Non-Solvable Equation Systems with Graphs Embedded in $\mathbb{R}^n$

Linfan Mao

(*Beijing University of Civil Engineering and Architecture, Beijing 100045*)

Beijing, P.R.China

June 28, 2013

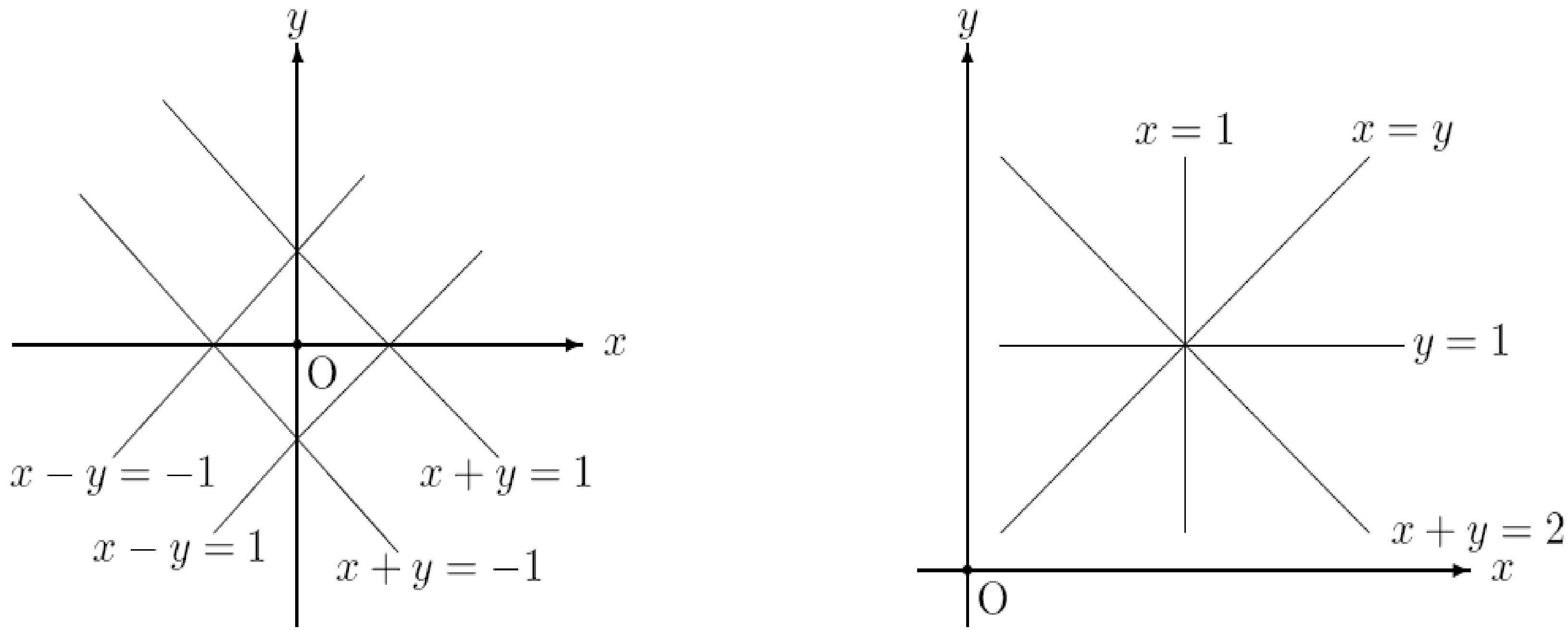
§1. **Introduction**

Consider two systems of linear equations following:

$$(LES_4^N) \quad \begin{cases} x+y &= 1 \\ x+y &= -1 \\ x-y &= -1 \\ x-y &= 1 \end{cases} \qquad\qquad (LES_4^S) \quad \begin{cases} x=y \\ x+y=2 \\ x=1 \\ y=1 \end{cases}$$

(LES_4^N) is non-solvable $\qquad$ (LES_4^S) is solvable

What is the geometrical essence of a non-solvable or solvable system of linear equations?



(LES_4^N) (LES_4^S)

Fig.1

(LES_4^n) is non-solvable but (LES_4^S) solvable in sense because of

$$L_{x+y=1} \bigcap L_{x+y=-1} \bigcap L_{x-y=1} \bigcap L_{x-y=-1} = \emptyset$$

and

$$L_{x=y} \bigcap L_{x=1} \bigcap L_{y=1} \bigcap L_{x+y=2} = \{(1,1)\}$$

Generally,

$$(ES_m) \quad \begin{cases} f_1(x_1, x_2, \cdots, x_n) = 0 \\ f_2(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_m(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

(ES_m) is solvable or not dependent on $\bigcap_{i=1}^{m} S_{f_i} = \emptyset$ or $\neq \emptyset$.

Proposition 1.1 *Any system (ES_m) of algebraic equations with each equation solvable posses a geometrical figure in $\mathbb{R}^n$, no matter it is solvable or not.*

Conversely, for a geometrical figure $\mathscr{G}$ in $\mathbb{R}^n, n \geq 2$,

how can we get an algebraic representation for geometrical figure $\mathscr{G}$?

As a special case, let G be a graph embedded in Euclidean space $\mathbb{R}^n$ and

$$(ES_e) \quad \begin{cases} f_1^e(x_1, x_2, \cdots, x_n) = 0 \\ f_2^e(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{n-1}^e(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

be a system of equations for determining an edge $e \in E(G)$ in $\mathbb{R}^n$.

