Equation reconstruction of prime sequence

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Abstract The theorem for equation reconstruction of prime sequence is presented and proved. This theorem is that the prime sequence could have the determined general term formula through diophantine equation reconstruction of prime number. Using the theorem, the Goldbach Conjecture and Twin Primes Conjecture are proved.

1 Introduction

Because it can not be divisible except 1 and itself, primes are difficult to be described by appropriate expressions. This property makes prime sequence be difficult to be described such as arithmetic progression, geometric progression with the determined term formula. However, this property can make prime number establish some diophantine equations. And prime numbers can be decided by whether there is positive whole number solutions of these diophantine equations. Therefore, the expressions for solutions of these diophantine equations and its transform are used to describe the divisible property of prime number, and forming an equivalent sequence for the property. Thus, this will be easy to find the key node and the law implied to solve the problem. To this end, the theorem for equation reconstruction of prime sequence is presented and proved in this paper. And according to the theorem, the Goldbach Conjecture and Twin Primes Conjecture are proved. It could be hope to provide an idea and methods to solve similar problems.

In this paper, all parameters are positive whole number except where stated.

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2 Proof of the theorem for equation reconstruction of prime sequence

The theorem for equation reconstruction of prime sequence: The prime sequence could be equivalent to the sequence with the determined general term formula through equation reconstruction of prime number for the divisible property.

Proof.

Any prime number c could be expressed as $3a \pm 1$ (a is an positive even), $4a \pm 1$ or $6a \pm 1$.

Proof is carried out the following in the case of $3a \pm 1$ first.

If $3a \pm 1$ is a prime number, it certainly can not be written $3a \pm 1 = (3x_1 \pm 1)(3x_2 \pm 1)$, otherwise, and vice versa.

Case 1: $3a + 1$

$$3a + 1 = (3x_1 + 1)(3x_2 + 1) = 9x_1x_2 + 3(x_1 + x_2) + 1$$

or

$$3a + 1 = (3x_1' - 1)(3x_2' - 1) = 9x_1'x_2' - 3(x_1' + x_2') + 1$$

Where, let $-x_1' = x_1$, $-x_2' = x_2$.

Then there is $a = 3x_1x_2 + (x_1 + x_2)$.

It is easy to see that whether $3a + 1$ is a prime number depends entirely on the a.

Namely $3a + 1$ is a prime number that is equivalent x_1 and x_2 are both positive whole number in $a = 3x_1x_2 + (x_1 + x_2)$.

Let $x_1 + x_2 = -q$, $x_1x_2 = p$

According to Vieta's formulas, equation (1) is established.

$$x^2 + qx + p = 0 \tag{1}$$

Then there is $x_{1,2} = \dfrac{-q \pm \sqrt{q^2 - 4p}}{2}$

Therefore, if x_1 and x_2 of equation (1) roots are not both positive whole number, $3a + 1$ must be a prime number. Otherwise, it will be a composite number.

There is

$$a = 3p - q$$

Obviously, if $3a+1$ is a prime number, q and $\sqrt{q^2-4p}$ are not both even numbers.

Therefore, in the divisible property of prime number, c_i in prime sequence $\{c_n\}$ is equivalent to $a_i = 3p_i - q_i$ in sequence $\{a_n\}$, namely prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n\}$.

Here q_i and $\sqrt{q_i^2-4p_i}$ are not both even numbers.

In order to facilitate the expression, let $q = 2s$, $p = 2r$.

Here s and r are real numbers.

$\therefore x_{1,2} = s \pm \sqrt{s^2-2r}$

Let $\sqrt{s^2-2r} = t$

There is

$$a = 12st - 12t^2 - 2s$$

Therefore, $a_i = 3p_i - q_i$ in sequence $\{a_n\}$ (q_i and $\sqrt{q_i^2-4p_i}$ are not both even numbers) is equivalent to $a_i' = 12s_it_i - 12t_i^2 - 2s_i$ in sequence $\{a_n'\}$ (s_i and t_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n'\}$.

It is obvious that

$$t = \frac{s \pm \sqrt{s^2 + \frac{2s-a}{3}}}{2}$$

Let $s^2 + \frac{2s-a}{3} = e^2$

Then there is

$$a = 3s^2 + 2s - 3e^2$$

Therefore, $a_i' = 12s_it_i - 12t_i^2 - 2s_i$ in sequence $\{a_n'\}$ (s_i and t_i are not both positive whole number solutions) is equivalent to $a_i'' = 3s_i^2 + 2s_i - 3e_i^2$ in sequence $\{a_n''\}$ (s_i and e_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n''\}$.

