

**SIMULTANEOUS MAXIMISATION
IN
ECONOMIC THEORY**

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Simultaneous Maximisation in Economic Theory

This document provides translations of two important but neglected articles of Bruno de Finetti, *Problemi di “optimum”* and *Problemi di “optimum” vincolato*, which deal with the problem of maximising several functions simultaneously.¹

Economic theory assumes that the participants in a social exchange economy are rational, optimising agents. Consumers maximise their utilities, producers maximise their profits; they all do so subject to whatever constraints are present. Many textbooks on microeconomic theory contain appendices dealing with the mathematical theory of constrained maximisation of one function (for example Malinvaud, 1972; Varian, 1992), and several textbooks on optimisation have been written especially for economists (for example Dixit, 1976, 1990; Léonard and Van Long, 1992).

The typical case in economics is that the actions of each agent affect the outcomes for all participants. The theory thus results in a mathematical problem in which a number of functions with the same list of arguments must be simultaneously maximal, a *simultaneous maximum problem* for short. Von Neumann and Morgenstern (1947, Section I.2) argued that, at the time of their writing, mathematical economics had not dealt adequately with this type of problem. They constructed game theory to remedy the error, confining their analysis to the case of discrete decision variables that may take only a finite number of values. A decade earlier De Finetti (1937a,b), motivated by the work of Pareto, had considered the problem of simultaneously maximising a number of smooth functions of continuous variables. In fact the notion of simultaneous maximum appeared in economics for the first time in Edgeworth’s (1881) *Mathematical Psychics*: It is the famous *contract curve*. But the notion is best known from the work of Pareto, whence the name of Pareto optimum.² As the term “Pareto optimum” has acquired a normative connotation in economics, I prefer the neutral term “simultaneous maximum” stemming from Zaccagnini (1947, 1951).

De Finetti (1940) himself has applied the notion of simultaneous maximum in his study of hedging the risk of a set of insurances when determining the optimal retention levels, which yield the best way of reinsuring parts of the insurances so as to reduce the risk (as measured by the variance of profit) within the desired limits while minimising the loss of mean profit.³ Zaccagnini (1947, 1951) has used the technique of simultaneous maximisation to solve the oligopoly problem and to derive Edgeworth’s contract curve. Much later, Smale (1975, 1974b) and others have revisited the subject of simultaneous maximisation. Smale has applied simultaneous maximisation to the study of general economic equilibrium using a calculus approach (for example Smale, 1974a, 1976).

A calculus approach to simultaneous maximum problems also clarifies the nature of the Nash equilibrium. In the simultaneous maximisation of a number of functions, as in cooperative game theory, all arguments are consistently treated as variables in all maximands. In non-cooperative game theory, however, the set of arguments is partitioned into a number, one for each maximand, of disjoint subsets; in each max-

¹I am grateful to prof. Pressacco of the University of Udine for checking an earlier version of the translations and saving me from some errors.

²Pareto (1909, Figure 50, p. 355) represents the contract curve graphically in a figure nowadays called the “Edgeworth Box.” It would be historically correct to speak of “the Edgeworth optimum in the Pareto Box” (cf. Hildenbrand, 1993).

³Only recently has this work been recognised as anticipating Markowitz (1952).

imand, only the arguments in the associated subset are treated as variables and the other arguments are treated as constants. “First-order conditions” are derived by varying the arguments in each subset only in the associated maximand and simultaneously holding them constant in all other maximands. We thus see that non-cooperative game theory splits the simultaneous maximum problem into a number of conditional maximum problems and, in so doing, disregards the interdependence of the maximum problems. The solution of the “first-order conditions,” with *all* arguments now treated as variables again in the partial first-order derivatives that are included in the system, is the Nash equilibrium. The inconsistent treatment of the arguments will manifest itself in contradictory results. The oligopoly problem provides an example (in the literature not recognised as such) with the Cournot equilibrium’s differing from the Bertrand equilibrium (for details and other examples, see Nieuwenhuis, 2013, available at https://www.dropbox.com/s/ctoxntaocy9tgz2/NotNash_vixra.pdf).⁴

The economics profession has been slow in adopting the technique of simultaneous maximisation, witness its absence from the textbooks mentioned above. Still, Smale’s work is increasingly being appreciated. It is just fair to point out that the first one to see the importance of simultaneous maximisation for economic theory is Bruno de Finetti.

⁴One may read Von Neumann and Morgenstern (1947) as arguing that the set of (constrained) maximum problems defined in mathematical economics ought to be treated as a simultaneous maximum problem, and Nash (1951) as promoting incorrect conditioning from vice to virtue.

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Appendix A

“Optimum” problems

by B. DE FINETTI*

ABSTRACT. — The notion of “optimum” as introduced in mathematical economics is clarified both by numerous examples of a geometrical and physical nature and by a sketch of what might be the systematic and general treatment of “optimum” problems.

A.1 Introduction

1. In my research in pure economics I have tried to elucidate the very essence of the notion of “optimum” that plays a role there, by showing how it differs from the usual notion of “maximum” in <mathematical> analysis. Indeed, this has become clear also from the works of Pareto, but the fact that he, by immediately adding to the system of equations expressing the conditions of “optimum” a second one consisting of balance identities, succeeds in determining a *unique* “optimum” point seems to have generated a certain confusion about the conceptual significance of the “optimum” problem. I think that the difficulty stems largely from the fact that such a notion and such a kind of problem has presented itself explicitly for the first time in mathematical economics, while one quite naturally wants to base this theory, the legitimacy of which has even been questioned, on notions already well-known in other fields, and particularly in geometry, which allows an intuitive view. In order not to leave the impression that the notion of “optimum” is some vague or artificial peculiarity of economics, I believe it is best to present, like I intend to do here, some simple examples of a geometrical nature that lead to “optimum” problems of the same type. After the treatment of several examples with direct methods suggested by the cases at hand, we shall study the “optimum” conditions in general, illustrating them with new examples. Not, incidentally, that it involves an absolutely new type of problems from an analytical point of view, because they reduce essentially to problems of “constrained maxima and minima;” however, they do differ from the latter, conceptually by the typical formulation of the problem

*Translation of B. de Finetti, *Problemi di “optimum”*, *Giornale dell’Istituto Italiano degli Attuari*, Anno VIII, n. 1, gennaio 1937-XV. I have corrected some misprints and obvious mistakes.

and analytically because, by consequence, the points of maximum of one function on the level curves of the other do not solve the problem unless they are simultaneously points of maximum of the second function on the level curves of the first one.

A.2 Preliminary examples

2. The nine roads along which to go to a given place, in order of decreasing panoramic value according to the taste of a given individual, are

A B C D E F G H I

and have the lengths in kilometres of respectively

57 59 55 43 50 45 42 48 42

If this individual wishes to choose at once so as to have minimal length and maximal panoramic beauty, he can be in doubt only between the four routes A, C, D and G, which satisfy the condition of "optimum" in that they are not preceded (in order of beauty) by any other shorter one. Whether the saving of a single kilometre is worth the loss <of panoramic beauty> from choosing the seventh route (G) instead of the fourth (D), or whether the supposedly slight scenic superiority of the third route (C) over the fourth (D) justifies twelve additional kilometres are questions that surpass the boundaries of the problem as it has been posed; for in it only the desirability according to two different criteria <applied> simultaneously enters, which precludes answering the two questions just posed, but still permits, for example, to exclude the choice of route F, given that it is preceded in beauty by D, which is shorter.

3. In some plane one wants to choose a point P , and one wishes it to be as close as possible to each of the two given points A and B . Only all points of the line segment AB are solutions of this "optimum" problem. For, if P is a point not belonging to the line segment AB , by drawing the two circles passing through P with centres in A and B one obtains a part of the plane (a *lunula*) inside both circles each point Q of which is closer to both A and B than point P is; therefore the latter is not an "optimum" point. Inversely, each point P of AB is an "optimum" point, because $\overline{AP} + \overline{PB} = \overline{AB}$, and hence it is not possible to diminish one of the two distances any further without augmenting the other. From a geometrical-analytical point of view, one observes that the circles considered, with centres in A and B , respectively, are the level curves of the two functions $\varphi(P) = \overline{AP}$ and $\psi(P) = \overline{BP}$ that one wants to minimise; the "optimum" points are the minimum points of φ on $\psi = \text{constant}$, or the minimum points of ψ on $\varphi = \text{constant}$, and that means they are also, given the "regularity" of the level curves (continuously varying tangent lines), the tangent points of $\varphi = \varphi_0$ and $\psi = \psi_0$ for some φ_0 and ψ_0 .

