PRIMES IN THE INTERVALS $[(k+1)n, kn]$

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Abstract. In this paper, we prove: (a) for every integer $n > 1$ and a fixed integer $k \leq n$, there exists a prime number p such that $kn \leq p \leq (k+1)n$, and (b) conjectures of Legendre, Oppermann, Andrica, Brocard, and Improved version of Legendre conjecture as a particular case of (a).

1. Introduction

In 1845, J. Bertrand conjectured that for every positive integer n, there is always atleast one prime p such that $n < p \leq 2n$. This was first proved by P. Chebyshev in 1850, therefore it is also called the Bertrand-Chebyshev Theorem(B.C.T). S. Ramanujan provided a very simple proof to B.C.T using elementary properties of Gamma function(see [7]). In conformity with S. Ramanujan, J. Nagura established the following:

Theorem 1.1 ([5]). *There is atleast one prime number between n and $\frac{6n}{5}$ for $n \geq 25$.*

Recently an interesting generalization of B.C.T was proposed by M.El.Bachraoui as an open problem: "Is it true for all integer $n > 1$ and a fixed integer $k \leq n$, there exists a prime number p such that $kn \leq p \leq (k+1)n$?", and proved that this is true for $k = 2$, whereas B.C.T answers this question affirmatively for $k = 1$. Latter, he concluded that a positive answer to this problem for every positive integer k with $k = n$ would prove Legendre conjecture(see [6]).
The purpose of this note is to provide a positive answer to the problem posed by M.El.Bachraoui. Consequently, we show that the conjectures of Legendre, Oppermann, Andrica, Brocard and Improved Version of Legendre conjecture are true.

2. Main Results

In what follows $\mathbb{Z}_+$ denote the set of postive integers.
Let $\pi(n)$ denote the number of prime numbers less than or equal to n, known as prime-counting function. Then one can restate B.C.T in terms of π as "For every $n \in \mathbb{Z}_+, \pi(2n) - \pi(n) \geq 1$".

Lemma 2.1 ([3],pp.427). *The function*

$$f(z) = \frac{e^{\frac{2\pi i \Gamma(z)}{z}} - 1}{e^{-\frac{2\pi i}{z}} - 1} \tag{2.1}$$

equals 0 or 1 according as z is composite or prime.

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Lemma 2.2 ([4])**.** *For $n \geq 5$ and $n \in \mathbb{Z}_+$, then*

$$\pi(n) = 2 + \sum_{q=5}^{n} \frac{e^{\frac{2\pi i \Gamma(q)}{q}} - 1}{e^{-\frac{2\pi i}{q}} - 1}. \tag{2.2}$$

The following theorem proves the claim made in (a).

Theorem 2.3. *For every integer $n > 1$ and a fixed integer $k \leq n$, there exists a prime number p such that $kn \leq p \leq (k+1)n$.*

Proof. We prove this theorem in two cases.
Case1: Let $k \in \mathbb{Z}_+$ and $1 \leq k \leq 5$.
It is clear that for all integers $n \geq 25$, $(k+1)n - \frac{6}{5}(kn) \geq 0$. By Theorem1.1, there exists atleast one prime in between kn and $(k+1)n$ for all $n \geq 25$.
By actual verification, we find that it is true for smaller values of n with $k \leq n$.
Case2: Let $k \in \mathbb{Z}_+$ and $k > 5$.
For each $q \in \mathbb{Z}_+$, we write $u_q = \frac{e^{\frac{2\pi i \Gamma(q)}{q}} - 1}{e^{-\frac{2\pi i}{q}} - 1}$. By Lemma2.2, for all $n \geq k$

$$\begin{aligned} \pi((k+1)n) &= 2 + \sum_{q=5}^{(k+1)n} u_q \\ &= 2 + \sum_{q=5}^{2k} u_q + \sum_{q=2k+1}^{(k+1)n} u_q \\ &= \pi(2k) + \sum_{q=2k+1}^{(k+1)n} u_q, \end{aligned} \tag{2.3}$$

and

$$\pi(kn) = \pi(k) + \sum_{q=k+1}^{kn} u_q. \tag{2.4}$$

Therefore,

$$\pi((k+1)n) - \pi(kn) = \pi(2k) - \pi(k) + \sum_{q=kn+1}^{(k+1)n} u_q - \sum_{q=k+1}^{2k} u_q. \tag{2.5}$$

Now,
$\pi(2k) - \pi(k) + \sum_{q=kn+1}^{(k+1)n} \min_q u_q - \sum_{q=k+1}^{2k} \min_q u_q \leq \pi((k+1)n) - \pi(kn) \leq \pi(2k) - \pi(k) + \sum_{q=kn+1}^{(k+1)n} \max_q u_q - \sum_{q=k+1}^{2k} \max_q u_q$ for all $n \geq k$.