It is obvious that

$$s = \frac{-1 \pm \sqrt{9e^2 + 3a + 1}}{3}$$

Let $9e^2 + 3a + 1 = (3h+1)^2$

Then there is

$$3a + 1 = (3h+1)^2 - (3e)^2$$

$$a = 3h^2 + 2h - 3e^2$$

Therefore, $a_i'' = 3s_i^2 + 2s_i - 3e_i^2$ in sequence $\{a_n''\}$ (s_i and e_i are not both positive whole number solutions) is equivalent to $3a_n''' + 1 = (3h_i + 1)^2 - (3e_i)^2$ in sequence $\{a_n''\}$ (e_i and h_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n'''\}$.

Case 2: $3a - 1$

$$3a - 1 = (3x_1' + 1)(3x_2' - 1) = 9x_1'x_2' + 3(x_2' - x_1') - 1$$

Where, let $-x_1' = x_1$, $x_2' = x_2$.

Then there is $a = -3x_1x_2 + (x_1 + x_2)$.

Namely $3a - 1$ is a prime number that is equivalent x_1 and x_2 are both positive whole number in $a = -3x_1x_2 + (x_1 + x_2)$.

Let $x_1 + x_2 = -q$, $x_1 x_2 = p$. Here p is negative whole number.

According to Vieta's formulas, equation (2) is established.

$$x^2 + qx + p = 0 \tag{2}$$

Then there is $x_{1,2} = \dfrac{-q \pm \sqrt{q^2 - 4p}}{2}$.

Therefore, if x_1 and x_2 of equation (2) roots are not both positive whole number, $3a-1$ must be a prime number. Otherwise, it will be a composite number.

There is

$$a = -3p - q$$

Obviously, if $3a-1$ is a prime number, q and $\sqrt{q^2 - 4p}$ are not both even numbers.

Therefore, in the divisible property of prime number, c_i in prime sequence $\{c_n\}$ is equivalent to $a_i = -3p_i - q_i$ in sequence $\{a_n\}$, namely prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n\}$.

Using the same argument as in the case 1, we can easily get

$$a = 12t^2 - 12st - 2s$$

Therefore, $a_i = -3p_i - q_i$ in sequence $\{a_n\}$ (q_i and $\sqrt{q_i^2 - 4p_i}$ are not both even numbers) is equivalent to $a_i' = 12t_i^2 - 12s_i t_i - 2s_i$ in sequence $\{a_n'\}$ (s_i and t_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a_n'\}$.

It is obvious that

$$t = \frac{s \pm \sqrt{s^2 + \dfrac{2s+a}{3}}}{2}$$

Let $s^2 + \dfrac{2s+a}{3} = e^2$

Then there is

$$a = 3e^2 - 3s^2 - 2s$$

Therefore, $a'_i = 12t_i^2 - 12s_i t_i - 2s_i$ in sequence $\{a'_n\}$ (s_i and t_i are not both positive whole number solutions) is equivalent to $a''_i = 3e_i^2 - 3s_i^2 - 2s$ in sequence $\{a''_n\}$ (s_i and e_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a''_n\}$.

It is obvious that

$$s = \frac{-1 \pm \sqrt{9e^2 - 3a + 1}}{3}$$

Let $9e^2 - 3a + 1 = (3h+1)^2$

Then there is

$$3a - 1 = (3e)^2 - (3h+1)^2$$

$$a = 3e^2 - 3h^2 - 2h$$

Therefore, $a''_i = 3e_i^2 - 3s_i^2 - 2s$ in sequence $\{a''_n\}$ (s_i and e_i are not both positive whole number solutions) is equivalent to $3a'''_n - 1 = (3e_i)^2 - (3h_i + 1)^2$ in sequence $\{a''_n\}$ (e_i and h_i are not both positive whole number solutions) .

Namely, in the divisible property of prime number, prime sequence $\{c_n\}$ is equivalent to sequence $\{a'''_n\}$.

The prime sequence that prime number c could be expressed as $4a \pm 1$ or $6a \pm 1$, have equivalent methods that are similar to the case of $3a \pm 1$. It can be proved in the same way as shown in the case of $3a \pm 1$ before. Of course, some new equivalent sequences are reconstructed through establishing other forms equations.

This completes the proof.

According to above proof, in the divisible property of prime number, the prime sequence $\{c_n\}$ without term formula is analyzed by using the sequence $\{a_n\}$、 $\{a'_n\}$、

$\{a''_n\}$、$\{a'''_n\}$ with term formula. This will be easy to find the key node and the law implied to solve the problem.

3 Proof of the Goldbach Conjecture

The Goldbach Conjecture：

Conjecture(A): Every even integer greater than 4 can be expressed as the sum of two primes.

Conjecture(B): Every odd integer greater than 7 can be expressed as the sum of three primes.

The proof of conjecture(A).

For even less than 10, there are $4 = 2 + 2$, $6 = 3 + 3$, $8 = 5 + 3$.

Therefore, Conjecture (A) holds for even less than 10.

For even greater than 10, it proves Conjecture (A) with the reduction to absurdity follows.