Then the system

$$\left.\begin{array}{r} f_1^e(x_1, x_2, \cdots, x_n) = 0 \\ f_2^e(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{n-1}^e(x_1, x_2, \cdots, x_n) = 0 \end{array}\right\} \ \forall e \in E(G)$$

is a non-solvable system of equations.

For example, let G be a planar graph, shown in Fig.2.

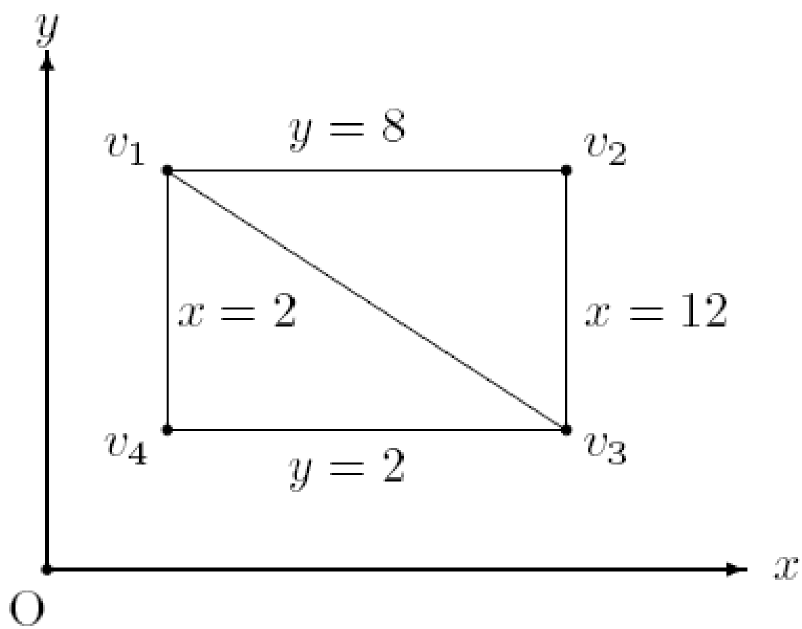


Fig.2

$$\begin{cases} x = 2 \\ y = 8 \\ x = 12 \\ y = 2 \\ 3x + 5y = 46. \end{cases}$$

Proposition 1.2 *Any geometrical figure $\mathscr{G}$ consisting of m parts, each of which is determined by a system of algebraic equations in $\mathbb{R}^n, n \geq 2$ posses an algebraic representation by system of equations, solvable or not in $\mathbb{R}^n$.*

§2. Smarandache Systems with Labeled Topological Graphs

Definition 2.1([5-7]) *A rule $\mathcal{R}$ in a mathematical system $(\Sigma;\mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set Σ, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma;\mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule $\mathcal{R}$.

Definition 2.2([5-7],[11]) *Let $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ be $m \geq 2$ mathematical spaces, different two by two. A Smarandache multi-space $\widetilde{\Sigma}$ is a union $\bigcup\limits_{i=1}^{m} \Sigma_i$ with rules $\widetilde{\mathcal{R}} = \bigcup\limits_{i=1}^{m} \mathcal{R}_i$ on $\widetilde{\Sigma}$, denoted by $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$.*

Such a typical example is the proverb of blind men with an elephant.

Definition 2.3(([5-7])) *Let* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ *be a Smarandache multi-space with* $\widetilde{\Sigma}=\bigcup\limits_{i=1}^{m}\Sigma_i$ *and* $\widetilde{\mathcal{R}}=\bigcup\limits_{i=1}^{m}\mathcal{R}_i$. *Then a inherited graph* $G\left[\widetilde{\Sigma},\widetilde{R}\right]$ *of* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ *is a labeled topological graph defined by*

$$V\left(G\left[\widetilde{\Sigma},\widetilde{R}\right]\right)=\{\Sigma_1,\Sigma_2,\cdots,\Sigma_m\},$$

$$E\left(G\left[\widetilde{\Sigma},\widetilde{R}\right]\right)=\{\ (\Sigma_i,\Sigma_j)\mid\Sigma_i\bigcap\Sigma_j\neq\emptyset,1\leq i,j\leq m\}$$

with an edge labeling

$$l^E:\ (\Sigma_i,\Sigma_j)\in E\left(G\left[\widetilde{S},\widetilde{R}\right]\right)\to l^E(\Sigma_i,\Sigma_j)=\varpi\left(\Sigma_i\bigcap\Sigma_j\right),$$

where ϖ *is a characteristic on* $\Sigma_i\bigcap\Sigma_j$ *such that* $\Sigma_i\bigcap\Sigma_j$ *is isomorphic to* $\Sigma_k\bigcap\Sigma_l$ *if and only if* $\varpi(\Sigma_i\bigcap\Sigma_j)=\varpi\left(\Sigma_k\bigcap\Sigma_l\right)$ *for integers* $1\leq i,j,k,l\leq m$.

For example, let $S_1 = \{a, b, c\}$, $S_2 = \{c, d, e\}$, $S_3 = \{a, c, e\}$ and $S_4 = \{d, e, f\}$. Then the multi-space $\widetilde{S} = \bigcup_{i=1}^{4} S_i = \{a, b, c, d, e, f\}$ with its labeled topological graph $G[\widetilde{S}]$ is shown in Fig.4.

