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A fast and effective approximation for primes

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Abstract

A computational fast and effective approximation heuristic for the prime counting function $\pi(x)$ is presented. As shown in Figure 10, the presented approximation heuristic is on average as good as $\int_2^x 1/\log(t)\ dt\ -\ \frac{1}{2}\ \int_2^{\sqrt{x}} 1/\log(t)\ dt$ for x values up to 100,000. The main advantage of the heuristic is, that it does not require an integral to be evaluated. The main disadvantage of the heuristic is, that it gives bad approximations for $x \in \{1, 2, 3\}$. The heuristic is directly presented in mathematical and source code form (Matlab/Octave). Its effectiveness is visually illustrated by some plots.

The approximation heuristic for $\pi(x)$ is as follows:

$$\pi(x) \quad \approx \quad x \cdot (\frac{\frac{1}{H_x} + \gamma}{H_x - e\gamma} + \frac{1 - \gamma - \frac{1}{H_x}}{H_x - \pi\gamma}) - e$$

where

$$H_x = \sum_{i=1}^{x} \frac{1}{i} \qquad \text{(Harmonic number)}$$

and

$$\begin{aligned} \gamma &= 0.577... && \text{(Euler-Mascheroni constant)} \\ e &= 2.718... && \text{(Euler number)} \\ \pi &= 3.141... && \text{(Circle constant)} \end{aligned}$$

A computational effective implemenation is as follows (approximating H_x by $log(x) + \gamma$)

```
function main
for x=4:100
       % Calculate prime counting function Pi(x)
       Pi(x) = size(primes(x),2);

       % Calculate approximation heuristic A(x)
       H = log(x) + 0.577; h = 1/H;
       A(x) = x*((h+0.577)/(H-1.569)+(0.423-h)/(H-1.813))-2.718;
end
% Plot Pi(x) [BLUE] and its Approximation A(x) [RED]
plot(Pi);hold on; plot(A,'r'); legend('Pi(x)','A(x)','Location','SouthEast');hold off
```

Figure 1: Plot of $\pi(x)$ (blue) and $A(x)$ (red) for $x = 4, ..., 100$.

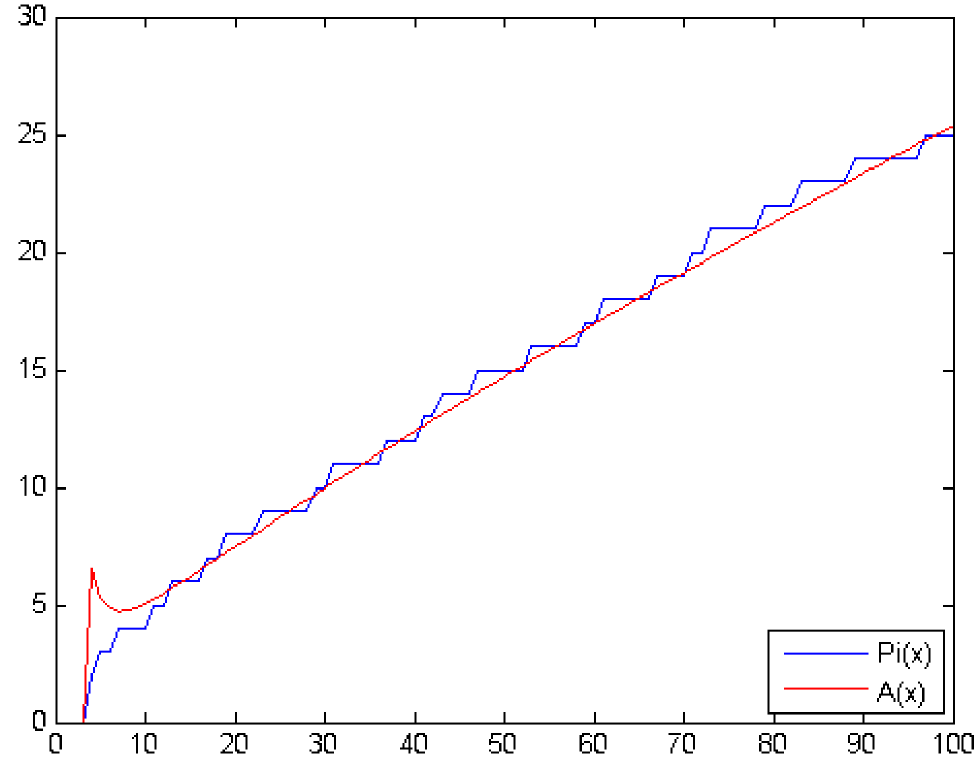


Figure 2: Plot of $\pi(x)$ (blue) and $A(x)$ (red) for $x = 4, ..., 500$.