If Conjecture (A) is not true, it becomes: there is at least one of even $2n$ greater than 10 that can not be expressed as the sum of two primes.

For even greater than 10, there is

$$2n = (3a \pm 1) + (3b \pm 1) = \begin{cases} 3a + 3b + 2 \\ 3a + 3b \\ 3a + 3b - 2 \end{cases}$$

Where, a and b are both positive even numbers, then $3a \pm 1$ and $3b \pm 1$ are odd numbers.

There are three cases for $2n$. Proof is carried out the following in the case 1.

Case 1: $2n = (3a + 1) + (3b + 1) = 3a + 3b + 2$

According to the theorem for equation reconstruction of prime sequence, let

$$3a + 1 = (3x_{a1} + 1)(3x_{a2} + 1) = 9x_{a1}x_{a2} + 3(x_{a1} + x_{a2}) + 1$$

or

$$3a + 1 = (3x'_{a1} - 1)(3x'_{a2} - 1) = 9x'_{a1}x'_{a2} - 3(x'_{a1} + x'_{a2}) + 1$$

$$3b+1=(3x_{b1}+1)(3x_{b2}+1)=9x_{b1}x_{b2}+3(x_{b1}+x_{b2})+1$$

or

$$3b+1=(3x'_{b1}-1)(3x'_{b2}-1)=9x'_{b1}x'_{b2}-3(x'_{b1}+x'_{b2})+1$$

Where, let $-x'_{a1}=x_{a1}$, $-x'_{a2}=x_{a2}$, $-x'_{b1}=x_{b1}$, $-x'_{b2}=x_{b2}$.

Then there is

$$a=3x_{a1}x_{a2}+(x_{a1}+x_{a2})$$

$$b=3x_{b1}x_{b2}+(x_{b1}+x_{b2})$$

Let

$$\begin{cases} x_{a1}+x_{a2}=-q_a \\ x_{a1}x_{a2}=p_a \end{cases}, \quad \begin{cases} x_{b1}+x_{b2}=-q_b \\ x_{b1}x_{b2}=p_b \end{cases}$$

Then there is

$$a=3p_a-q_a \tag{3}$$

$$b=3p_b-q_b \tag{4}$$

According to Vieta's formulas, equation (5) and equation (6) is established.

$$x_a^2+q_ax_a+p_a=0 \tag{5}$$

$$x_b^2+q_bx_b+p_b=0 \tag{6}$$

The roots of equation (5) are

$$x_{a1,2}=\frac{-q_a\pm\sqrt{q_a^2-4p_a}}{2}$$

The roots of equation (6) are

$$x_{b1,2}=\frac{-q_b\pm\sqrt{q_b^2-4p_b}}{2}$$

According to the theorem for equation reconstruction of prime sequence, if $3a+1$ and $3b+1$ are prime numbers, equation (5) and equation (6) have integer solutions. Therefore, at least one of equation (5) and equation (6) has integer solutions.

Since $3a\pm1$ and $3b\pm1$ are both odd numbers, and $a+b$ is an even number. a and b are both even numbers.

And according to equation (3) and equation (4), p_a and q_a are both even numbers, p_b and q_b are both even numbers.

Since q_a and $\sqrt{q_a^2-4p_a}$ are both even numbers, q_b and $\sqrt{q_b^2-4p_b}$ are both even numbers. x_{a1} and x_{a2} are both even numbers, x_{b1} and x_{b2} are both even numbers.

Let $q_a=2s_a$, $p_a=2r_a$, $q_b=2s_b$, $p_b=2r_b$.

Then, there are

$$x_{a1,2}=s_a\pm\sqrt{s_a^2-2r_a}$$

$$x_{b1,2}=s_b\pm\sqrt{s_b^2-2r_b}$$

And s_a and $\sqrt{s_a^2-2r_a}$ are both even numbers or both odd numbers, s_b and $\sqrt{s_b^2-2r_b}$ are both even numbers or both odd numbers.

Therefore, s_a^2 and $(s_a^2-2r_a)$ are both even numbers or both odd numbers, s_b^2 and $(s_b^2-2r_b)$ are both even numbers or both odd numbers.

Let $\sqrt{s_a^2-2r_a}=t_a$, $\sqrt{s_b^2-2r_b}=t_b$, $a+b=2m$.

Then, it is easy to see that r_a and r_b must be even numbers.

And there are

$$0<t_a\le\sqrt{(3s_a-1)^2-1}+\frac{s_a}{2},\quad t_{a\max}=\left[\sqrt{(3s_a-1)^2-1}+\frac{s_a}{2}\right]$$

$$0<t_b\le\sqrt{(3s_b-1)^2-1}+\frac{s_b}{2},\quad t_{b\max}=\left[\sqrt{(3s_b-1)^2-1}+\frac{s_b}{2}\right]$$

$$d=(6s_at_a-6t_a^2-s_a)+(6s_bt_b-6t_b^2-s_b),\quad s_a>t_a,\quad s_b>t_b$$

$$\frac{a}{2}=6s_at_a-6t_a^2-s_a,\quad \frac{b}{2}=6s_bt_b-6t_b^2-s_b$$

Set sequence $\{J_n\}$, $J_i=6s_it_i-6t_i^2-s_i$, $i=1,2,3,\cdots n$.