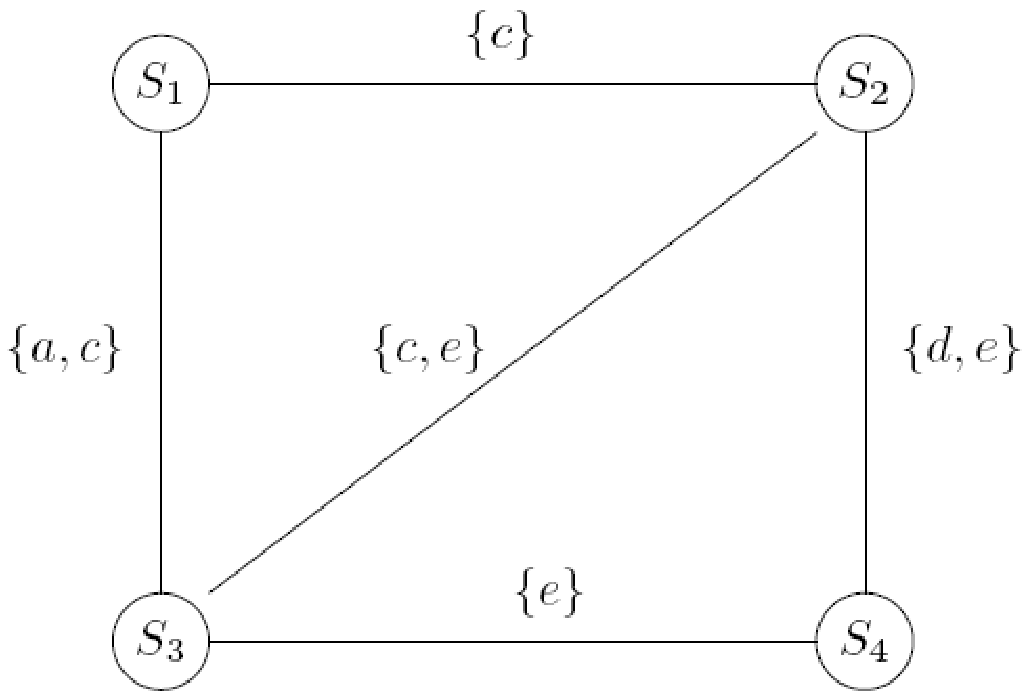


Fig.4

The labeled topological graph $G\left[\widetilde{\Sigma}, \widetilde{R}\right]$ reflects the notion that there exists linkage between things in philosophy. In fact, each edge $(\Sigma_i, \Sigma_j) \in E\left(G\left[\widetilde{\Sigma}, \widetilde{R}\right]\right)$ is such a linkage with coupling $\varpi(\Sigma_i \bigcap \Sigma_j)$. For example, let $a = \{\text{tusk}\}$, $b = \{\text{nose}\}$, $c_1, c_2 = \{\text{ear}\}$, $d = \{\text{head}\}$, $e = \{\text{neck}\}$, $f = \{\text{belly}\}$, $g_1, g_2, g_3, g_4 = \{\text{leg}\}$, $h = \{\text{tail}\}$ for an elephant $\mathscr{C}$. Then its labeled topological graph is shown in Fig.5,

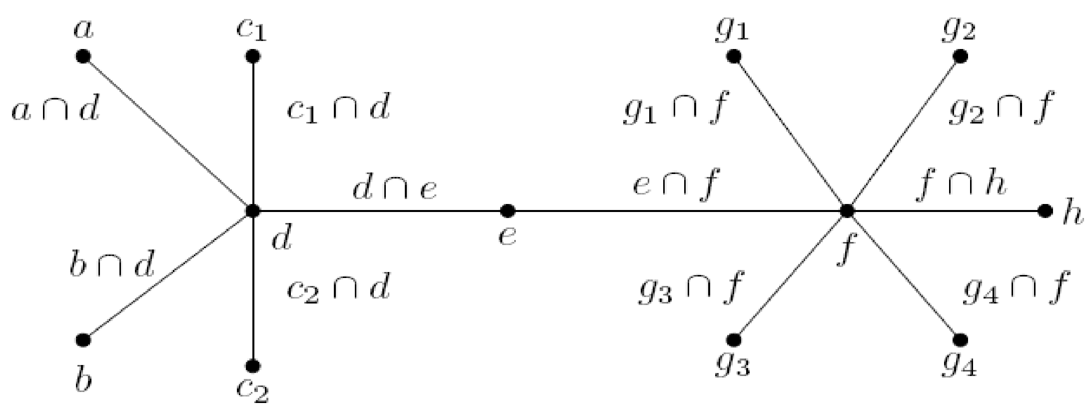


Fig.5

which implies that one can characterizes the geometrical behavior of an elephant combinatorially.

§3. Non-Solvable Systems of Ordinary Differential Equations

3.1 Linear Ordinary Differential Equations

For integers $m, \ n \geq 1$, let

$$\dot{X} = F_1(X), \ \dot{X} = F_2(X), \cdots, \dot{X} = F_m(X) \qquad (DES_m^1)$$

be a differential equation system with continuous $F_i : \mathbf{R}^n \to \mathbf{R}^n$ such that $F_i(\overline{0}) = \overline{0}$, particularly, let

$$\dot{X} = A_1 X, \cdots, \dot{X} = A_k X, \cdots, \dot{X} = A_m X \qquad (LDES_m^1)$$

be a linear ordinary differential equation system of first order with

$$A_k = \begin{bmatrix} a_{11}^{[k]} & a_{12}^{[k]} & \cdots & a_{1n}^{[k]} \\ a_{21}^{[k]} & a_{22}^{[k]} & \cdots & a_{2n}^{[k]} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}^{[k]} & a_{n2}^{[k]} & \cdots & a_{nn}^{[k]} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}$$

where each $a_{ij}^{[k]}$ is a real number for integers $0 \leq k \leq m, \ 1 \leq i, j \leq n$.