4. Let there be given, somewhere in a plane, two line segments, AB and CD . One wants to choose a point P , wishing that from there one sees both line segments under the maximally possible angle. The two functions to be maximised are $\varphi(P) = \sphericalangle APB$ and $\psi(P) = \sphericalangle CPD$; obviously the level curves of φ and ψ are the circles passing through A and B and through C and D , respectively, which results from the well-known theorem of elementary geometry according to which the angle at the centre of the circle is twice the angle at the circumference (and therefore, given a circle passing through A and B with O denoting its centre, the angle APB is constant when P moves along

the circumference, because there must always hold $\sphericalangle APB = 1/2 \sphericalangle AOB$). The locus of “optimum” points is therefore given by the points P where the circles determined by A, B and P and by C, D and P , respectively, are tangent (and hence are tangent at P). In order to limit ourselves to the case in which the determination of this locus is rather elementary, let us suppose that the point R in which the lines through A and B and through C and D intersect has the same potential with respect to the two circles having AB and CD , respectively, as diameters (and hence $\overline{RA} \cdot \overline{RB} = \overline{RC} \cdot \overline{RD}$). The circle \mathcal{C} with centre R and radius $r = \sqrt{\overline{RA} \cdot \overline{RB}} = \sqrt{\overline{RC} \cdot \overline{RD}}$ intersects both circles mentioned orthogonally, and hence also all other circles of both bundles. From this it follows that in each point of the circle \mathcal{C} orthogonal to both bundles a circle of one bundle is tangent to one of the other bundle: the locus wanted is therefore the part of the circle \mathcal{C} that lies within the concave angle formed by the half-lines RA and RD (the whole circle if A, B, C and D are on a straight line).

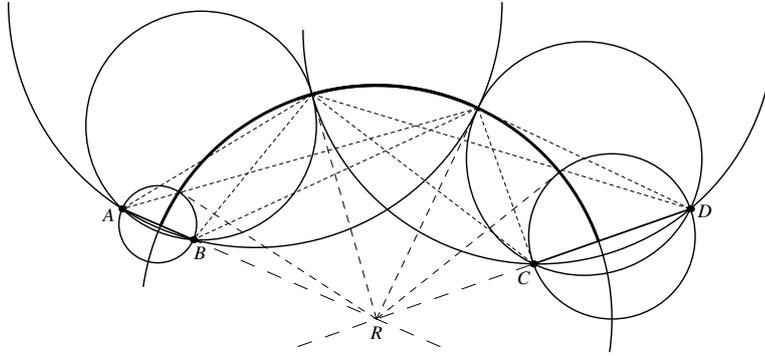


Figure A.1:

5. Given a Cartesian system x, y and a point A (which we suppose to lie in the first quadrant) one wants to choose a point P so as to minimise the distance \overline{AP} and to maximise the surface area of the rectangle formed by the axes and the parallel lines through P . We must consider the two functions $\varphi(P) = xy$ (to be maximised) and $\psi(P) = \overline{AP}$ (to be minimised), and look for the points of tangency between a hyperbola $\varphi = xy = \text{constant}$ and a circle with centre A , $\psi = \overline{AP} = \text{constant}$. Let a, b be the co-ordinates of A ; in the generic point $P(x, y)$, the inclination of the tangent line to the circle with centre A passing through P is $-(x-a)/(y-b)$, that of the tangent line to the hyperbola $y = k/x$ is $-y/x$. In order for the two curves to be tangent, that is for the two tangent lines to coincide, there must hold $y/x = (x-a)/(y-b)$, or $y(y-b) = x(x-a)$. As one sees more easily with the substitution

$$\xi = x - \frac{a}{2}, \quad \eta = y - \frac{b}{2},$$

which gives

$$\left(\eta - \frac{b}{2}\right)\left(\eta + \frac{b}{2}\right) = \left(\xi - \frac{a}{2}\right)\left(\xi + \frac{a}{2}\right)$$

or

$$\xi^2 - \eta^2 = \frac{1}{4}(a^2 - b^2),$$

the locus wanted is the equilateral hyperbola passing through A with the point $(\frac{a}{2}, \frac{b}{2})$ (which is the midpoint of the line segment OA) as its centre and with the axes parallel to the co-ordinate axes (the transverse axis is the one parallel to the x -axis if, as in Figure A.2, $a > b$, and inversely in the opposite case). Next, one sees easily that the part of the curve that forms the solution of the problem is the part of the branch departing from A that lies in the first quadrant, which is drawn with a solid curve in the figure.

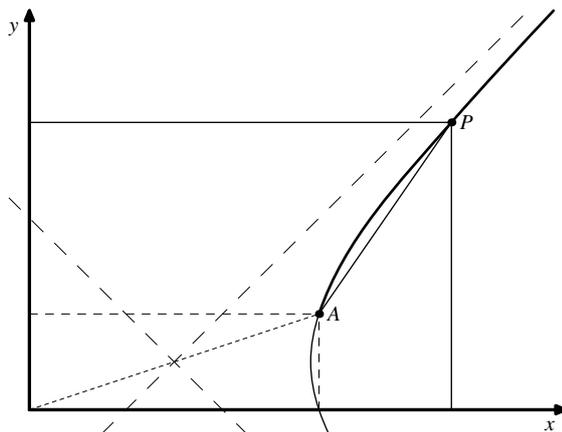


Figure A.2:

6. One wants to set up a game of chance in which there are three possible outcomes with probabilities u , v and w (hence $u + v + w = 1$). One wants to determine the values to be given to u , v and w so as to render as high as possible, in three independent trials, both the probability $\varphi = 6uvw$ of three *different* outcomes and the probability $\psi = u^3 + v^3 + w^3$ of three *equal* outcomes. If one wants a geometrical representation, one interprets u , v and w as the barycentric co-ordinates of the points inside a triangle, which for simplicity's sake may be taken to be equilateral.

We get

$$\begin{aligned} d\varphi &= 6d(uvw) = 6(vwdu + uwdv + uvdw), \\ d\psi &= d(u^3 + v^3 + w^3) = 3(u^2du + v^2dv + w^2dw); \end{aligned}$$

the latter expression equalised to zero gives, together with $du + dv + dw = 0$ (which holds identically because of $u + v + w = 1$), a system of linear homogeneous equations in du , dv and dw , the solution of which shows that a move along a line $\psi = \text{constant}$ can be written in the following form:

$$du = (w^2 - v^2)dt, \quad dv = (u^2 - w^2)dt, \quad dw = (v^2 - u^2)dt,$$

where one indicates with dt the common value of $du/(w^2 - v^2)$, etc. By the move just indicated, the increment of φ is

$$\begin{aligned} d\varphi &= 6d(uvw) = 6(vw(w^2 - v^2) + uw(u^2 - w^2) + uv(v^2 - u^2))dt \\ &= 6(v - w)(w - u)(u - v)dt \end{aligned}$$

as results by keeping in mind that $u + v + w = 1$ and simplifying appropriately. Let us now assume $w > v > u$; then $du > 0$, $dv < 0$, $dw > 0$, and $d\varphi > 0$, and this means that,

by moving along the line $\psi = \text{constant}$ in the direction that lowers v and raises u and w , ϕ increases. Therefore ϕ reaches the maximum of its values on $\psi = \text{constant}$ when u and v will have become equal, and the “optimum” points will be those where $w \geq v = u$. That each of those points is really acceptable as “optimum” appears readily from the observation that if $u = v$, then $\phi = u^2(1 - 2u)$ and $\psi = 2u^3 + (1 - 2u)^3$; on the trajectory of interest ($0 \leq u \leq 1/3$, because if $u > 1/3$ one could rewrite $w < 1/3 < u$ contrary to the hypothesis), one sees immediately by taking derivatives that ϕ is increasing while ψ is decreasing in u . By symmetry, abandoning the hypothesis that w is the largest of the three values one arrives easily at the following conclusion: the locus of “optimum” points consists of the three segments joining the centre of the triangle ($u = v = w = 1/3$) to each of the three vertices ($u = 1; v = 1; w = 1$).

7. One may think of the same ternary diagram (that is, the triangle) as representing a series of “optimum” problems that actually occur in the technical sciences. Note that one usually represents, in the ternary diagram, the various alloys of three metals A , B and C , indicating the alloy that contains them in the fractions u , v and w ($u + v + w = 1$) by the point with barycentric co-ordinates u , v and w (the three vertices thus representing the three metals in pure form). If one wants to obtain an alloy that, to the greatest possible extent, has two different properties (for example, lightness and resistance, or flexibility and fusibility, etc. etc.), the solutions of the “optimum” problem are precisely the points defined in the familiar way: the level curves of the two properties being $\phi = \text{constant}$ and $\psi = \text{constant}$, one must look for the point of contact of a level curve of one property with the highest level curve of the other property among those with which it has some point(s) in common. Because these level curves, for the physical properties mentioned, are in general determined experimentally (except for the specific gravity, for which one obviously has $\phi = au + bv + cw$, with a , b and c the specific gravities of A , B and C , so that the level curves are parallel straight lines), we cannot solve the problem analytically, as in the preceding example. Rather, the tangency of the two level curves may simply fail to hold, a property that up to now was always true given that the curves had continuously varying tangent lines. It may even be the case that no tangent line exists; take for example the level curves of the fusion temperature, which will exhibit cusps corresponding to the “eutectics;” if such a cusp touches a level curve of the other property, one may here have an “optimum” point without tangency of the two curves.

A.3 Sketch of a systematic treatment

8. In all preceding examples (except the first one, which is very elementary for a first orientation) we have considered problems involving the desire to maximise $n = 2$ quantities, and, as the locus satisfying the condition, we have found *curves*, or sets of $n - 1 = 1$ dimensions. Moreover, in the preceding examples we have, case by case, looked for a solution method suggested by the particular problem, and this would probably almost never be equally successful in the case of problems posed in a space involving more than two variables. Therefore we intend to sketch a systematic treatment of the problem, supposing that we want to maximise n functions in a space of q variables (n not larger than q); we shall assume the functions to be differentiable and shall find that, “in general,” the result of the foregoing particular case always holds good: the locus of “optimum” points is a variety of $n - 1$ dimensions.