In view of Lemma2.1, we have $\pi(2k) - \pi(k) + \sum_{q=kn+1}^{(k+1)n} 0 - \sum_{q=k+1}^{2k} 0 \leq \pi((k+1)n) - \pi(kn) \leq \pi(2k) - \pi(k) + \sum_{q=kn+1}^{(k+1)n} 1 - \sum_{q=k+1}^{2k} 1$
$\Rightarrow \pi(2k) - \pi(k) \leq \pi((k+1)n) - \pi(kn) \leq \pi(2k) - \pi(k) + (n-k)$.
Since $n \geq k$ and $\pi(2k) - \pi(k) \geq 1$ for all $k > 5$, $\pi((k+1)n) - \pi(kn) \geq 1$. This completes the proof. □

Corollary 2.4. *For each $n \in \mathbb{Z}_+$ and $n > 5$, $\pi(2n) - \pi(n) = \pi(n^2+n) - \pi(n^2)$.*

Proof. Follows from case 2. of Theorem2.3 by taking $k = n$. □

Now we can prove a few well-known conjectures in number theory as a special case of Theorem2.3. Most of them are still unsolved.

Corollary 2.5. *(Oppermann's Conjecture is true) For each $n \in \mathbb{Z}_+$ and $n > 1$, $\pi(n^2) - \pi(n^2 - n) \geq 1$ and $\pi(n^2 + n) - \pi(n^2) \geq 1$.*

Proof. Follows from Theorem2.3. □

Corollary 2.6. *For each $n \in \mathbb{Z}_+$, $\pi((n+1)^2) - \pi(n^2) \geq 2$*
(This conjecture is due to Adway Mitra et.al [1], called the Improved version of Legendre Conjecture).

Proof. Let $n \in \mathbb{Z}_+$. Then there exist primes p and q such that $n^2 \leq p \leq n(n+1)$ and $n(n+1) \leq q \leq (n+1)^2$. (By Theorem2.3)
Hence $\pi((n+1)^2) - \pi(n^2) \geq 2$. □

Corollary 2.7. *(Legendre's Conjecture is true) For each $n \in \mathbb{Z}_+$, $\pi((n+1)^2) - \pi(n^2) \geq 1$.*

Proof. Follows from Corollary2.6 □

Corollary 2.8. *(Brocard's Conjecture is true) For each integer $n > 1$, $\pi((p_{n+1})^2) - \pi((p_n)^2) \geq 4$ where p_n is the nth prime number.*

Proof. Let p_n, p_{n+1} be consecutive primes with $n > 1$.
Then $(p_n + 2)^2 \leq (p_{n+1})^2$ since the minimum gap between consecutive primes is 2. By applying Theorem2.3 repeatedly, there exist primes p, q, r, s such that $(p_n)^2 < p < p_n(p_n + 1)$, $p_n(p_n + 1) < q < (p_n + 1)(p_n + 1)$, $(p_n + 1)(p_n + 1) < r < (p_n + 2)(p_n + 1)$, $(p_n + 1)(p_n + 2) < s < (p_n + 2)^2$.
Hence $\pi((p_{n+1})^2) - \pi((p_n)^2) \geq 4$. □

Corollary 2.9. *(Andrica's Conjecture is true) The inequality $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all n, where p_n is the nth prime number.*

Proof. Let p_n, p_{n+1} be two consecutive prime numbers.
In view of Theorem 2.3 we can find a $k \in \mathbb{Z}_+$ such that p_n is in any one of these intervals $I_1 = [k(k-1), k^2]$, $I_2 = [k^2, k(k+1)]$, $I_3 = [k(k+1), (k+1)^2]$.
Case 1: Suppose p_n, $p_{n+1} \in I_2 \cup I_3$ then $k^2 < p_n < p_{n+1} < (k+1)^2$
$\Rightarrow \sqrt{p_{n+1}} - \sqrt{p_n} < 1$.
Case 2: Suppose p_n, $p_{n+1} \in I_1 \cup I_2$ then $k^2 - k + \frac{1}{4} < p_n < p_{n+1} < k^2 + k + \frac{1}{4}$
$\Rightarrow \sqrt{p_{n+1}} - \sqrt{p_n} < 1$.
Case3: Suppose $p_n \in I_3$ and $p_{n+1} \notin I_1 \cup I_2 \cup I_3$
Since p_n, p_{n+1} are consecutive primes,we have $(k+1)^2 < p_{n+1} < (k+1)(k+2)$ (in view of Theorem2.3),
$\Rightarrow k^2 + k + \frac{1}{4} < p_n < p_{n+1} < k^2 + 3k + \frac{9}{4}$
$\Rightarrow \sqrt{p_{n+1}} - \sqrt{p_n} < 1$. This completes the proof. □

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