Definition 3.1 *An ordinary differential equation system* (DES_m^1) *or* $(LDES_m^1)$ *are called non-solvable if there are no function* $X(t)$ *hold with* (DES_m^1) *or* $(LDES_m^1)$ *unless the constants.*

As we known, the general solution of the ith differential equation in $(LDES_m^1)$ is a linear space spanned by the elements in the solution basis

$$\mathscr{B}_i = \{ \overline{\beta}_k(t)e^{\alpha_k t} \mid 1 \leq k \leq n \}$$

for integers $1 \leq i \leq m$, where

$$\alpha_i = \begin{cases} \lambda_1, & if\ 1 \leq i \leq k_1; \\ \lambda_2, & if\ k_1 + 1 \leq i \leq k_2; \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots ; \\ \lambda_s, & if\ k_1 + k_2 + \cdots + k_{s-1} + 1 \leq i \leq n, \end{cases}$$

λ_i is the k_i-fold zero of the characteristic equation

$$\det(A - \lambda I_{n\times n}) = |A - \lambda I_{n\times n}| = 0$$

with $k_1 + k_2 + \cdots + k_s = n$ and $\overline{\beta}_i(t)$ is an n-dimensional vector consisting of polynomials in t with degree$\leq k_i - 1$.

In this case, we can simplify the labeled topological graph $G\left[\widetilde{\sum},\widetilde{R}\right]$ replaced each $\sum_i$ by the solution basis $\mathscr{B}_i$ and $\sum_i\bigcap\sum_j$ by $\mathscr{B}_i\bigcap\mathscr{B}_j$ if $\mathscr{B}_i\bigcap\mathscr{B}_j\neq\emptyset$ for integers $1\leq i,j\leq m$, denoted by $G[LDES_m^1]$.

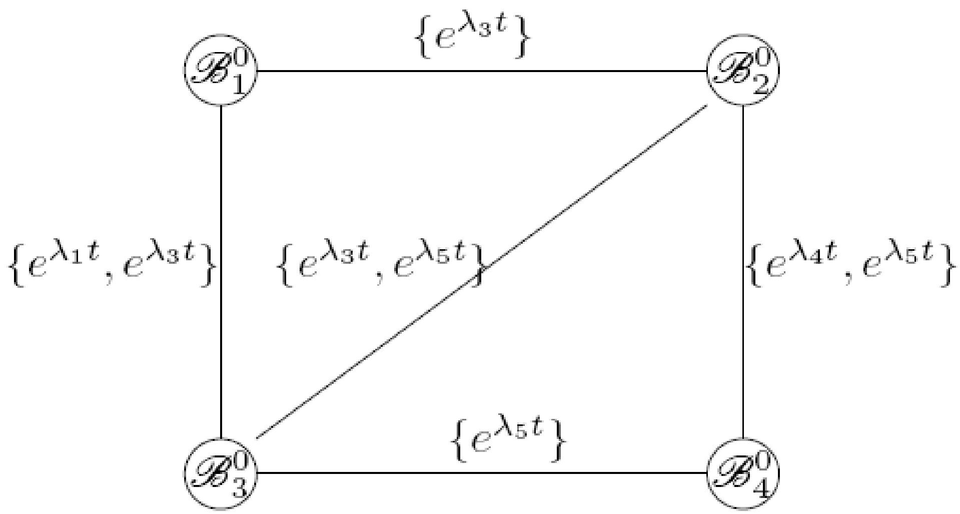
For example, let $m=4$ and

$$\mathscr{B}_1^0=\{e^{\lambda_1 t},e^{\lambda_2 t},e^{\lambda_3 t}\},\ \mathscr{B}_2^0=\{e^{\lambda_3 t},e^{\lambda_4 t},e^{\lambda_5 t}\},\ \mathscr{B}_3^0=\{e^{\lambda_1 t},e^{\lambda_3 t},e^{\lambda_5 t}\}$$

$\mathscr{B}_4^0=\{e^{\lambda_4 t},e^{\lambda_5 t},e^{\lambda_6 t}\}$, where λ_i, $1\leq i\leq 6$ are real numbers different two by two.

Then $G[LDES_m^1]$ is shown in Fig.6.

Fig.6



Theorem 3.2([10]) *Every linear homogeneous differential equation system* $(LDES_m^1)$ *uniquely determines a basis graph* $G[LDES_m^1]$ *inherited in* $(LDES_m^1)$. *Conversely, every basis graph* G *uniquely determines a homogeneous differential equation system* $(LDES_m^1)$ *such that* $G[LDES_m^1] \simeq G$.

Such a basis graph $G[LDES_m^1]$ is called the G-solution of $(LDES_m^1)$.

Theorem 3.3([10]) *Every linear homogeneous differential equation system* $(LDES_m^1)$ *has a unique* G*-solution, and for every basis graph* H*, there is a unique linear homogeneous differential equation system* $(LDES_m^1)$ *with* G*-solution* H.