9. Let one have q independent variables x_1, x_2, \dots, x_q (or a space S_q of q dimensions), and n functions ($n \leq q$) of x_1, x_2, \dots, x_q , which we shall indicate by $\varphi_1, \varphi_2, \dots, \varphi_n$. One has to choose a point (x_1, x_2, \dots, x_q) , and one wants that all the functions φ_h take a value as large as possible there. We can limit ourselves to posing the problem this way, saying "as large as possible," because if instead in an application the case of "as small as possible" would occur, or of "as large as possible" for some of the functions and "as small as possible" for the other ones, one would only have to change the signs of the φ_h or of a subset of them, respectively.

Like in the preceding examples, when choosing at random a point (x_1, x_2, \dots, x_q) we shall in general find that points preferable to it exist, the φ_h all having a higher value; however, there may be points from which it is not possible to move without diminishing the value of at least one of the φ_h , and we intend precisely to determine the locus of these points, which we shall call the "optimum".

We shall suppose that the φ_h are differentiable, so that, by moving from the point P with co-ordinates x_k to the point $P + dP$ with co-ordinates $x_k + dx_k$, the increase of φ_h will be in first approximation

$$d\varphi_h = \sum_{k=1}^q \frac{\partial \varphi_h}{\partial x_k} dx_k \quad (h = 1, 2, \dots, n).$$

If, by a suitable choice of the dx_k , all $d\varphi_h$ turn out positive, the point P cannot be an "optimum", because, for a sufficiently small ε , in the point $P + \varepsilon dP$ the φ_h certainly take values all larger than in P : $\varphi_h(P + \varepsilon dP) > \varphi_h(P)$ ($h = 1, 2, \dots, n$). A necessary condition for a point to be an "optimum" is therefore that the n linear expressions

$$\sum_k \frac{\partial \varphi_h}{\partial x_k} y_k \quad (h = 1, 2, \dots, n),$$

the coefficients of which form the $n \times q$ matrix $\Phi := [\partial \varphi_h / \partial x_k]$, cannot all be rendered positive simultaneously. First of all, therefore, the matrix must have deficient rank, else one will always have, by putting the n linear expressions equal to arbitrary (in particular, all positive) values, a *compatible* system of n linear equations in n unknowns. The condition that Φ have deficient rank is equivalent to $q - n + 1$ scalar equations (as many as there are linearly independent minors (*Jacobians*) of order n , which the condition requires to be zero), and hence, "in general," it defines a variety of $q - (q - n + 1) = n - 1$ dimensions in the space S_q , which variety, by consequence, is or contains the locus of "optimum" points. We shall shortly see that the ulterior conditions have the character of inequalities, so that in general they will not determine a variety of fewer dimensions, but will only delimit a portion of this variety as the locus of "optimum" points.

Given that the matrix has deficient row rank, the n linear expressions will be connected by some linear relationship, and hence coefficients $\lambda_1, \lambda_2, \dots, \lambda_n$ will exist such that

$$\sum_{h=1}^n \lambda_h \frac{\partial \varphi_h}{\partial x_k} = 0 \quad \text{for } k = 1, 2, \dots, q.$$

Let us first suppose that the matrix Φ has rank $n - 1$, so that $\lambda_1, \lambda_2, \dots, \lambda_n$ are determined uniquely (up to an arbitrary multiplicative constant). The λ_h thus appear proportional to the n cofactors of the elements of any one column in any one of the square submatrices of order n of the matrix. If the λ_h do not all have the same sign, it is

obviously always possible to choose positive numbers c_h such that $\lambda_1 c_1 + \lambda_2 c_2 + \dots + \lambda_n c_n = 0$, which condition is necessary and sufficient for the system of linear equations

$$\sum_{k=1}^q \frac{\partial \varphi_h}{\partial x_k} y_k = c_h \quad (h = 1, 2, \dots, n)$$

to have a solution. If instead all λ_h have the same sign (while some of them may be zero), one cannot find positive numbers c_h for which the system admits a solution, because $\lambda_1 c_1 + \lambda_2 c_2 + \dots + \lambda_n c_n = 0$ cannot hold good.

If the rank of the matrix is less than $n - 1$, the $\lambda_1, \lambda_2, \dots, \lambda_n$ are no longer determined uniquely (to be precise, if the rank of the matrix is $n - r$, there exist, as is well-known, r linearly independent n -tuples $\lambda_1, \lambda_2, \dots, \lambda_n$). The preceding conclusion extends to the general case in the sense that the necessary and sufficient condition for the existence of positive numbers c_h that make the system solvable is the non-existence of a positive (not necessarily strictly positive) n -tuple $\lambda_1, \lambda_2, \dots, \lambda_n$; let this statement alone suffice, because it seems superfluous to burden the discussion, deliberately kept brief, by lingering over the exceptional case in which even all minors of order $n - 1$ are zero.

Let us therefore recapitulate our conclusions in the following way:

The "optimum" points belong to the variety, generally of $n - 1$ dimensions, on which the matrix of <first-order> partial derivatives has deficient rank. Knowing the values of the n cofactors $\lambda_1, \lambda_2, \dots, \lambda_n$, we can exclude that the case of an "optimum" applies if two of them have opposite signs; if, on the contrary, they are all of the same sign, or maybe some of them zero, this means that, as far as can be deduced from the mere knowledge of the first derivatives (that is, from the first-order approximation of the φ_h), the case of an "optimum" may apply (if all λ_h are zero, the knowledge of the λ_h does not suffice for the conclusion whether the case of an "optimum" may apply or not by referring to the first-order approximation).

Naturally, as in all maximum and minimum problems, the conditions relating to the first derivatives cannot be sufficient but only necessary. For a complete elaboration of the general treatment it would be necessary to examine the conditions relating to the second derivatives, and possibly to the successive derivatives, in the case that the behaviour of the second derivatives still left ambiguity; generally however, in concrete examples the examination of the conditions relating to the second and higher derivatives is either practically superfluous by the very nature of the question, or may be replaced by more intuitive considerations suggested case by case.

10. Let us write the result more explicitly for the most simple cases ($n = 2$ and 3).

For $n = 2$:

The condition that Φ have deficient rank means now (with $\varphi_1 = \varphi$ and $\varphi_2 = \psi$):

$$\frac{\frac{\partial \varphi}{\partial x_1}}{\frac{\partial \psi}{\partial x_1}} = \frac{\frac{\partial \varphi}{\partial x_2}}{\frac{\partial \psi}{\partial x_2}} = \dots = \frac{\frac{\partial \varphi}{\partial x_k}}{\frac{\partial \psi}{\partial x_k}} = \dots = \frac{\frac{\partial \varphi}{\partial x_q}}{\frac{\partial \psi}{\partial x_q}},$$

and hence is equivalent to $q - 1$ equations. The condition on the cofactors reduces to the very simple condition that the common value of the q fractions written above be positive (if it proves to be zero or infinite, the case is ambiguous).

For $n = 3$:

The condition that Φ have deficient rank means now (with $\varphi_1 = \varphi$, $\varphi_2 = \psi$ and $\varphi_3 = \chi$):

$$\begin{vmatrix} \frac{\partial \varphi}{\partial x_1} & \frac{\partial \varphi}{\partial x_2} & \frac{\partial \varphi}{\partial x_k} \\ \frac{\partial \psi}{\partial x_1} & \frac{\partial \psi}{\partial x_2} & \frac{\partial \psi}{\partial x_k} \\ \frac{\partial \chi}{\partial x_1} & \frac{\partial \chi}{\partial x_2} & \frac{\partial \chi}{\partial x_k} \end{vmatrix} = 0 \quad \text{for } k = 3, 4, \dots, q,$$

or

$$\begin{aligned} & \frac{\partial \varphi}{\partial x_k} \left(\frac{\partial \psi}{\partial x_1} \frac{\partial \chi}{\partial x_2} - \frac{\partial \chi}{\partial x_1} \frac{\partial \psi}{\partial x_2} \right) + \frac{\partial \psi}{\partial x_k} \left(\frac{\partial \chi}{\partial x_1} \frac{\partial \varphi}{\partial x_2} - \frac{\partial \varphi}{\partial x_1} \frac{\partial \chi}{\partial x_2} \right) + \\ & \frac{\partial \chi}{\partial x_k} \left(\frac{\partial \varphi}{\partial x_1} \frac{\partial \psi}{\partial x_2} - \frac{\partial \psi}{\partial x_1} \frac{\partial \varphi}{\partial x_2} \right) = A \frac{\partial \varphi}{\partial x_k} + B \frac{\partial \psi}{\partial x_k} + C \frac{\partial \chi}{\partial x_k} = 0 \\ & \text{for } k = 3, 4, \dots, q, \end{aligned}$$

which is equivalent to $q - 2$ equations. The condition on the cofactors can be expressed by saying that A , B and C must prove to be of the same sign (while some of them may be zero; if they are all zero, one is in an ambiguous case). Naturally, it is inessential to which two of the co-ordinates one attributes the role we have given here to x_1 and x_2 .

