Lorentz Violation and Modified Geodesics

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Abstract

We propose a modification of proper time, which is dependent on vierbein and spin connection. It explicitly breaks local Lorentz gauge symmetry, while preserving diffeomorphism invariance. In the non-relativistic limit, the geodesics are consistent with galactic rotation curves without invoking dark matter.

Keywords. Lorentz violation, modified geodesics, dark matter.

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1 Introduction

Astronomical evidence indicates[1, 2] that observable mass can not provide sufficient gravitational attraction within galaxies and galaxy clusters. The dark matter hypothesis states that there is a vast amount of unseen mass in the universe. An alternative to dark matter is the modification of Newtonian dynamics[3] (MOND). It is a classical dynamics theory, which explains the mass discrepancies in galactic systems without resorting to dark matter.

We propose a relativistic theory with modified proper time. In the non-relativistic limit, the circular motion of test body is congruent with Tully-Fisher law[4] of galactic rotation curves. The dynamics is different from MOND in general.

This paper is structured as follows: Section 2 introduces local gauge transformation properties of vierbein and spin connection. In section 3, proper time interval with Lorentz violation is defined. In section 4, we discuss the motion of test body in the non-relativistic limit. In the last section we draw our conclusions.

2 Gauge Symmetry

The similarity between gravity field and non-Abelian Yang-Mills field inspired gauge theory of gravity[5, 6, 7]. Gravity and Yang-Mills actions can be formulated as different order terms in a generalized action[8, 9]. They show disparate dynamics, due to symmetry breaking via Higgs fields and non-degenerate vacuum expectation value (VEV) of gravity field.

In de Sitter gauge theory of gravity[7], gravity field can be written as a Clifford-valued 1-form[8, 9] on 4-dimensional space-time manifold

$$A = \frac{1}{l}e + \omega, \tag{1}$$

$$e = e_\mu dx^\mu = e_\mu^a \gamma_a dx^\mu, \tag{2}$$

$$\omega = \omega_\mu dx^\mu = \frac{1}{2}\omega_\mu^{ab}\gamma_{ab}dx^\mu, \tag{3}$$

where e is vierbein, ω is spin connection, l is a constant related to Minkowskian VEV magnitude of gravity gauge field $\bar{A} = \frac{1}{l}\delta_\mu^a \gamma_a dx^\mu$, $\mu, a, b = 0, 1, 2, 3$, $\omega_\mu^{ab} = -\omega_\mu^{ba}$, and $\gamma_{ab} = \gamma_a\gamma_b$. Here we adopt the summation convention for repeated indices. Clifford algebra vetors γ_a observe anticommutation relations

$$\{\gamma_a, \gamma_b\} \equiv \frac{1}{2}(\gamma_a\gamma_b + \gamma_b\gamma_a) = \eta_{ab}, \tag{4}$$

where η_{ab} is of signature $(+, -, -, -)$.

Local de Sitter gauge transformation is characterized by[1]

$$\mathbb{R}_{dS}(x) = e^{\frac{1}{2}[\epsilon^a(x)\gamma_a + \epsilon^{ab}(x)\gamma_{ab}]}, \tag{5}$$

where $\epsilon^{ab}(x) = -\epsilon^{ba}(x)$, and (γ_a, γ_{ab}) are generators of de Sitter algebra.

Gauge field A obeys local de Sitter gauge transformation law

$$A(x) \quad \to \quad \mathbb{R}_{dS}(x)A(x)\mathbb{R}_{dS}(x)^{-1} - d\mathbb{R}_{dS}(x)\mathbb{R}_{dS}(x)^{-1}. \tag{6}$$

One can write down an action for general relativity using gauge field A for de Sitter group, but invariant only under Lorentz group[7]. Under Lorentz gauge transformation

$$\mathbb{R}_L(x) = e^{\frac{1}{2}[\epsilon^{ab}(x)\gamma_{ab}]}, \tag{7}$$

gauge fields transform as

$$e(x) \quad \to \quad \mathbb{R}_L(x)e(x)\mathbb{R}_L(x)^{-1}, \tag{8}$$

$$\omega(x) \quad \to \quad \mathbb{R}_L(x)\omega(x)\mathbb{R}_L(x)^{-1} - d\mathbb{R}_L(x)\mathbb{R}_L(x)^{-1}. \tag{9}$$

The standard proper time interval is defined as

$$d\tau^2 = \langle e_\mu e_\nu \rangle \, dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu, \tag{10}$$

where $\langle \cdots \rangle$ means Clifford scalar part of enclosed expression, and $g_{\mu\nu}$ is metric. The proper time interval is invariant under local Lorentz gauge transformation, thanks to the transform property of vierbein (8).