Example 3.4

Let (LDE_m^n) be the following linear homogeneous differential equation system

$$\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{cases}$$

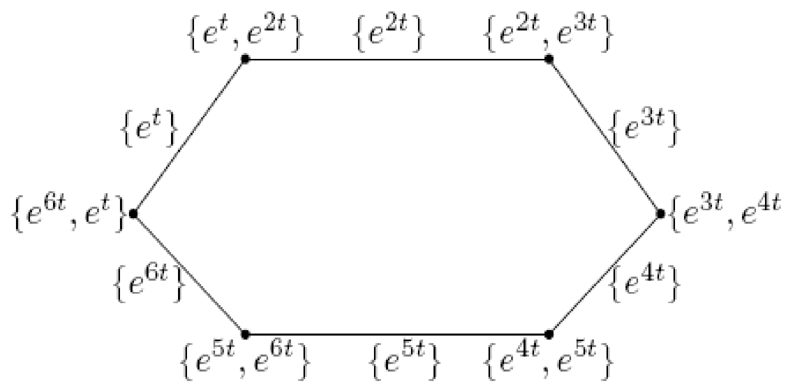


Fig.7 A basis graph

3.2 Combinatorial Characteristics of Linear Differential Equations

Definition 3.5 *Let* $(LDES^1_m)$, $(LDES^1_m)'$ *be two linear homogeneous differential equation systems with G-solutions* H, H'. *They are called combinatorially equivalent if there is an isomorphism* $\varphi : H \to H'$, *thus there is an isomorphism* $\varphi : H \to H'$ *of graph and labelings* θ, τ *on* H *and* H' *respectively such that* $\varphi\theta(x) = \tau\varphi(x)$ *for* $\forall x \in V(H)\bigcup E(H)$, *denoted by* $(LDES^1_m) \stackrel{\varphi}{\simeq} (LDES^1_m)'$.

Definition 3.6 *Let* G *be a simple graph. A vertex-edge labeled graph* $\theta : G \to \mathbb{Z}^+$ *is called integral if* $\theta(uv) \leq \min\{\theta(u), \theta(v)\}$ *for* $\forall uv \in E(G)$, *denoted by* G^{I_θ}.

Let $G_1^{I_\theta}$ *and* $G_2^{I_\tau}$ *be two integral labeled graphs. They are called identical if* $G_1 \stackrel{\varphi}{\simeq} G_2$ *and* $\theta(x) = \tau(\varphi(x))$ *for any graph isomorphism* φ *and* $\forall x \in V(G_1)\bigcup E(G_1)$, *denoted by* $G_1^{I_\theta} = G_2^{I_\tau}$.

For example, these labeled graphs shown in Fig.8 are all integral on $K_4 - e$, but $G_1^{I_\theta} = G_2^{I_\tau}$, $G_1^{I_\theta} \neq G_3^{I_\sigma}$.

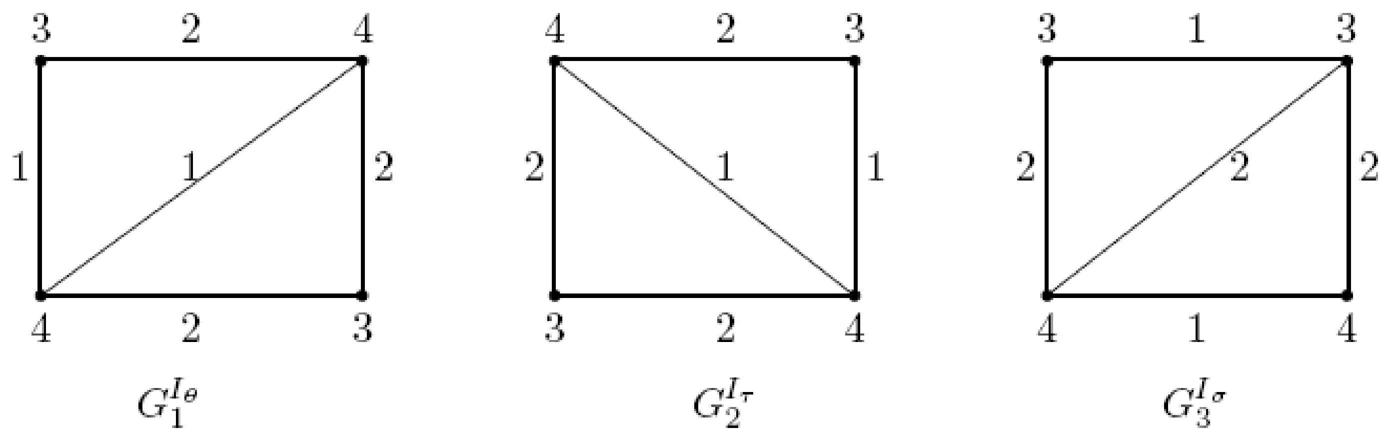


Fig.8

Theorem 3.5([10]) *Let* $(LDES_m^1)$, $(LDES_m^1)'$ *be two linear homogeneous differential equation systems with integral labeled graphs* H, H'. *Then* $(LDES_m^1) \overset{\varphi}{\simeq} (LDES_m^1)'$ *if and only if* $H = H'$.

3.3 Non-Linear Ordinary Differential Equations

If some functions $F_i(X),\ 1 \leq i \leq m$ are non-linear in (DES^1_m), we can linearize these non-linear equations $\dot{X} = F_i(X)$ at the point $\overline{0}$, i.e., if

$$F_i(X) = F_i'(\overline{0})X + R_i(X),$$

where $F_i'(\overline{0})$ is an $n \times n$ matrix, we replace the ith equation $\dot{X} = F_i(X)$ by a linear differential equation

$$\dot{X} = F_i'(\overline{0})X$$

in (DES^1_m).