Superfluous to repeat that the conditions just reported in full for $n = 2$ and $n = 3$ are only those relating to the first derivatives, and that for convincing oneself that one has really obtained an "optimum", either a further analysis of second (and possibly even higher) derivatives would be necessary, or else some supplementary consideration suggested by the problem.

And let us end with the following observation, which is often useful in practice for reaching the solution of concrete problems by working with less complex expressions:

In the matrix Φ one may always suppress positive factors common to all elements of one and the same row or of one and the same column (because it does not affect the sign or the vanishing of the determinants of the various submatrices).

In the next example we shall see the usefulness of this possibility of simplification, which exists of course not only for the special cases of this paragraph ($n = 2$ and 3), but quite generally.

11. Let us immediately apply the treatment just developed to an example.

On a plane (x, y) one has three points A_1 , A_2 and A_3 (not lying on a straight line); where above the plane must one place a spotlight P so that the illumination of the plane is as strong as possible in the points A_1 , A_2 and A_3 ? With z denoting the height of P above the plane and r its distance to an arbitrary point A of the plane, the illumination in A will be proportional to z/r^3 ; for it is known to be given by $I \cos \gamma / r^2$, where I is the intensity of the light source, γ the angle of incidence of the <light> ray on the plane, and r the distance, and evidently $\cos \gamma = z/r$. Hence, with r_1 , r_2 and r_3 denoting the distances of P to A_1 , A_2 and A_3 , the three functions to be maximised are zr_1^{-3} , zr_2^{-3} and zr_3^{-3} ; given the fact that the resulting expressions are rational and have simpler derivatives, we prefer to say that we want to minimise the reciprocals of the squares of the three functions:

$$\varphi_1 = \varphi = r_1^6 z^{-2}, \quad \varphi_2 = \psi = r_2^6 z^{-2}, \quad \varphi_3 = \chi = r_3^6 z^{-2}.$$

Let $\varphi_i = r_i^6 z^{-2}$ be any one of φ , ψ and χ ; the derivatives will be

$$\begin{aligned}\frac{\partial \varphi_i}{\partial x} &= 3z^{-2} r_i^4 \frac{\partial r_i^2}{\partial x}, & \frac{\partial \varphi_i}{\partial y} &= 3z^{-2} r_i^4 \frac{\partial r_i^2}{\partial y}, \\ \frac{\partial \varphi_i}{\partial z} &= 3z^{-2} r_i^4 \frac{\partial r_i^2}{\partial z} - 2z^{-3} r_i^6,\end{aligned}$$

but, with x_i, y_i the co-ordinates of A_i ($i = 1, 2, 3$), and x, y, z those of P , there holds

$$r_i^2 = (x - x_i)^2 + (y - y_i)^2 + z^2,$$

and hence

$$\frac{\partial r_i^2}{\partial x} = 2(x - x_i), \quad \frac{\partial r_i^2}{\partial y} = 2(y - y_i), \quad \frac{\partial r_i^2}{\partial z} = 2z,$$

so that

$$\begin{aligned}\frac{\partial \varphi_i}{\partial x} &= 6z^{-2} r_i^4 (x - x_i), & \frac{\partial \varphi_i}{\partial y} &= 6z^{-2} r_i^4 (y - y_i), \\ \frac{\partial \varphi_i}{\partial z} &= 6z^{-2} r_i^4 z - 2z^{-3} r_i^6.\end{aligned}$$

By eliminating the positive factor $6z^{-2} r_i^4$, the three derivatives appear proportional to

$$x - x_i, \quad y - y_i, \quad z - r_i^2/3z,$$

and in writing the matrix <of first derivatives> we can further simplify the last column because we may multiply it by the positive common factor $3z$, so that we get the following matrix:

$$\begin{bmatrix} x - x_1 & y - y_1 & 3z^2 - r_1^2 \\ x - x_2 & y - y_2 & 3z^2 - r_2^2 \\ x - x_3 & y - y_3 & 3z^2 - r_3^2 \end{bmatrix}$$

or, by developing the r_i^2 (and writing the typical row, instead of repeating it three times with $i = 1, 2, 3$):

$$[x - x_i \quad y - y_i \quad 2z^2 - (x - x_i)^2 - (y - y_i)^2]$$

or also

$$[x - x_i \quad y - y_i \quad 2z^2 + (x^2 + y^2) - (x_i^2 + y_i^2) - 2x(x - x_i) - 2y(y - y_i)].$$

The matrix is square (for $q = n = 3$) and it suffices that its determinant equals zero; to that effect, one may suppress the terms in $(x - x_i)$ and $(y - y_i)$ in the third column, because they are proportional to the first and second column, and finally one may put $x_i^2 + y_i^2 = R^2$, because it is allowed to suppose, without loss of generality and with the advantage of simplification we are about to see, that the origin of the co-ordinate system x, y is situated in the centre of the circle passing through A_1, A_2 and A_3 ; thence $x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2 = R^2$, the geometrical meaning of R being just that of the radius of this circle. But then all terms in the third column appear equal to one another, and, to be precise, equal to $2z^2 + x^2 + y^2 - R^2$, and the equation reduces to

$$(2z^2 + x^2 + y^2 - R^2) \begin{vmatrix} x - x_1 & y - y_1 & 1 \\ x - x_2 & y - y_2 & 1 \\ x - x_3 & y - y_3 & 1 \end{vmatrix} = 0$$

or $2z^2 + x^2 + y^2 = R^2$, given that the determinant simply represents twice the area of the triangle <formed by> A_1 , A_2 and A_3 , which points do not lie on a straight line by assumption.

The variety on which the determinant vanishes is thus (because one may consider z essentially positive) the round semi-ellipsoid $z = \frac{1}{\sqrt{2}} \sqrt{R^2 - (x^2 + y^2)}$, which one may simply think of as being obtained from the hemisphere resting on the circle determined by the three given points by reducing the ordinate z in the ratio of $\sqrt{2}$ to 1, that is, by somewhat flattening it. It remains to see the signs of the three cofactors, which are

$$\begin{vmatrix} x-x_1 & y-y_1 \\ x-x_2 & y-y_2 \end{vmatrix}, \quad \begin{vmatrix} x-x_2 & y-y_2 \\ x-x_3 & y-y_3 \end{vmatrix}, \quad \begin{vmatrix} x-x_3 & y-y_3 \\ x-x_1 & y-y_1 \end{vmatrix},$$

and hence represent (twice) the areas of the three triangles $A_1A_2P_0$, $A_2A_3P_0$ and $A_3A_1P_0$, where P_0 is the image of P on the plane $z=0$. The equality condition on the signs holds if the three areas are equally oriented, that is if P_0 lies inside the triangle $A_1A_2A_3$. That we really have an "optimum" seems intuitive from the very nature of the problem; that the locus of "optimum" points is the whole surface indicated (and not, because of ulterior conditions in the second derivatives, only part of it) will appear from the considerations of the next two paragraphs.

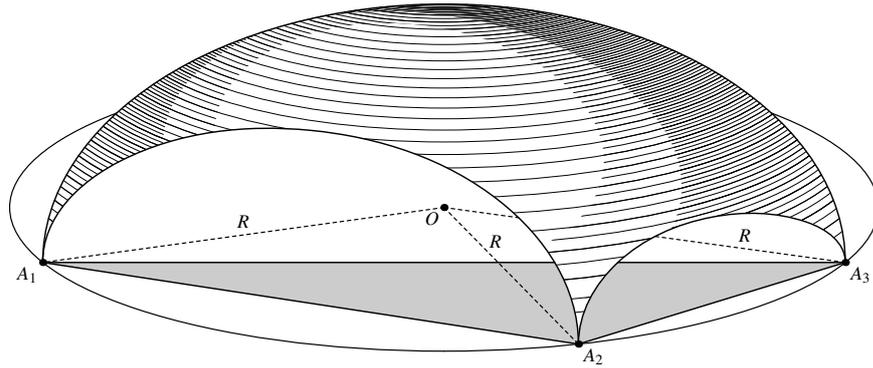


Figure A.3:

Hence one may conclude: if one wants to maximise the illumination of a plane in three of its points A_1 , A_2 and A_3 , the locus of "optimum" points for the spot in which to place the light source is represented by the part above the triangle $A_1A_2A_3$ of the semi-ellipsoid of revolution resting on the circle passing through A_1 , A_2 and A_3 , and with height reduced in the ratio of $\sqrt{2}$ to 1 compared to the hemisphere.

12. On this example one can make an almost banal, but interesting observation, which may often turn out rather useful.

If one tries to solve the same "optimum" problem with respect to just two points A_1 and A_2 , it is easy to see that the solution is given by the semi-ellipse that rests on the line segment A_1A_2 and that one obtains from the circle by flattening it in the familiar ratio of $\sqrt{2}$ to 1. So, it is the border of the locus of "optimum" points of the problem relating to three points A_1 , A_2 and A_3 , and precisely the intersection of the semi-ellipsoid with the vertical plane through A_1 and A_2 . This holds analogously for the two other sides, from A_1 to A_3 and from A_2 to A_3 . The points A_1 , A_2 and A_3 , vertices common to two of

the sides, are, if one likes to say so, the solution of the “optimum” problem relative to the single point A_1 , or to the single point A_2 , or to the single point A_3 .