Then, there are $J_i=\dfrac{a_i}{2}$, $J_{n-i}=\dfrac{b_i}{2}$.

And there is

$$d = J_i + J_{n-i} = J_{i+1} + J_{n-i-1} = \cdots = J_{i+\Delta} + J_{n-i-\Delta} = J_0 + J_n$$

Where, J_0 can be seen as a constant.

It is easy to see that only s_i and t_i corresponds to a unique J_i.

It is also easy to see that there are

$$d = \frac{a_i}{2} + \frac{b_i}{2} = J_i + J_{n-i}$$

$$d = \frac{a_i + 2}{2} + \frac{b_i - 2}{2} = \frac{a_{i+1}}{2} + \frac{b_{i-1}}{2} = (J_i + 1) + (J_{n-i} - 1)$$

And the like, there is

$$d = \frac{a_i + 2\Delta}{2} + \frac{b_i - 2\Delta}{2} = \frac{a_{i+\Delta}}{2} + \frac{b_{i-\Delta}}{2} = (J_i + \Delta) + (J_{n-i} - \Delta)$$

Since at least one of equation (5) and equation (6) respectively corresponding to $a_{i+\Delta}$ and $b_{i-\Delta}$ must has integer solutions, at least one of $J_i + \Delta = J_{i+\Delta}$ and $J_{n-i} - \Delta = J_{n-i-\Delta}$ in sequence $\{J_n\}$ must hold.

Namely, for sequence $\{H_m\}$ consists of $\left\{J_{\frac{n}{2}}\right\}$ and $\left\{J_n - \frac{J_1 + J_n}{2}\right\}$ together, there are $H_j = \begin{cases} J_i \\ J_{n-i} - \frac{J_n}{2} \end{cases}$ ($i < \frac{n+1}{2}$, $J_i \neq J_{n-i} - \frac{J_n}{2}$, $H_j \neq H_{j+\Delta}$) .

Then, sequence $\{H_m\}$ must have $H_j + \Delta = H_{j+\Delta}$.

And sequence $\{H_m\}$ must be an arithmetic progression.

Therefore, sequence $\{H_m\}$ corresponding to equation (5) and equation (6) must be an arithmetic progression.

If n is an even number, $\sum_{j=1}^{m} H_j = \sum_{i=1}^{n} J_i - \sum_{i=\frac{n}{2}+1}^{n} \frac{J_n}{2} - \sum_{i=1}^{n'} J_i'$.

If n is an odd number, $\sum_{j=1}^{m} H_j = \sum_{i=1}^{n} J_i - \sum_{i=\frac{n+1}{2}}^{n} \frac{J_n}{2} - \sum_{i=1}^{n'} J_i'$.

Where, $\sum_{i=1}^{n'} J_i'$ is that the sum of all term $J_i = J_{n-i} - \frac{J_n}{2}$.

According to sequence $\{H_m\}$, it follows

$$\sum_{j=1}^{m} H_j = \frac{J_n^2 + 10J_n + 24}{8} - \frac{(s^2 - 2s - 8)\cdot t_{\max}}{8} - \frac{(s^2 - 2s - 8)\cdot t_{\max}\cdot J_n}{32}$$

$$\sum_{j=1}^{m} H_j = \frac{3}{16}s^2 t_{\max}^3 - \frac{3}{16}s^3 t_{\max}^2 + \frac{1}{32}s^3 t_{\max} + \frac{259}{8}s^2 t_{\max}^2 - \frac{579}{8}st_{\max}^3 + 36t_{\max}^4 - \frac{195}{16}s^2 t_{\max}$$
$$+ \frac{27}{2}st_{\max}^2 - \frac{3}{2}t_{\max}^3 + s^2 + 60st - 60t_{\max}^2 + 10s + t_{\max} + 24$$

Here $t_{\max} = \left[\sqrt{(3s-1)^2 - 1} + \frac{s}{2}\right]$

Therefore, the highest degree of the sum of the first m terms in sequence $\{H_m\}$ is fifth degree.

It is easy to see that this is in contradiction with the necessary and sufficient condition of the arithmetic progression, in which the highest degree of the sum of the first m terms is second degree.

Therefore, Conjecture (A) is true for Case 1.

This completes the proof for Case 1.