§4. Cauchy Problem on Non-Solvable Partial Differential Equations

Let $(PDES_m)$ be a system of partial differential equations with

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ F_m(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \end{cases}$$

on a function $u(x_1, \cdots, x_n, t)$. Then its *symbol* is determined by

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0, \end{cases}$$

i.e., substitute $p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_n^{\alpha_n}$ into $(PDES_m)$ for the term $u_{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}}$, where $\alpha_i \geq 0$ for integers $1 \leq i \leq n$.

Definition 4.1 *A non-solvable $(PDES_m)$ is algebraically contradictory if its symbol is non-solvable. Otherwise, differentially contradictory.*

Theorem 4.2([11]) *A Cauchy problem on systems*

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \end{cases}$$

of partial differential equations of first order is non-solvable with initial values

$$\begin{cases} x_i|_{x_n = x_n^0} = x_i^0(s_1, s_2, \cdots, s_{n-1}) \\ u|_{x_n = x_n^0} = u_0(s_1, s_2, \cdots, s_{n-1}) \\ p_i|_{x_n = x_n^0} = p_i^0(s_1, s_2, \cdots, s_{n-1}), \quad i = 1, 2, \cdots, n \end{cases}$$

if and only if the system

$$F_k(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0, \ 1 \leq k \leq m$$

is algebraically contradictory, in this case, there must be an integer k_0, $1 \leq k_0 \leq m$ such that

$$F_{k_0}(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) \neq 0$$

or it is differentially contradictory itself, i.e., there is an integer j_0, $1 \leq j_0 \leq n-1$ such that

$$\frac{\partial u_0}{\partial s_{j_0}} - \sum_{i=0}^{n-1} p_i^0 \frac{\partial x_i^0}{\partial s_{j_0}} \neq 0.$$

Corollary 4.3 *Let*

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \end{cases}$$

be an algebraically contradictory system of partial differential equations of first order. Then there are no values $x_i^0, u_0, p_i^0,\ 1 \leq i \leq n$ *such that*

$$\begin{cases} F_1(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) = 0, \\ F_2(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) = 0. \end{cases}$$

Corollary 4.4 *A Cauchy problem* $(LPDES_m^C)$ *of quasilinear partial differential equations with initial values* $u|_{x_n = x_n^0} = u_0$ *is non-solvable if and only if the system* $(LPDES_m)$ *of partial differential equations is algebraically contradictory.*

Denoted by $\widehat{G}[PDES_m^C]$ such a graph $G[PDES_m^C]$ eradicated all labels. Particularly, replacing each label $S^{[i]}$ by $S_0^{[i]} = \{u_0^{[i]}\}$ and $S^{[i]} \bigcap S^{[j]}$ by $S_0^{[i]} \bigcap S_0^{[j]}$ for integers $1 \leq i, j \leq m$, we get a new labeled topological graph, denoted by $G_0[PDES_m^C]$. Clearly, $\widehat{G}[PDES_m^C] \simeq \widehat{G}_0[PDES_m^C]$.

Theorem 4.5([11]) *For any system $(PDES_m^C)$ of partial differential equations of first order, $\widehat{G}[PDES_m^C]$ is simple. Conversely, for any simple graph G, there is a system $(PDES_m^C)$ of partial differential equations of first order such that $\widehat{G}[PDES_m^C] \simeq G$.*

Corollary 4.6 *Let $(LPDES_m)$ be a system of linear partial differential equations of first order with maximal contradictory classes $\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s$ on equations in $(LPDES)$. Then $\widehat{G}[LPDES_m^C] \simeq K(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s)$, i.e., an s-partite complete graph.*

Definition 4.7 *Let $(PDES_m^C)$ be the Cauchy problem of a partial differential equation system of first order. Then the labeled topological graph $G[PDES_m^C]$ is called its topological graph solution, abbreviated to G-solution.*

Combining this definition with that of Theorems 4.5, the following conclusion is holden immediately.

Theorem 4.8([11]) *A Cauchy problem on system $(PDES_m)$ of partial differential equations of first order with initial values $x_i^{[k^0]}, u_0^{[k]}, p_i^{[k^0]}$, $1 \leq i \leq n$ for the kth equation in $(PDES_m)$, $1 \leq k \leq m$ such that*

$$\frac{\partial u_0^{[k]}}{\partial s_j} - \sum_{i=0}^{n} p_i^{[k^0]} \frac{\partial x_i^{[k^0]}}{\partial s_j} = 0$$

is uniquely G-solvable, i.e., $G[PDES_m^C]$ is uniquely determined.

§5. Global Stability of Non-Solvable Differential Equations

Definition 5.1 *Let H be a spanning subgraph of $G[LDES_m^1]$ of systems $(LDES_m^1)$ with initial value $X_v(0)$. Then $G[LDES_m^1]$ is called sum-stable or asymptotically sum-stable on H if for all solutions $Y_v(t)$, $v \in V(H)$ of the linear differential equations of $(LDES_m^1)$ with $|Y_v(0) - X_v(0)| < \delta_v$ exists for all $t \geq 0$,*

$$\left| \sum_{v \in V(H)} Y_v(t) - \sum_{v \in V(H)} X_v(t) \right| < \varepsilon,$$

or furthermore,

$$\lim_{t \to 0} \left| \sum_{v \in V(H)} Y_v(t) - \sum_{v \in V(H)} X_v(t) \right| = 0.$$