It is now easy to understand in general that each “optimum” point relative to $m < n$ of the n functions $\varphi_1, \varphi_2, \dots, \varphi_n$ is *a fortiori* an “optimum” point with regard to all of them; that is what the very definition of “optimum” says, and moreover the examination of the condition on the matrix appears in accordance with this fact. Thereupon the spontaneous idea arises that, under certain general conditions, it is possible to enunciate the following result: *The locus of “optimum” points with respect to n functions is, topologically, a simplex of $n - 1$ dimensions, the n faces of which are the loci of “optimum” with respect to $n - 1$ <of the> functions, the $\binom{n}{2}$ edges of which those for $n - 2$ <of the> functions, and so on, up to the n vertices, “optimum” points with respect to the n functions separately.* We shall occupy ourselves shortly, if merely cursorily, with this matter; for now we note only that the property just pointed out allows one often to find “optimum” points, curves, etc. in a more direct and easy way before solving the problem completely; in relation to the preceding example (n. 11) this <property> allows one in particular to convince oneself that the whole portion of the ellipsoid up to the determined limits is effectively to be considered as satisfying the problem.

13. The most spontaneous idea for trying to examine the validity of the topological hypothesis just advanced consists in observing that if a linear combination with positive coefficients (which in the sequel we shall briefly call “a positive linear combination”) of the φ_h , with $\varphi = \sum_h \rho_h \varphi_h$ ($\rho_h \geq 0$), has its absolute maximum in a point P , such a point is necessarily an “optimum”. For if there would exist a point Q where all φ_h would take a larger value than in P , also $\varphi = \sum_h \rho_h \varphi_h$ would have a larger value in Q than in P , against the hypothesis. Let us now suppose that all positive linear combinations of such type admit a unique absolute maximum and have partial derivatives all zero there and in no other point, which occurs in particular if the φ_h (and hence all their positive linear combinations) are concave and differentiable functions, and let us show that in that case also the inverse conclusion holds good: if P is an “optimum” point, it is the absolute maximum of one of the positive linear combinations $\varphi = \sum_h \rho_h \varphi_h$. In fact, let P be an “optimum”, and let the n usual cofactors $\lambda_1, \lambda_2, \dots, \lambda_n$ have the values $\rho_1, \rho_2, \dots, \rho_n$ there: the relationship between the n rows of the matrix Φ then means that the following q relationships exist:

$$\sum_h \rho_h \frac{\partial \varphi_h}{\partial x_k} = 0 \quad (k = 1, 2, \dots, q)$$

or

$$\frac{\partial}{\partial x_k} \sum_h \rho_h \varphi_h = \frac{\partial \varphi}{\partial x_k} = 0 \quad (k = 1, 2, \dots, q) \quad \text{when } \varphi = \sum_h \rho_h \varphi_h.$$

By assumption, such a relationship cannot exist but in the point of absolute maximum of φ , which therefore must coincide with P , *q.e.d.*; moreover, it appears that different “optimum” points correspond to distinct (and not simply proportional) linear combinations. We can easily eliminate functions that are simply proportional by limiting ourselves to linear combinations for which $\sum \rho_h = 1$: in this way, such functions correspond to all points of a simplex of $n - 1$ dimensions (that is, a line segment for $n - 1 = 1$, a triangle for $n - 1 = 2$, a tetrahedron for $n - 1 = 3$, and their evident generalisations in the spaces of 4, 5, etc. dimensions for $n - 1 = 4, 5$, etc.), where one can make a one-to-one correspondence between $\varphi = \sum \rho_h \varphi_h$ and the point $A = \rho_1 A_1 + \rho_2 A_2 + \dots + \rho_n A_n$ with barycentric co-ordinates $\rho_1, \rho_2, \dots, \rho_n$ with respect to the n vertices of the simplex A_1, A_2, \dots, A_n .

For demonstrating the validity, under the stated conditions, of our topological hypothesis, it remains to demonstrate the continuity of the just established correspondence: let us therefore show that, when one alters the $\rho_1, \rho_2, \dots, \rho_n$ a little, also the point of absolute maximum of $\varphi = \sum \rho_h \varphi_h$ can move only a small distance, or, in more precise terms, that, given an arbitrarily small distance θ , one can always determine ε so that, from the coexistence of the inequalities $|\bar{\rho}_h - \rho_h| < \varepsilon$ ($h = 1, 2, \dots, n$), it follows necessarily that the maximum point \bar{P} of $\bar{\varphi} = \sum \bar{\rho}_h \varphi_h$ is no farther than θ away from the maximum point P of $\varphi = \sum \rho_h \varphi_h$. Let us write $\bar{\rho}_h = \rho_h + \varepsilon_h$ ($|\varepsilon_h| < \varepsilon$, $\sum \varepsilon_h = 0$), so that $\bar{\varphi} = \varphi + \sum \varepsilon_h \varphi_h$; let $M = \varphi(P)$ be the maximum of φ , and M_θ the maximum of φ for the points not within distance θ from P , and let M' be the largest among the (absolute) maxima M_1, M_2, \dots, M_n of $\varphi_1, \varphi_2, \dots, \varphi_n$. In the point P we then get

$$\bar{\varphi}(P) = \varphi(P) + \sum \varepsilon_h \varphi_h(P) > M - n\varepsilon H$$

where H is the largest of the n values $|\varphi_1(P)|, |\varphi_2(P)|, \dots, |\varphi_n(P)|$, while for each point Q not within distance θ from P we get

$$\bar{\varphi}(Q) = \varphi(Q) + \sum \varepsilon_h \varphi_h(Q) < M_\theta + n\varepsilon M'.$$

Hence, if $\varepsilon < \frac{1}{n} \frac{M - M_\theta}{M' + H}$, it follows that $\bar{\varphi}(P) > \bar{\varphi}(Q)$ for all Q not within distance θ from P , and this suffices to prove that it is impossible for the point \bar{P} in which $\bar{\varphi}$ reaches its maximum not to have a distance from P less than θ , for else one would have $\bar{\varphi}(P) > \bar{\varphi}(\bar{P})$, against the definition of \bar{P} . We can summarise and render intuitive the meaning of the proof in the following way: with a slight variation of the coefficients ρ_h , φ varies slightly, too, and hence it cannot decrease by so much in P and increase by so much in a point Q external to a neighbourhood of P as to take in Q a value larger than in P , let alone its maximum value.

Hence the topological property stated at the end of the previous subsection exists certainly if the φ_h are concave,[†] differentiable functions, or, more generally, are such that all their positive linear combinations have derivatives equal to zero in a unique point (absolute maximum). As is easy to see, one may express this condition by saying that among all points satisfying the necessary conditions established for the "optimum" (matrix of deficient rank, equally signed cofactors[‡]) there do not exist two in which the n cofactors take the same values or proportional values (that is, given $\lambda_1, \lambda_2, \dots, \lambda_n$ and $\lambda'_1, \lambda'_2, \dots, \lambda'_n$, respectively, the values of the cofactors in the two points of possible "optimum", P and P' , it may not be true that $\lambda_1/\lambda'_1 = \lambda_2/\lambda'_2 = \dots = \lambda_n/\lambda'_n$). If the condition of concavity, or the other less restrictive one is not satisfied, it may get satisfied by substituting for the n functions $\varphi_1, \varphi_2, \dots, \varphi_n$ other functions $f_1(\varphi_1), f_2(\varphi_2), \dots, f_n(\varphi_n)$, where the f_h are increasing functions, or by mapping the space S_q with co-ordinates x_1, x_2, \dots, x_q onto the space S'_q with new co-ordinates y_1, y_2, \dots, y_q , or by applying both these possibilities simultaneously; given the intrinsic character of the notion of "optimum", invariant with respect to each system of reference, and the obvious possibility of replacing the functions φ_h by increasing but otherwise arbitrary functions of them, the demonstrated property holds good also in this new, extended case.

[†]In the Italian original: *convessa*.

[‡]In the Italian original: *minori*, and likewise for the next two occurrences of "cofactor" in the translation.

14. Let us examine the condition just found in the case of the example of n. 11.

We have seen that the three cofactors represent the areas of the three triangles $A_1A_2P_0$, $A_2A_3P_0$ and $A_3A_1P_0$, where P_0 is the projection of P on the plane of the three points A_1 , A_2 and A_3 . The proportionality (and even equality) occurs therefore only for points P with the same projection P_0 , that is, for points on the same perpendicular to the given plane, or also, in terms of the co-ordinates used before, with equal x and equal y , and differing only in z . Because, when one considers z essentially positive, the equation obtained from the condition that the Jacobian have deficient rank represents a semi-ellipsoid and hence z appears a single-valued function of x and y , $z = f(x, y)$, the condition is satisfied. If one would not specify which side of the plane one wants to illuminate, and if one would hence consider both signs admissible for z , the condition would no longer be satisfied (in fact, one would have $z = \pm f(x, y)$), and the locus of "optimum" points would effectively split itself in two, each topologically of the type considered (to be precise, the locus found for positive z and its mirror image with respect to the plane).