Case 2: $2n = (3a-1) + (3b-1) = 3a + 3b - 2$

According to the theorem for equation reconstruction of prime sequence, let

$$3a - 1 = (3x_{a1}' + 1)(3x_{a2}' - 1) = 9x_{a1}'x_{a2}' + 3(x_{a2}' - x_{a1}') - 1$$

$$3b - 1 = (3x_{b1}' + 1)(3x_{b2}' - 1) = 9x_{b1}'x_{b2}' + 3(x_{b2}' - x_{b1}') - 1$$

Where, let $-x_{a1}' = x_{a1}$, $x_{a2}' = x_{a2}$, $-x_{b1}' = x_{b1}$, $x_{b2}' = x_{b2}$.

Using the same argument as in the case 1, we can easily get

$$\frac{a}{2} = 6t_a^2 - 6s_a t_a - s_a, \quad \frac{b}{2} = 6t_b^2 - 6s_b t_b - s_b$$

And there are

$$0 < s_a \le \frac{6t_a^2}{6t_a + 1}, \quad s_{a\max} = \left[\frac{6t_a^2}{6t_a + 1}\right]$$

$$0 < s_b \le \frac{6t_b^2}{6t_b+1}, \quad s_{b\max} = \left[\frac{6t_b^2}{6t_b+1}\right].$$

Set sequence $\{J_n\}$, $J_i = 6t_i^2 - 6s_it_i - s_i$, $i = 1,2,3,\cdots n$.

Therefore, at least one of $J_i + \Delta = J_{i+\Delta}$ and $J_{n-i} - \Delta = J_{n-i-\Delta}$ in sequence $\{J_n\}$ must hold.

Namely, for sequence $\{H_m\}$ consists of $\left\{J_{\frac{n}{2}}\right\}$ and $\left\{J_n - \frac{J_1+J_n}{2}\right\}$ together, there are $H_j = \begin{cases} J_i \\ J_{n-i} - \frac{J_n}{2} \end{cases}$ （$i < \frac{n+1}{2}$, $J_i \ne J_{n-i} - \frac{J_n}{2}$, $H_j \ne H_{j+\Delta}$）.

Therefore, sequence $\{H_m\}$ must be an arithmetic progression.

If n is an even number, $\sum_{j=1}^{m} H_j = \sum_{i=1}^{n} J_i - \sum_{i=\frac{n}{2}+1}^{n} \frac{J_n}{2} - \sum_{i=1}^{n'} J_i'$.

If n is an odd number, $\sum_{j=1}^{m} H_j = \sum_{i=1}^{n} J_i - \sum_{i=\frac{n+1}{2}}^{n} \frac{J_n}{2} - \sum_{i=1}^{n'} J_i'$.

Where, $\sum_{i=1}^{n'} J_i'$ is that the sum of all term $J_i = J_{n-i} - \frac{J_n}{2}$.

According to sequence $\{H_m\}$, it follows

$$\sum_{j=1}^{m} H_j = \frac{J_n^2 + 10J_n + 24}{8} - \frac{(s_{\max}^2 - 2s_{\max} - 8)\cdot t}{8} - \frac{(s_{\max}^2 - 2s_{\max} - 8)\cdot t \cdot J_n}{32}$$

$$\sum_{j=1}^{m} H_j = \frac{3}{16}s_{\max}^3 t^2 - \frac{3}{16}s_{\max}^2 t^3 + \frac{1}{32}s_{\max}^3 t + \frac{33}{8}s_{\max}^2 t^2 - \frac{81}{8}s_{\max} t^3 + \frac{9}{2}t^4 + \frac{21}{16}s_{\max}^2 t$$
$$+ \frac{3}{2}s_{\max} t^2 + \frac{3}{2}t^3 + \frac{1}{8}s_{\max}^2 - \frac{15}{2}s_{\max} t + \frac{15}{2}t^2 - \frac{5}{4}s_{\max} + t + 3$$

Therefore, the highest degree of the sum of the first m terms in sequence $\{H_m\}$ is fifth degree.

It is easy to see that this is also in contradiction with the necessary and sufficient condition of the arithmetic progression, in which the highest degree of the sum of the first m terms is second degree.

Therefore, Conjecture (A) is true for Case 2.

This completes the proof for Case 2.

Case 3: $2n = (3a+1)+(3b-1) = 3a+3b$

The equation reconstruction of $3a+1$ can be gotten in the same way as Case 1.

There are

$$3a+1 = (3x_{a1}+1)(3x_{a2}+1) = 9x_{a1}x_{a2} + 3(x_{a1}+x_{a2}) + 1$$

or

$$3a+1 = (3x'_{a1}-1)(3x'_{a2}-1) = 9x'_{a1}x'_{a2} - 3(x'_{a1}+x'_{a2}) + 1$$

The equation reconstruction of $3b-1$ can be gotten in the same way as Case 2.