Similarly, a system $(PDES_m^C)$ *is sum-stable if for any number* $\varepsilon > 0$ *there exists* $\delta_v > 0,\ v \in V(\widehat{G}[0])$ *such that each* $G(t)$*-solution with* $\left|{u'}_0^{[v]} - u_0^{[v]}\right| < \delta_v, \forall v \in V(\widehat{G}[0])$ *exists for all* $t \geq 0$ *and with the inequality*

$$\left|\sum_{v \in V(\widehat{G}[t])} {u'}^{[v]} - \sum_{v \in V(\widehat{G}[t])} u^{[v]}\right| < \varepsilon$$

holds, denoted by $G[t] \overset{\Sigma}{\sim} G[0]$. *Furthermore, if there exists a number* $\beta_v > 0,\ v \in V(\widehat{G}[0])$ *such that every* $G'[t]$*-solution with* $\left|{u'}_0^{[v]} - u_0^{[v]}\right| < \beta_v, \forall v \in V(\widehat{G}[0])$ *satisfies*

$$\lim_{t \to \infty} \left|\sum_{v \in V(\widehat{G}[t])} {u'}^{[v]} - \sum_{v \in V(\widehat{G}[t])} u^{[v]}\right| = 0,$$

then the $G[t]$*-solution is called asymptotically stable, denoted by* $G[t] \xrightarrow{\Sigma} G[0]$.

Theorem 5.2([10]) *A zero G-solution of linear homogenous differential equation systems* $(LDES_m^1)$ *is asymptotically sum-stable on a spanning subgraph* H *of* $G[LDES_m^1]$ *if and only if* $\mathrm{Re}\alpha_v < 0$ *for each* $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ *in* $(LDES^1)$ *hold for* $\forall v \in V(H)$.

Example 5.3 Let a G-solution of $(LDES_m^1)$ or (LDE_m^n) be the basis graph shown in Fig.4.1, where $v_1 = \{e^{-2t}, e^{-3t}, e^{3t}\}$, $v_2 = \{e^{-3t}, e^{-4t}\}$, $v_3 = \{e^{-4t}, e^{-5t}, e^{3t}\}$, $v_4 = \{e^{-5t}, e^{-6t}, e^{-8t}\}$, $v_5 = \{e^{-t}, e^{-6t}\}$, $v_6 = \{e^{-t}, e^{-2t}, e^{-8t}\}$. Then the zero G-solution is sum-stable on the triangle $v_4v_5v_6$, but it is not on the triangle $v_1v_2v_3$. In fact, it is prod-stable on the triangle $v_1v_2v_3$.

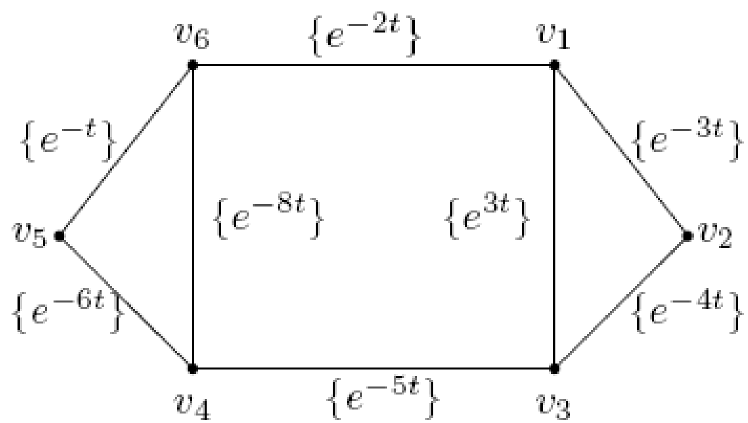


Fig.9

For partial differential equations, let the system $(PDES_m^C)$ be

$$\left.\begin{array}{l}\dfrac{\partial u}{\partial t}=H_i(t,x_1,\cdots,x_{n-1},p_1,\cdots,p_{n-1})\\ u|_{t=t_0}=u_0^{[i]}(x_1,x_2,\cdots,x_{n-1})\end{array}\right\}\quad 1\leq i\leq m \qquad (APDES_m^C)$$

A point $X_0^{[i]}=(t_0,x_{10}^{[i]},\cdots,x_{(n-1)0}^{[i]})$ with $H_i(t_0,x_{10}^{[i]},\cdots,x_{(n-1)0}^{[i]})=0$ for $1\leq i\leq m$ is called an *equilibrium point* of the ith equation in $(APDES_m)$. Then we know that

Theorem 5.4([11]) *Let $X_0^{[i]}$ be an equilibrium point of the ith equation in $(APDES_m)$ for each integer $1\leq i\leq m$. If $\sum\limits_{i=1}^{m}H_i(X)>0$ and $\sum\limits_{i=1}^{m}\frac{\partial H_i}{\partial t}\leq 0$ for $X\neq\sum\limits_{i=1}^{m}X_0^{[i]}$, then the system $(APDES_m)$ is sum-stability, i.e., $G[t]\overset{\Sigma}{\sim}G[0]$. Furthermore, if $\sum\limits_{i=1}^{m}\frac{\partial H_i}{\partial t}<0$ for $X\neq\sum\limits_{i=1}^{m}X_0^{[i]}$, then $G[t]\overset{\Sigma}{\to}G[0]$.*