15. To have a simple example of the application of the general procedure in the case of $n < q$ as well, let us generalise the problem of n. 5, by searching, in the space of three or more dimensions, for the locus of points P from which it is not possible to move so as to diminish the distance to a given point A and to augment the volume (or hypervolume) of the prism between the co-ordinate planes and the parallel planes through P . In the space of three dimensions ($q = 3$, $n = 2$), the two functions to be maximised are

$$\varphi = xyz, \quad \psi = -\frac{1}{2} \left((x-a)^2 + (y-b)^2 + (z-c)^2 \right),$$

with a, b, c the co-ordinates of A (which we shall suppose positive). The matrix of derivatives is

$$\begin{bmatrix} yz & xz & xy \\ x-a & y-b & z-c \end{bmatrix},$$

and the equations that result from the condition of deficient rank are

$$\frac{x-a}{yz} = \frac{y-b}{xz} = \frac{z-c}{xy}$$

or

$$x(x-a) = y(y-b) = z(z-c),$$

while the supplementary condition is that the three terms are not negative. Remembering the result of n. 5, one sees that here (and also, as one can easily check, in the case of more than three dimensions: $q = \text{any integer}$, $n = 2$), the solution is given by the curve that has as its projection on each of the co-ordinate planes the curve that constitutes the solution in two dimensions, which is a branch of an equilateral hyperbola, running from the projection of A on the plane to infinity. The curve wanted departs therefore from A and tends asymptotically to the straight line

$$x - \frac{a}{2} = y - \frac{b}{2} = z - \frac{c}{2}.$$

16. In a subsequent paper we shall elucidate and study the problems of "constrained optimum," in which the problem of economic "optimum" returns more properly; we shall see how this and its well-known solution in the form of the equations of Jevons-Walras fit in the general framework, and we shall give a generalisation.

to treat the problem directly in the new form. Naturally, instead of $n \leq q$ there must now hold $n \leq q - s$.

One would have such a problem—and we shall examine it once we are able to—for example by modifying the problem considered earlier concerning the “optimal” illumination of a plane in three of its points in the following way: instead of three points there are only two, but the light source is constrained to a given surface (for example, in a concrete case, to the ceiling).

Turning to the general problem, let us re-assume the framework of n. 8 of the preceding paper, supposing that also the functions G_j are differentiable. We shall always have

$$d\varphi_h = \sum_k \frac{\partial \varphi_h}{\partial x_k} dx_k \quad (h = 1, 2, \dots, n),$$

but the dx_k will be bound by the s conditions

$$dG_j = \sum_k \frac{\partial G_j}{\partial x_k} dx_k = 0 \quad (j = 1, 2, \dots, s).$$

As necessary condition for the “optimum” we find, like before, that the matrix below must have deficient rank:

$$\begin{bmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \dots & \frac{\partial \varphi_1}{\partial x_q} \\ \dots & \dots & \dots & \dots \\ \frac{\partial \varphi_n}{\partial x_1} & \frac{\partial \varphi_n}{\partial x_2} & \dots & \frac{\partial \varphi_n}{\partial x_q} \\ \frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial x_2} & \dots & \frac{\partial G_1}{\partial x_q} \\ \dots & \dots & \dots & \dots \\ \frac{\partial G_s}{\partial x_1} & \frac{\partial G_s}{\partial x_2} & \dots & \frac{\partial G_s}{\partial x_q} \end{bmatrix}.$$

The matrix has $n + s$ rows and q ($q \geq n + s$) columns, so that the condition of deficient rank is equivalent to $q - n - s + 1$ equations; by adding the s equations $G_j = 0$ one has again $q - n + 1$ equations, and hence, in general, a variety of $q - (q - n + 1) = n - 1$ dimensions, as must be (by the equivalence, as noted, of the constrained “optimum” problem and the <unconstrained> “optimum” problem in a space of $q - s$ dimensions).

The condition on the cofactors remains essentially unchanged, too: we shall limit ourselves also here to the case that the matrix has rank $n + s - 1$, and we shall even suppose that the matrix formed solely by the derivatives of the G_j (that is, by the last s rows of the matrix above) has full rank. We must see if *positive* numbers c_1, c_2, \dots, c_n exist for which the following system of $n + s$ equations has a solution:

$$\begin{aligned} \sum_k \frac{\partial \varphi_h}{\partial x_k} y_k &= c_h & (h = 1, 2, \dots, n) \\ \sum_k \frac{\partial G_j}{\partial x_k} y_k &= 0 & (j = 1, 2, \dots, s). \end{aligned}$$

We know, incidentally, that coefficients $\lambda_1, \lambda_2, \dots, \lambda_n, \mu_1, \mu_2, \dots, \mu_s$ (determined uniquely up to an inessential multiplicative constant) exist such that for each $k = 1, 2, \dots, q$,

$$\sum_{h=1}^n \lambda_h \frac{\partial \varphi_h}{\partial x_k} + \sum_{j=1}^s \mu_j \frac{\partial G_j}{\partial x_k} = 0.$$

The compatibility condition on the c_1, c_2, \dots, c_n is therefore also here that $\lambda_1 c_1 + \lambda_2 c_2 + \dots + \lambda_n c_n = 0$ (the other terms would be $\mu_1 0 + \mu_2 0 + \dots + \mu_n 0$), and hence, in order that the c_h can all be positive, it is necessary and sufficient that among the λ_h there are two of opposite sign.

2. Let us turn to the example indicated above: one wants to illuminate, as intensely as possible, the plane $z = 0$ in the two points $A_1 (x_1, y_1)$ and $A_2 (x_2, y_2)$, but the light source is constrained to a given surface $G(x, y, z) = 0$. We shall assume that also here the origin has equal distances to A_1 and A_2 : $x_1^2 + y_1^2 = x_2^2 + y_2^2 = R^2$. We have to make equal to zero the following determinant:

$$\begin{aligned} & \begin{vmatrix} x-x_1 & y-y_1 & 3z^2 - r_1^2 \\ x-x_2 & y-y_2 & 3z^2 - r_2^2 \\ G'_x & G'_y & 3zG'_z \end{vmatrix} = \\ & = \begin{vmatrix} x-x_1 & y-y_1 & 2z^2 + (x^2 + y^2) - R^2 \\ x-x_2 & y-y_2 & 2z^2 + (x^2 + y^2) - R^2 \\ G'_x & G'_y & 3zG'_z + 2xG'_x + 2yG'_y \end{vmatrix} = \\ & = \begin{vmatrix} x-x_1 & y-y_1 & 2z^2 + (x^2 + y^2) - R^2 \\ x_1-x_2 & y_1-y_2 & 0 \\ G'_x & G'_y & 3zG'_z + 2xG'_x + 2yG'_y \end{vmatrix} = \\ & = (2z^2 + (x^2 + y^2) - R^2)((x_1 - x_2)G'_y - (y_1 - y_2)G'_x) + \\ & + (3zG'_z + 2xG'_x + 2yG'_y)(x(y_1 - y_2) - y(x_1 - x_2) + x_1 y_2 - y_1 x_2) = 0. \end{aligned}$$

The single steps are not explained here, but will not be difficult to reconstruct by confronting them with those of n. 11 of the preceding paper. The supplementary conditions reduce to just one: the two determinants

$$\begin{vmatrix} x-x_1 & y-y_1 \\ G'_x & G'_y \end{vmatrix}, \quad \begin{vmatrix} x-x_2 & y-y_2 \\ G'_x & G'_y \end{vmatrix}$$

or the two expressions

$$(x-x_1)G'_y - (y-y_1)G'_x, \quad (x-x_2)G'_y - (y-y_2)G'_x$$

must have opposite signs.

Let us consider the particular case in which the constraint is formed by a planar surface. If the plane were horizontal ($z = \text{constant}$), the solution would be obvious (the projection of the line segment $A_1 A_2$); let it, then, not be parallel to the xy -plane, and, without loss of generality (except for the exclusion of the other trivial case in which the straight line through A_1 and A_2 is orthogonal to the intersection of the two planes),

let the x -axis be the intersection of the plane $G = 0$ with the plane $z = 0$. Hence the constraint will be $G = y - az = 0$ (with a the cotangens of the angle formed by the two planes $G = 0$ and $z = 0$; so $a = 0$ if, in particular, the two planes are orthogonal, like in the practical case that the constraint is a vertical wall). Hence $G'_x = 0$, $G'_y = 1$, $G'_z = -a$, and the equation becomes

$$(2z^2 + (x^2 + y^2) - R^2)(x_1 - x_2) + (2y - 3az)(x(y_1 - y_2) - y(x_1 - x_2) + x_1y_2 - y_1x_2) = 0,$$

which, together with $G = y - az$, or equivalently $az = y$, gives

$$2z^2 = R^2 - x^2 - 2y^2 + xy \frac{y_1 - y_2}{x_1 - x_2} + y \frac{x_1y_2 - x_2y_1}{x_1 - x_2}.$$

This is the equation of an ellipsoid, symmetrical with respect to the plane $z = 0$, which it intersects along the ellipse the equation of which one obtains by equalising the second member to zero. As one easily verifies, this ellipse passes through A_1 and A_2 and through the two points on the x -axis at distance $\pm R$ from the origin (which is, as we will remember, the point on the axis at equal distance—to be precise, at distance R —from A_1 and A_2), and has the straight line $2(x_1 - x_2)x = (y_1 - y_2)y$ as conjugated diameter with respect to the one parallel to the x -axis (which enables one to determine, by symmetry, two new points A'_1 and A'_2 , and thus to characterise the ellipse completely, maybe in an easier way). Once the ellipse is determined, so is the ellipsoid provided one makes just one more observation, for example that $z = R/\sqrt{2}$ for $x = y = 0$.