3 Lorentz Violation

In the absence of Lorentz symmetry, we study the remaining symmetry under local gauge transformation

$$\mathbb{R}_S(x) = e^{\frac{1}{2}\epsilon^{ij}(x)\gamma_{ij}}, \tag{11}$$

where $i, j = 1, 2, 3$. Gravity gauge fields

$$A = \frac{1}{l}e + \omega = \frac{1}{l}(e_T + e_S + \omega_T + \omega_S), \tag{12}$$

$$e_T = e_{T\mu} dx^\mu = e^0_\mu \gamma_0 dx^\mu, \tag{13}$$

$$e_S = e_{S\mu} dx^\mu = e^i_\mu \gamma_i dx^\mu, \tag{14}$$

$$\omega_T = \omega_{T\mu} dx^\mu = \frac{l}{2}(\omega^{0i}_\mu \gamma_{0i} + \omega^{i0}_\mu \gamma_{i0}) dx^\mu = l\omega^{0i}_\mu \gamma_{0i} dx^\mu, \tag{15}$$

$$\omega_S = \omega_{S\mu} dx^\mu = \frac{l}{2}\omega^{ij}_\mu \gamma_{ij} dx^\mu, \tag{16}$$

[1]See e.g. chapter 2.1.3 of [10] for discussions about local gravity(Lorentz) gauge transformations in addition to diffeomorphism transformations.

transform as

$$
\begin{aligned}
e_T(x) &\rightarrow \mathbb{R}_S(x)e_T(x)\mathbb{R}_S(x)^{-1}, &(17)\\
e_S(x) &\rightarrow \mathbb{R}_S(x)e_S(x)\mathbb{R}_S(x)^{-1}, &(18)\\
\omega_T(x) &\rightarrow \mathbb{R}_S(x)\omega_T(x)\mathbb{R}_S(x)^{-1}, &(19)\\
\omega_S(x) &\rightarrow \mathbb{R}_S(x)\omega_S(x)\mathbb{R}_S(x)^{-1} - d\mathbb{R}_S(x)\mathbb{R}_S(x)^{-1}. &(20)
\end{aligned}
$$

Spin conections ω_T and ω_S are defined to be dimensionless in the same way as vierbeins $e = e_T + e_S$, via rescaling factor $\frac{1}{l}$ in (12).

Now one has the freedom to define gauge (11) and diffeomorphism invariant proper time interval as dependent on $e_T(x), e_S(x)$ and $\omega_T(x)$, respectively. Specifically we propose[2]

$$
d\tau^2 = \{\langle e_{T\mu}e_{T\nu}\rangle + \langle e_{S\mu}e_{S\nu}\rangle\, f([\langle \omega_{T\alpha}\omega_{T\beta}\rangle \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}]^{\frac{1}{4}})\}dx^\mu dx^\nu, \tag{21}
$$

with interpolation function

$$
\begin{aligned}
f(z) &\rightarrow 1 \qquad\qquad\quad for \quad z \gg 1, &(22)\\
f(z) &\rightarrow c_1 z - c_2 z^2 \quad for \quad z \ll 1, &(23)
\end{aligned}
$$

where c_1 and c_2 are dimensionless positive coefficients. In the limit $\langle \omega_{T\alpha}\omega_{T\beta}\rangle \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \gg 1$, the proper time interval is reduced to the standard one (10), restoring local Lorentz gauge invariance.

4 Modified Geodesics

Geodesics are obtained by minimizing action

$$
S = -mc\int d\tau, \tag{24}
$$

where m is the mass of test body, and c is the speed of light.