§6. Applications

6.1 Application to Geometry

Theorem 6.1([11]) *Let the Cauchy problem be $(PDES_m^C)$. Then every connected component of $\Gamma[PDES_m^C]$ is a differentiable n-manifold with atlas $\mathscr{A} = \{(U_v, \phi_v)|v \in V(\widehat{G}[0])\}$ underlying graph $\widehat{G}[0]$, where U_v is the n-dimensional graph $G[u^{[v]}] \simeq \mathbb{R}^n$ and ϕ_v the projection $\phi_v : ((x_1, x_2, \cdots, x_n), u(x_1, x_2, \cdots, x_n))) \rightarrow (x_1, x_2, \cdots, x_n)$ for $\forall (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$.*

Theorem 6.2([11]) *For any integer $m \geq 1$, let $U_i, 1 \leq i \leq m$ be open sets in $\mathbb{R}^n$ underlying a connected graph defined by*

$$V(G) = \{U_i|1 \leq i \leq m\}, \quad E(G) = \{(U_i, U_j)|U_i \bigcap U_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

If X_i is a vector field on U_i for integers $1 \leq i \leq m$, then there always exists a differentiable manifold $M \subset \mathbb{R}^n$ with atlas $\mathscr{A} = \{(U_i, \phi_i)|1 \leq i \leq m\}$ underlying graph G and a function $u_G \in \Omega^0(M)$ such that

$$X_i(u_G) = 0, \quad 1 \leq i \leq m.$$

6.2 Global Control of Infectious Diseases

Consider two cases of virus for infectious diseases:

Case 1 *There are m known virus $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ with infected rate k_i, heal rate h_i for integers $1 \leq i \leq m$ and an person infected a virus $\mathscr{V}_i$ will never infects other viruses $\mathscr{V}_j$ for $j \neq i$.*

Case 2 *There are m varying $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ from a virus $\mathscr{V}$ with infected rate k_i, heal rate h_i for integers $1 \leq i \leq m$.*

We are easily to establish a non-solvable differential model for the spread of infectious viruses by applying the SIR model of one infectious disease following:

$$\left\{\begin{array}{l} \dot{S} = -k_1 SI \\ \dot{I} = k_1 SI - h_1 I \\ \dot{R} = h_1 I \end{array}\right. \quad \left\{\begin{array}{l} \dot{S} = -k_2 SI \\ \dot{I} = k_2 SI - h_2 I \\ \dot{R} = h_2 I \end{array}\right. \quad \cdots \quad \left\{\begin{array}{l} \dot{S} = -k_m SI \\ \dot{I} = k_m SI - h_m I \\ \dot{R} = h_m I \end{array}\right. \qquad (DES_m^1)$$

Conclusion 6.3([10]) *For m infectious viruses $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ in an area with infected rate k_i, heal rate h_i for integers $1 \leq i \leq m$, then they decline to 0 finally if $0 < S < \sum\limits_{i=1}^{m} h_i \Big/ \sum\limits_{i=1}^{m} k_i$, i.e., these infectious viruses are globally controlled. Particularly, they are globally controlled if each of them is controlled in this area.*

6.3 Flows in Network

How can we characterize the behavior of flow F?

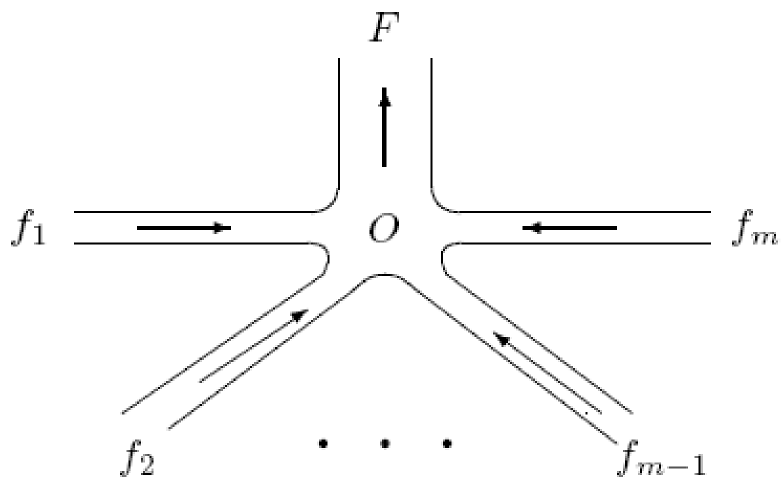


Fig.10

Denote the rate, density of flow f_i by $\rho^{[i]}$ for integers $1 \le i \le m$ and that of F by $\rho^{[F]}$

$$\frac{\partial \rho^{[i]}}{\partial t} + \phi_i(\rho^{[i]})\frac{\partial \rho^{[i]}}{\partial x} = 0,\ 1 \le i \le m.$$

Replacing each $\rho^{[i]}$ by ρ, $1 \le i \le m$ enables one getting a non-solvable system

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t} + \phi_i(\rho)\frac{\partial \rho}{\partial x} = 0 \\ &\rho\,|_{t=t_0} = \rho^{[i]}(x, t_0) \end{aligned}\right\} 1 \le i \le m.$$

Applying Theorem 5.4, if

$$\sum_{i=1}^{m} \phi_i(\rho) < 0 \ \text{ and } \ \sum_{i=1}^{m} \phi_(\rho) \left[\frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2\right] \ge 0$$

for $X \ne \sum_{k=1}^{m} \rho_0^{[i]}$, then we know that the flow F is stable and furthermore, if

$$\sum_{i=1}^{m} \phi_(\rho) \left[\frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2\right] < 0$$

for $X \ne \sum_{k=1}^{m} \rho_0^{[i]}$, then it is also asymptotically stable.

Thanks for your Attention！