There is

$$3b-1 = (3x'_{b1}+1)(3x'_{b2}-1) = 9x'_{b1}x'_{b2} + 3(x'_{b2}-x'_{b1}) - 1$$

Using the same argument as shown before, we can easily get

$$\frac{a}{2} = 6s_a t_a - 6t_a^2 - s_a \tag{7}$$

$$\frac{b}{2} = 6t_b^2 - 6s_b t_b - s_b \tag{8}$$

For equation (7), there are

$$t_a = \frac{s_a \pm \sqrt{s_a^2 + \frac{2s_a - a}{3}}}{2}$$

Let $s_a^2 + \frac{2s_a - a}{3} = e_a^2$

Then, there is

$$s_a = \frac{-1 \pm \sqrt{9e_a^2 + 3a + 1}}{3}$$

Let $9e_a^2 + 3a + 1 = (3h_a + 1)^2$. Namely it makes s_a be a positive whole number.

Then, there is

$$3a+1 = (3h_a+1)^2 - 9e_a^2$$

$$a = 3h_a^2 + 2h_a - 3e_a^2$$

Set sequence $\{H_{an}\}$, $H_{ai} = 3h_{ai}^2 + 2h_{ai} - 3e_{ai}^2$, $i = 1,2,3,\cdots n$.

For equation (8), there are

$$t_b = \frac{s_b \pm \sqrt{s_b^2 + \frac{2s_b + b}{3}}}{2}$$

Let $s_b^2 + \frac{2s_b + b}{3} = e_b^2$

Then, there is

$$s_b = \frac{-1 \pm \sqrt{9e_b^2 - 3b + 1}}{3}$$

Let $9e_b^2 - 3b + 1 = (3h_b + 1)^2$. Namely it makes s_a be a positive whole number.

Then, there is

$$3b - 1 = 9e_b^2 - (3h_b + 1)^2$$

$$b = 3e_b^2 - 3h_b^2 - 2h_b$$

Set sequence $\{H_{bm}\}$, $H_{bm} = 3e_{bj}^2 - 3h_{bj}^2 - 2h_{bj}$, $i = 1,2,3,\cdots m$.

Then, there are $H_{ai} = \frac{a_i}{2}$, $H_{b(m-j)} = \frac{b_j}{2}$.

Therefore, sequence $\{H_{an}\}$ and sequence $\{H_{bm}\}$ merge into sequence $\{H_N\}$. Here

$H_k = \left|3h_k^2 + 2h_k - 3e_k^2\right|$, $H_k \neq H_{k+\Delta}$.

Since $d = \frac{a_i}{2} + \frac{b_j}{2}$, there is $i + j = N$.

Then, there is $d = H_k + H_{N-k} = H_{k+1} + H_{N-k-1)} = \cdots = H_{k+\Delta} + H_{N-k-\Delta)} = H_0 + H_N$.

Where H_0 is a constant.

It is easy to see that only h_k and e_k corresponds to a unique H_k.

It is also easy to see that there are

$$d = \frac{a_i}{2} + \frac{b_j}{2} = H_k + H_{N-k}$$

$$d=\frac{a_i+2}{2}+\frac{b_j-2}{2}=\frac{a_{i+1}}{2}+\frac{b_{j-1}}{2}=(H_k+1)+(H_{N-k}-1)$$

And the like, there is

$$d=\frac{a_i+2\Delta}{2}+\frac{b_j-2\Delta}{2}=\frac{a_{i+\Delta}}{2}+\frac{b_{j-\Delta}}{2}=(H_k+\Delta)+(H_{N-k}-\Delta)$$

Since at least one of reconstruction equations respectively corresponding to $a_{i+\Delta}$ and $b_{i-\Delta}$ must has integer solutions, at least one of $H_k+\Delta=H_{k+\Delta}$ and $H_{N-k}-\Delta=H_{N-k-\Delta}$ in sequence $\{H_N\}$ must hold.

Then, sequence $\{H_N\}$ must be an arithmetic progression.

Therefore, sequence $\{H_N\}$ corresponding to reconstruction equations must be an arithmetic progression.

According to sequence $\{H_N\}$, it follows

$$\sum_{k=1}^{N}H_k=\sum_{i=1}^{n}H_{ai}+\sum_{j=1}^{m}H_{bj}-\sum_{f=1}^{F}H_f'''$$

$$\sum_{i=1}^{n}H_{ai}=\sum_{h_a=1}^{h_a}\sum_{e_a=1}^{e_a}\left(3h_a^2+2h_a-3e_a^2\right)=h_a^3e_a+\frac{5}{2}h_a^2e_a+\frac{3}{2}h_ae_a-e_a^3-\frac{3}{2}e_a^2-\frac{1}{2}e_a$$

$$\sum_{j=1}^{m}H_{bj}=\sum_{e_b=3}^{e_b}\sum_{h_b=1}^{h_b}\left(3e_b^2-3h_b^2-2h_b\right)=h_be_b^3-3h_b^3e_b+\frac{3}{2}h_be_b^2-\frac{15}{2}h_b^2e_b-4h_be_b-5h_b$$

Where, H_f''' is term $H_{ai}=H_{bj}$ of sequence $\{H_{an}\}$ and sequence $\{H_{bm}\}$. That is $H_f'''=H_a=H_b$.