In the non-relativistic ($dx^0 = cdt \gg dx^i$) and weak field (gravity gauge field almost Minkowskian $A \approx \bar{A} = \frac{1}{l}\bar{e} = \frac{1}{l}\delta^a_\mu\gamma_a dx^\mu$) limit , the action is approximated by

$$
\begin{aligned}
S &\approx -mc\int\sqrt{(1+2\Delta e^0_0)dx^0dx^0 - f([l\omega^{0j}_0 l\omega^{0j}_0]^{\frac{1}{4}})dx^idx^i} &(25)\\
&\approx \int dt\mathcal{L}, &(26)\\
\mathcal{L} &= -mc^2 - mV + \frac{1}{2}m\dot{x}^i\dot{x}^i f([l\omega^{0j}_0 l\omega^{0j}_0]^{\frac{1}{4}}), &(27)
\end{aligned}
$$

[2]In this paper, we are assuming that gravity fields are not changed (or rather not changed much) by Lorentz violation. Thus they satisfy the standard Einstein equations(or Einstein-Cartan equations in the presence of spin currents).

where $\frac{V}{c^2} = \Delta e_0^0 = e_0^0 - \bar{e}_0^0 = e_0^0 - 1$, and $\dot{x}^i = dx^i/dt$.

When spin current is negligible, spin connection is determined by zero torsion condition

$$T = de + \omega \wedge e + e \wedge \omega = 0, \tag{28}$$

where $\wedge$ denotes exterior product of forms. For static and weak gravity field, one has

$$\omega_0^{0i} \approx \frac{1}{2}\partial_i e_0^0. \tag{29}$$

Thus we can write the lagrangian as

$$\mathcal{L} = -mc^2 - mV + \frac{1}{2}mv^2 f([\frac{l}{2c^2}|\nabla V|]^{\frac{1}{2}}), \tag{30}$$

where $|\nabla V| = \sqrt{\partial_i V \partial_i V}$ and $v^2 = \dot{x}^i \dot{x}^i$. The Euler-Lagrange equation reads

$$\ddot{x}^i f = -\partial_i V - \frac{1}{2}v^2 \partial_i f. \tag{31}$$

Newtonian dynamics is recovered in the limit $|\nabla V| \gg \frac{2c^2}{l}$.

For spherically symmetric potential $V(r)$, the lagrangian admits two constants of motion as angular momentum and energy (rescaled by m)

$$L = r^2 \dot{\phi} f, \tag{32}$$

$$E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) f + V + c^2 \tag{33}$$

$$= \frac{1}{2}\dot{r}^2 f + (V + \frac{1}{2}\frac{L^2}{r^2 f}) + c^2 \tag{34}$$

$$= \frac{1}{2}\dot{r}^2 f + V_{eff} + c^2, \tag{35}$$

where effective potential $V_{eff} = V + \frac{1}{2}\frac{L^2}{r^2 f}$, r and ϕ are spherical coordinates, and $\theta = \pi/2$ is assumed. Circular orbit is determined by the condition

$$\partial_r V_{eff}|_{r=r_0} = 0. \tag{36}$$

For Newtonian potential

$$V = -\frac{GM}{r}, \tag{37}$$

one can calculate rotation velocity as

$$v^2|_{r=r_0} = r^2\dot{\phi}^2|_{r=r_0} = \frac{GM}{\partial_r(\frac{1}{2}r^2 f)}|_{r=r_0}. \tag{38}$$

In the limit $|\nabla V| = \frac{GM}{r^2} \ll \frac{2c^2}{l}$, one has

$$v^4|_{r=r_0} = \frac{8c^2}{c_1^2 l} GM = a_0 GM. \tag{39}$$

It is r_0 independent, which is congruent with Tully-Fisher law[4] of galactic rotation curves where constant $a_0 = \frac{8c^2}{c_1^2 l}$ is estimated as

$$a_0 \approx 10^{-8} cm/s^2 \approx \frac{c^2}{6} (\frac{\Lambda}{3})^{\frac{1}{2}}. \tag{40}$$

Here Λ is cosmological constant. Thus we have

$$c_1^2 \approx \frac{48}{l} (\frac{\Lambda}{3})^{-\frac{1}{2}}. \tag{41}$$

We know that $\frac{1}{l}$ is of order $\Lambda^{\frac{1}{2}}$ [8, 9]. Therefore, coefficient c_1 is close to order 1.

The coefficient c_2 in the interpolation function (23) does not appear in rotation velocity (39). The coefficient c_2 is essential for ensuring

$$\partial_r^2 V_{eff}|_{r=r_0} > 0, \tag{42}$$

so that the circular orbit is stable.

5 Conclusion

We propose a modification of proper time interval (21), which is dependent on vierbein as well as spin connection. In the non-relativistic limit, the modified geodesics are consistent with galactic rotation curves without invoking dark matter.

In the case of galaxy clusters, further data analysis is needed to validate our theory. Since the modified proper time interval is relativistic in nature, one can potentially apply it in cosmological settings as well.

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