (columns) exist in the adjacency matrix.

Proof

1. An adjacency matrix is a square matrix representing the combination of edges of a directed graph.
2. The determinant of a square matrix gives a non-zero value when there no zero rows (columns) and no similar rows (columns).
3. A directed Hamiltonian path is a sequence of one-way compatible edges in an adjacency matrix, a representation of a directed graph.
4. A sequence of one-way compatible edges is applied in an arbitrary adjacency matrix ensuring that there are no zero rows (columns) and no rows (columns) are similar.
5. In this adjacency matrix, a non-zero determinant value verifies the existence of a directed Hamiltonian path.

A similar proof is provided for the directed Hamiltonian cycle.

Proposition 2: In an adjacency matrix which encodes for a directed Hamiltonian cycle, a non-zero determinant value certifies the existence of a directed Hamiltonian cycle when no zero rows (columns) and no similar rows (columns) exist in the adjacency matrix.

Proof

1. An adjacency matrix is a square matrix and a representation of a directed graph.
2. The determinant of a square matrix is non-zero if there are no zero rows (columns) and no similar rows (columns) in the adjacency matrix.
3. A Hamiltonian cycle is applied in the adjacency matrix in such a way that there are no zero rows (columns) and no similar rows (columns).
4. In this adjacency matrix, a non-zero determinant value certifies the existence of a Hamiltonian cycle.

III. CONCLUSION: IS P = NP?

The P vs NP question has been open for four decades. In a poll (Gasarch, 2002) conducted on a sample of 100 experts, a majority (61) thought P ≠ NP and only 9 thought P = NP. A few noted that a solution might lie in the application of unconventional wisdom.

In using a non-zero determinant value to decide whether an adjacency matrix (a representation of a directed graph) encodes a directed Hamiltonian path under the two conditions stated above, a solution is found albeit restrictive. There is merit, I suppose, even in solutions that work in restrictive conditions: not least they often deepen our insights into aspects of the problem. Reductions to the particular instances of some other NP-complete problems might however prove daunting.

This solution does not imply P = NP, indeed, a solution which works in restrictive conditions might indicate that P ≠ NP. It is hoped this work is a contribution towards solving the problem in some instances of practical importance.

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