Namely, $3(h-\Delta_1)^2+2(h-\Delta_1)-3(e+\Delta_2)^2=-\left(3h^2+2h-3e^2\right)$.

Then, there is $H_f'''=3h_f\Delta_{1f}-\frac{3}{2}\Delta_{1f}^2+\Delta_{1f}+3e_f\Delta_{2f}+\frac{3}{2}\Delta_{2f}^2$.

Where, Δ_1 and Δ_2 are both even numbers.

Here

$$\sum_{f=1}^{F} H'''_f = \sum_{\Delta_1=2}^{\Delta_1}\sum_{\Delta_2=2}^{\Delta_2}\sum_{h=1}^{h}\sum_{e=1}^{e}\left(3h\Delta_1 - \frac{3}{2}\Delta_1^2 + \Delta_1 + 3e\Delta_2 + \frac{3}{2}\Delta_2^2\right)$$

$$= \frac{3}{16}h^2 e\Delta_1^2\Delta_2 + \frac{3}{16}he^2\Delta_1\Delta_2^2 + \frac{3}{8}he^2\Delta_1\Delta_2$$

$$-\frac{1}{8}he\Delta_1^3\Delta_2 + \frac{1}{8}he\Delta_1\Delta_2^3 + \frac{9}{16}he\Delta_1\Delta_2^2 - \frac{3}{16}he\Delta_1^2\Delta_2 + \frac{7}{48}he\Delta_1\Delta_2$$

Therefore, the highest degree of the sum of the first N terms in sequence $\{H_N\}$ is sixth degree.

It is easy to see that this is also in contradiction with the necessary and sufficient condition of the arithmetic progression, in which the highest degree of the sum of the first m terms is second degree.

Therefore, Conjecture (A) is true for Case 3.

This completes the proof for Case 3.

Taking above three cases, Conjecture (A) is true.

The proof of Conjecture (A) is now completed.

The proof of conjecture(B).

Since Conjecture (A) is true, every even integer greater $2n$ than 4 can be expressed as the sum of prime number $3a \pm 1$ and prime number $3b \pm 1$.

It is easy to see that every odd integer greater $2n+3$ than 7 can be expressed as the sum of 3, prime number $3a \pm 1$ and prime number $3b \pm 1$.

Namely $2n+3 = 3 + (3a \pm 1) + (3b \pm 1)$.

Therefore, Conjecture (B) is true.

The proof of the Goldbach Conjecture is now completed.

4 Proof of the Twin Primes Conjecture

The Twin Primes Conjecture: There are infinitely many primes that differs from another prime number by two.

Proof.

It proves the Twin Primes Conjecture with the reduction to absurdity follows.

If Conjecture is not true, it becomes: there is an even number a large enough that makes at least one of every twin odd numbers no less than $3a \pm 1$ be a composite number.

According to the theorem for equation reconstruction of prime sequence, there are

$$3a - 1 = (3x_1 + 1)(3x_2 - 1) \tag{9}$$

$$3a + 1 = (3x_1' + 1)(3x_2' + 1) \text{ or } 3a + 1 = (3x_1' - 1)(3x_2' - 1) \tag{10}$$

Where, $a = 2l$, l is positive whole number.

Therefore, when a is large enough, at least one of equation (9) and equation (10) has integer solutions.

Using the same argument as in the proof of the Goldbach Conjecture, we can easily get this statement fellows.

For equation (9), there are

$$\frac{a}{2} = 6t_1^2 - 6s_1t_1 - s_1$$

$$t_1 = \frac{s_1 \pm \sqrt{s_1^2 + \frac{2s_1 + a}{3}}}{2}$$

Let $s_1^2 + \frac{2s_1 + a}{3} = e_1^2$

Then, there is

$$s_1 = \frac{-1 \pm \sqrt{9e_1^2 - 6l + 1}}{3}$$

Let $9e_1^2 - 6l + 1 = (3h_1 + 1)^2$, Namely it makes s_1 be a positive whole number.

Then, there is

$$6l - 1 = 9e_1^2 - (3h_1 + 1)^2$$

$$a = 2l = 3e_1^2 - 3h_1^2 - 2h_1$$

Set sequence $\{H_n'\}$, $H_i' = 3e_{1i}^2 - 3h_{1i}^2 - 2h_{1i}$, $i = 1, 2, 3, \cdots n$.