For each value of a , the intersection of this ellipsoid with the plane $G = y - az = 0$ gives an ellipse, which is the locus wanted for which the matrix has deficient rank. The acceptable part is the segment for which $x - x_1$ and $x - x_2$ have opposite signs, or the segment between x_1 and x_2 (certainly, $x_1 \neq x_2$, because $x_1 = x_2$ would mean the excluded borderline case, incompatible with the choice of the origin at equal distances from A_1 and A_2 , in which the straight line $G = z = 0$ is orthogonal to the line connecting A_1 and A_2).

3. A particularly noteworthy case of constrained “optimum” occurs in what one might call the “allocation problem,” which constitutes the simplest “optimum” problem of economics. One may solve it in a very direct way, but it will be useful to see how it fits in the general treatment, also because only in this way will one be able to turn to other, less easy cases.

One has fixed quantities x_1, x_2, \dots, x_m of m goods, and one wants to allocate them to n individuals so as to maximise for each of them a function of the quantities received, which will represent the utility, or rather ophelimity, that the bundle received of the goods has for him. There are $q = nm$ variables (the quantities of each good for each single individual), and, to keep the meaning in mind, we shall not indicate them with a unique progressive index $(x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_{2m}, x_{2m+1}, \dots, x_{nm})$ but with two indices (where the superscript is not to be confused with an exponent!), that is, with

$$\begin{matrix} x_1^1 & x_1^2 & \cdots & x_1^n \\ x_2^1 & x_2^2 & \cdots & x_2^n \\ \dots & \dots & \dots & \dots \\ x_m^1 & x_m^2 & \cdots & x_m^n. \end{matrix}$$

The constraints express that the total quantity of every good is given, and are

$$\begin{aligned} G_1 &= x_1^1 + x_1^2 + \dots + x_1^n - X_1 = 0 \\ G_2 &= x_2^1 + x_2^2 + \dots + x_2^n - X_2 = 0 \\ &\dots\dots\dots \\ G_m &= x_m^1 + x_m^2 + \dots + x_m^n - X_m = 0, \end{aligned}$$

and the opheimities are given by the n functions

$$\begin{aligned} \varphi_1 &= \varphi_1(x_1^1, x_2^1, \dots, x_m^1) \\ \varphi_2 &= \varphi_2(x_1^2, x_2^2, \dots, x_m^2) \\ &\dots\dots\dots \\ \varphi_n &= \varphi_n(x_1^n, x_2^n, \dots, x_m^n); \end{aligned}$$

one observes that the G s contain each the variables of one row of the array of the x_j^h , the φ s each those of one column. Let \mathbf{I}_m be the unit matrix of order m and \mathbf{F}^h ($h = 1, 2, \dots, n$) the $n \times m$ matrix consisting of zeros except for its h -th row, which contains the derivatives of φ_h :

$$\mathbf{F}^h = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ \frac{\partial \varphi_h}{\partial x_1^h} & \frac{\partial \varphi_h}{\partial x_2^h} & \dots & \frac{\partial \varphi_h}{\partial x_m^h} \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then the matrix <of first derivatives>, with $n + m$ rows and nm columns, has the following structure:

$$\begin{bmatrix} \mathbf{F}^1 & \mathbf{F}^2 & \dots & \mathbf{F}^n \\ \mathbf{I}_m & \mathbf{I}_m & \dots & \mathbf{I}_m \end{bmatrix}.$$

For this matrix to have deficient rank, the familiar coefficients $\lambda_1, \lambda_2, \dots, \lambda_n, \mu_1, \mu_2, \dots, \mu_m$ must exist such that the sum of the $n + m$ rows, multiplied by them, be zero, and moreover, for the case of an "optimum" to obtain, the λ s must have the same sign. But in each column we have just two non-zero elements, so that each column gives us an equation of the type

$$\lambda_h \frac{\partial \varphi_h}{\partial x_j^h} + \mu_j = 0$$

or

$$\frac{\partial \varphi_h}{\partial x_j^h} = -\frac{\mu_j}{\lambda_h}.$$

One thus obtains, from the nm derivatives $\partial \varphi_h / \partial x_j^h$, $q - m - n + 1 = nm - m - n + 1 = (n - 1)(m - 1)$ equations expressing that in the array

$$\begin{bmatrix} \frac{\partial \varphi_1}{\partial x_1^1} & \frac{\partial \varphi_1}{\partial x_2^1} & \dots & \frac{\partial \varphi_1}{\partial x_m^1} \\ \frac{\partial \varphi_2}{\partial x_1^2} & \frac{\partial \varphi_2}{\partial x_2^2} & \dots & \frac{\partial \varphi_2}{\partial x_m^2} \\ \dots & \dots & \dots & \dots \\ \frac{\partial \varphi_n}{\partial x_1^n} & \frac{\partial \varphi_n}{\partial x_2^n} & \dots & \frac{\partial \varphi_n}{\partial x_m^n} \end{bmatrix}$$

all the rows (and hence the columns) are proportional to one another (with positive proportionality coefficients). This conclusion constitutes, in the case of the economic allocation problem, the classical result of Jevons–Walras, basis of the masterly treatment of Vilfredo Pareto.

4. The same allocation problem may show up, however, in very different cases, in which the functions to be maximised may have a physical meaning, more concrete and by many better accepted than that of opheimity. Therefore I think it opportune to give such an example, which clarifies the matter without any reference to the economic problem.

Let us suppose to have at our disposal quantities X_1, X_2, \dots, X_m of m metals and that one must make n objects, for each of which one wants to render maximal a certain physical property that is a function of the amounts of the various metals it contains. How to allocate the m metals to the n objects?

Let us consider a realistic example of this kind. One has two metals, in the respective quantities of $1 - \gamma$ and γ (it will be convenient to measure the quantities in volume terms and to put the total volume equal to one, like we have done: $1 - \gamma + \gamma = 1$) with the specific gravities of 1 and $1 - a$, respectively. One wants to construct three bodies, of a shape that we shall specify shortly, so that an inertial moment is as large as possible, and one asks how one must, to that end, allocate the quantities of the two metals, given that $x_1 + x_2 + x_3 = 1 - \gamma$ and $y_1 + y_2 + y_3 = \gamma$ (please note that the indices, in this example written as subscripts for the sake of ease, are those which according to the notation of the foregoing subsection would have been superscripts). The shapes of the three bodies must be the following: 1) *Ball*; inside, a ball of the lighter metal; outside, a hollow ball of the heavier metal. 2) *Cylinder* of a given height; internal cylinder of the lighter metal; hollow cylinder of the heavier metal. 3) *Ball*; an ellipsoid of revolution about the longer axis—which shares its maximal diameter with the ball—of the lighter metal; the residual surrounding volume of the heavier metal. The inertial moment to be maximised is in each case that relative to the axis of rotation.

The inertial moment of a ball of a given metal is proportional to r^5 (with r its radius), and therefore to $v^{5/3}$ (with v its volume); for the cylinder, however, to r^4 , and therefore to v^2 ; for the ellipsoid, finally, to $r\rho^4$ (with r the longer radius (along the axis of revolution), and ρ the shorter radius) and hence to $u^2v^{-1/3}$ (with u the volume of the ellipsoid (proportional to $r\rho^2$), and v the volume of the ball with radius r). Now, in the three cases one has $v = x_j + y_j$; furthermore, one obtains the inertial moment of each body by supposing it to be made entirely of the heavier metal with specific weight 1 and subtracting the inertial moment of the lighter volume multiplied by a , which part has the volume $u = y_j$ and the same shape <as the total body> in the cases 1 and 2, that of an ellipsoid in case 3; therefore the three inertial moments are proportional to

$$\begin{aligned}\varphi_1 &= (x_1 + y_1)^{5/3} - ay_1^{5/3}, \\ \varphi_2 &= (x_2 + y_2)^2 - ay_2^2, \\ \varphi_3 &= (x_3 + y_3)^{5/3} - ay_3^2(x_3 + y_3)^{-1/3}.\end{aligned}$$