For equation (10), there are

$$\frac{a}{2} = 6s_2t_2 - 6t_2^2 - s_2$$

$$t_2 = \frac{s_2 \pm \sqrt{s_2^2 + \frac{2s_2 - a}{3}}}{2}$$

Let $s_2^2 + \frac{2s_2 - a}{3} = e_2^2$

Then, there is

$$s_2 = \frac{-1 \pm \sqrt{9e_2^2 + 6l + 1}}{3}$$

Let $9e_2^2 + 6l + 1 = (3h_2 + 1)^2$, Namely it makes s_2 be a positive whole number.

Then, there is

$$6l + 1 = (3h_2 + 1)^2 - 9e_2^2$$

$$a = 2l = 3h_2^2 + 2h_2 - 3e_2^2$$

Set sequence $\{H''_m\}$, $H''_m = 3h_{2j}^2 + 2h_{2j} - 3e_{2j}^2$, $i = 1,2,3,\cdots m$.

Therefore, at least one of $H'_i = a_g$ and $H''_j = a_g$ is true for a_g in sequence $\{a_M\}$ consists of a.

Then, sequence $\{H'_n\}$ and sequence $\{H''_m\}$ merge into sequence $\{H_N\}$. Here $H_k = \left|3h^2 + 2h - 3e^2\right|$, $H_k \neq H_{k+1}$. And H_1 is large engouh.

Therefore, there must be $H_k = a_g$ for any a_g large engouh.

Since $\{a_M\}$ is an arithmetic progression of 2, sequence $\{H_N\}$ is an arithmetic progression.

According to sequence $\{H_N\}$, it follows

$$\sum_{k=1}^{N} H_k = \sum_{i=1}^{n} H'_i + \sum_{j=1}^{m} H''_j - \sum_{f=1}^{F} H'''_f$$

$$\sum_{i=1}^{n} H'_i = \sum_{e_1=3}^{e_1}\sum_{h_1=1}^{h_1}\left(3e_1^2 - 3h_1^2 - 2h_1\right) - H'_1(g)$$

$$= h_1e_1^3 - h_1^3e_1 + \frac{3}{2}h_1e_1^2 - \frac{5}{2}h_1^2e_1 - \frac{22}{3}h_1e_1 - H'_1(g)$$

Where $H'_1(g)$ is the sum of all terms less than H_1 in sequence $\{H'_n\}$.

$$\sum_{j=1}^{m} H''_j = \sum_{h_2=1}^{h_2}\sum_{e_2=1}^{e_2}\left(3h_2^2 + 2h_2 - e_2^2\right) - H'''_1(g)$$
$$= h_2^3 e_2 + \frac{5}{2}h_2^2 e_2 + h_2 e_2 - h_2 e_2^3 - \frac{3}{2}h_2 e_2^2 - H'''_1(g)$$

Where $H'''_1(g)$ is the sum of all terms less than H_1 in sequence $\{H''_m\}$.

H'''_f is term $H'_i = H''_j$ of sequence $\{H'_n\}$ and sequence $\{H''_m\}$. That is $H'''_f = H'_i = H''_j = H_k = H_{k+1}$.

Namely, $3(h-\Delta_1)^2 + 2(h-\Delta_1) - 3(e+\Delta_2)^2 = -(3h^2 + 2h - 3e^2)$.

Then, there is $H'''_f = 3h_f\Delta_{1f} - \frac{3}{2}\Delta_{1f}^2 + \Delta_{1f} + 3e_f\Delta_{2f} + \frac{3}{2}\Delta_{2f}^2$.

Where, Δ_1 and Δ_2 are both even numbers.

Here

$$\sum_{f=1}^{F} H'''_f = \sum_{\Delta_1=2}^{\Delta_1}\sum_{\Delta_2=2}^{\Delta_2}\sum_{h=1}^{h}\sum_{e=1}^{e}\left(3h\Delta_1 - \frac{3}{2}\Delta_1^2 + \Delta_1 + 3e\Delta_2 + \frac{3}{2}\Delta_2^2\right) - H'''_1(g)$$
$$= \frac{3}{16}h^2 e\Delta_1^2\Delta_2 + \frac{3}{16}he^2\Delta_1\Delta_2^2 + \frac{3}{8}he^2\Delta_1\Delta_2$$
$$-\frac{1}{8}he\Delta_1^3\Delta_2 + \frac{1}{8}he\Delta_1\Delta_2^3 + \frac{9}{16}he\Delta_1\Delta_2^2 - \frac{3}{16}he\Delta_1^2\Delta_2 + \frac{7}{48}he\Delta_1\Delta_2 - H'''_1(g)$$

Where $H'''_1(g)$ is the sum of all terms less than H_1 and satisfied $H_k = \left|3h^2 + 2h - e^2\right|$ in sequence $\{H''_m\}$.

Therefore, the highest degree of the sum of the first N terms in sequence $\{H_N\}$ is sixth degree

It is easy to see that this is also in contradiction with the necessary and sufficient condition of the arithmetic progression, in which the highest degree of the sum of the first m terms is second degree.

Therefore, the Twin Primes Conjecture is true.

This completes the proof.

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