Taking the derivatives with respect to x and y one gets

$$\begin{aligned}\frac{\partial \varphi_1}{\partial x_1} &= \frac{5}{3}(x_1 + y_1)^{2/3}, \\ \frac{\partial \varphi_1}{\partial y_1} &= \frac{5}{3}(x_1 + y_1)^{2/3} - \frac{5}{3}ay_1^{2/3},\end{aligned}$$

$$\begin{aligned}\frac{\partial \varphi_2}{\partial x_2} &= 2(x_2 + y_2), \\ \frac{\partial \varphi_2}{\partial y_2} &= 2(x_2 + y_2) - 2ay_2, \\ \frac{\partial \varphi_3}{\partial x_3} &= \frac{5}{3}(x_3 + y_3)^{2/3} + \frac{1}{3}ay_3^2(x_3 + y_3)^{-4/3}, \\ \frac{\partial \varphi_3}{\partial y_3} &= \frac{5}{3}(x_3 + y_3)^{2/3} + \frac{1}{3}ay_3^2(x_3 + y_3)^{-4/3} - 2ay_3(x_3 + y_3)^{-1/3},\end{aligned}$$

and the condition of proportionality yields the two equations

$$\begin{aligned}\frac{ay_1^{2/3}}{(x_1 + y_1)^{2/3}} &= \frac{ay_2}{x_2 + y_2} = \frac{6ay_3(x_3 + y_3)}{5(x_3 + y_3)^2 + ay_3^2} \\ &\left(= \frac{6ay_3(x_3 + y_3)^{-1/3}}{5(x_3 + y_3)^{2/3} + ay_3^2(x_3 + y_3)^{-4/3}} \right)\end{aligned}$$

which, together with $x_1 + x_2 + x_3 = 1 - \gamma$ and $y_1 + y_2 + y_3 = \gamma$, yield four equations in the six variables x_j and y_j , thus defining the (two-dimensional) ‘‘optimum’’ surface. Please note that, when one indicates with

$$\gamma_1 = \frac{y_1}{x_1 + y_1}, \quad \gamma_2 = \frac{y_2}{x_2 + y_2}, \quad \gamma_3 = \frac{y_3}{x_3 + y_3}$$

the volume percentages of the lighter metal in the three bodies, one can write the equations as functions of the γ_j as follows:

$$\gamma_1^{2/3} = \gamma_2 = \frac{6}{\frac{5}{\gamma_3} + a\gamma_3}.$$

Let us write

$$f_1(z) = z^{2/3}, \quad f_2(z) = z, \quad f_3(z) = \frac{6}{\frac{5}{z} + az}$$

and let us draw, in one figure, the graphs of the three functions in the relevant interval ($0 < z < 1$): the conclusion found says that γ_1, γ_2 and γ_3 must be the abscissa of the points of intersection of the three curves with one and the same horizontal. We observe that f_1 and f_3 are always larger than f_2 ; that f_3 , initially smaller than f_1 , intersects this function in the point $z = \xi^3$, where ξ is the unique real root between 0 and 1 of the equation $6\xi - a\xi^6 = 5$, and that it reaches the value 1 in the point $z = \eta$, where

$$\eta = \frac{1}{a} \left(3 - \sqrt{9 - 5a} \right).$$

For the ultimate solution of the problem it remains only to determine the volumes V_1, V_2 and V_3 compatible with each admissible term γ_1, γ_2 and γ_3 , and to that end it suffices to observe that $V_1 + V_2 + V_3 = (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = 1$, while $V_1\gamma_1 + V_2\gamma_2 + V_3\gamma_3 = y_1 + y_2 + y_3 = \gamma$; geometrically the condition means that the weighted average of the three points of intersection, taken with the weights V_1, V_2 and V_3 , is on the vertical $z = \gamma$. More directly, considering V_1, V_2 and V_3 as barycentric co-ordinates

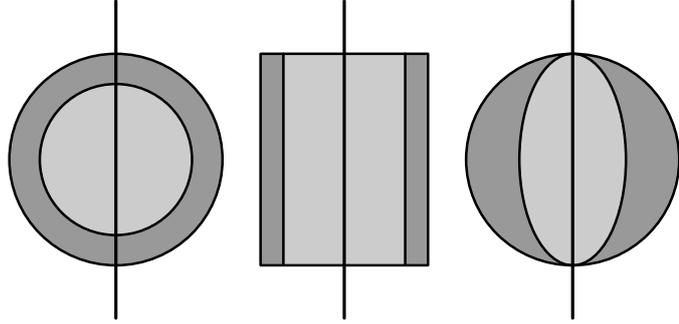


Figure B.1: Cross-sections of the three bodies 1), 2) and 3)

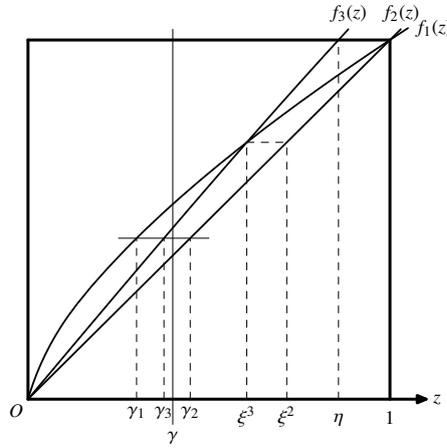


Figure B.2: The graphs of $f_1(z)$, $f_2(z)$ and $f_3(z)$

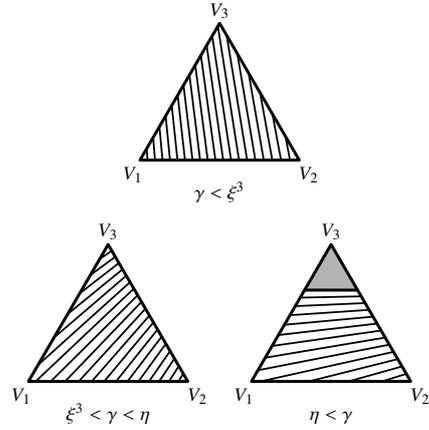


Figure B.3: The courses of the level lines $t = \text{constant}$ in the ternary diagram of V_1 , V_2 and V_3

in the familiar ternary diagram, the equation $V_1\gamma_1 + V_2\gamma_2 + V_3\gamma_3 = y_1 + y_2 + y_3 = \gamma$ represents a straight line, and, by varying the trio γ_1 , γ_2 and γ_3 , a family of straight lines. With $t = \gamma_2$ taken as parameter, one may write this equation explicitly in the following way:

$$V_1 t^{3/2} + V_2 t + V_3 \frac{1}{at} (3 - \sqrt{9 - 5at^2}) = \gamma,$$

given that one then obtains

$$\gamma_1 = t^{3/2}, \quad \gamma_2 = t, \quad \gamma_3 = \frac{1}{at} (3 - \sqrt{9 - 5at^2}).$$

For the behaviour of the family of straight lines in the triangle, and hence for the solution of our problem, it is necessary to distinguish three cases, according to $0 < \gamma < \xi^3$, $\xi^3 < \gamma < \eta$, or $\eta < \gamma < 1$ (plus the two borderline cases $\gamma = \xi^3$ and $\gamma = \eta$). The qualitative behaviour in the three cases is indicated in the figure, and we do not want to linger over it any longer; we only note that for $\gamma > \eta$ the points near Vertex 3 are excluded, due to the fact that the third body, in the allocations satisfying the “optimum” condition, can at most take the volume $V_3 = (1 - \gamma)/(1 - \eta)$.

by the cofactors of the latter matrix divided by the value Δ_h of the determinant, so that one can write the linear combination A_{hk} as

$$A_{hk} = \frac{1}{\Delta_h} \begin{vmatrix} \frac{\partial G_1}{\partial x_1^h} & \frac{\partial G_1}{\partial x_2^h} & \cdots & \frac{\partial G_1}{\partial x_m^h} \\ \dots & \dots & \dots & \dots \\ \frac{\partial G_{k-1}}{\partial x_1^h} & \frac{\partial G_{k-1}}{\partial x_2^h} & \cdots & \frac{\partial G_{k-1}}{\partial x_m^h} \\ \frac{\partial \varphi_h}{\partial x_1^h} & \frac{\partial \varphi_h}{\partial x_2^h} & \cdots & \frac{\partial \varphi_h}{\partial x_m^h} \\ \frac{\partial G_{k+1}}{\partial x_1^h} & \frac{\partial G_{k+1}}{\partial x_2^h} & \cdots & \frac{\partial G_{k+1}}{\partial x_m^h} \\ \dots & \dots & \dots & \dots \\ \frac{\partial G_m}{\partial x_1^h} & \frac{\partial G_m}{\partial x_2^h} & \cdots & \frac{\partial G_m}{\partial x_m^h} \end{vmatrix}.$$

To the elements thus transformed becomes applicable the conclusion of n. 3, which one can formulate in the following way: *From each of the n matrices G^h , the m determinants obtained by substituting the derivatives $\partial \varphi_h / \partial x_k^h$ for the first, second, . . . , m-th row follow; the n m-tuples appear proportional to one another, and the coefficient of proportionality between the two m-tuples with $h = h'$ and $h = h''$ is positive or negative according to the determinants $\Delta_{h'}$ and $\Delta_{h''}$ having the same or opposite signs.* Thus, all rows of the array

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{bmatrix}$$

appear proportional to each other (and with proportionality coefficients of the sign required by the preceding rule); hence the proportionality obtains, naturally, also between any two columns.

6. I hope that the notion of “optimum” as conceived and applied in mathematical economics will have become clarified with the treatment developed in this paper and the preceding one, and with the examples that illustrate it, chosen in various fields outside of economics. This seems to me particularly important because, like I have expounded in other works, this notion alone, together with the notion of “ophelimity” that is applied, should constitute the basis of economic theory. On this subject I plan to speak amply in other work; for now, it was important for me to clear the ground beforehand of any possible misunderstanding, doubt and distrust that the notion of “optimum” would maybe have allowed to persist if it would always have presented itself only in connection with economic problems, and would have been applied only to the notion of “ophelimity,” the meaning and importance of which, it seems, are not understood and appreciated by many at their proper value.