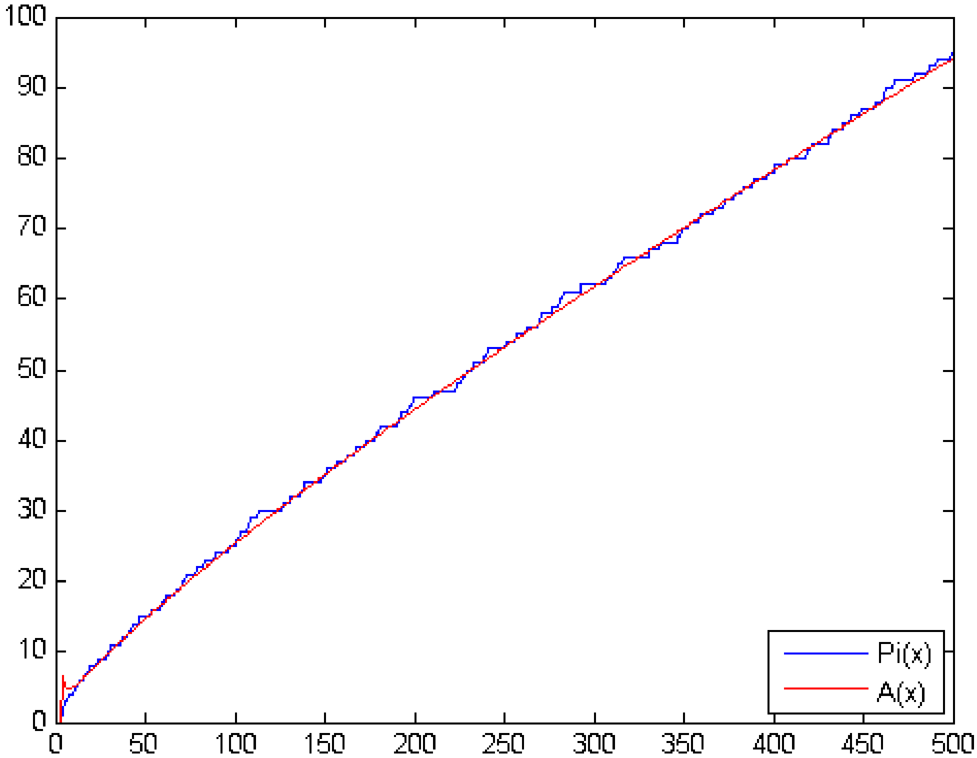


Figure 3: Plot of $\pi(x)$ (blue) and $A(x)$ (red) for $x = 4, ..., 1000$.

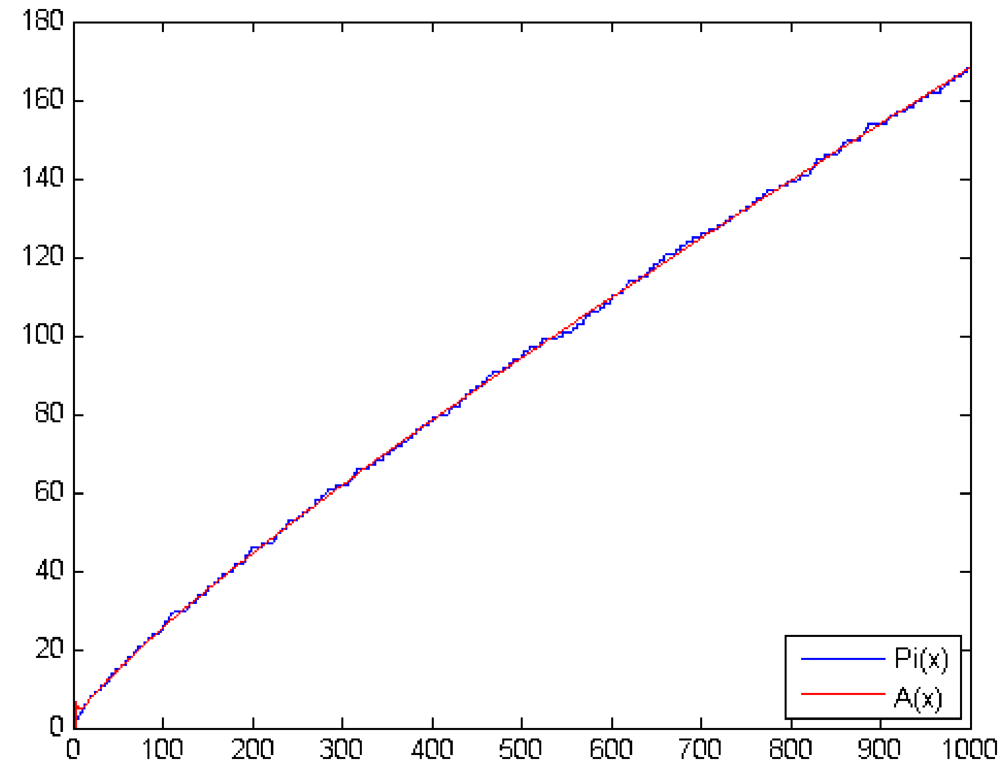


Figure 4: Plot of $\pi(x)$ (blue) and $A(x)$ (red) for $x = 4, ..., 10000$.

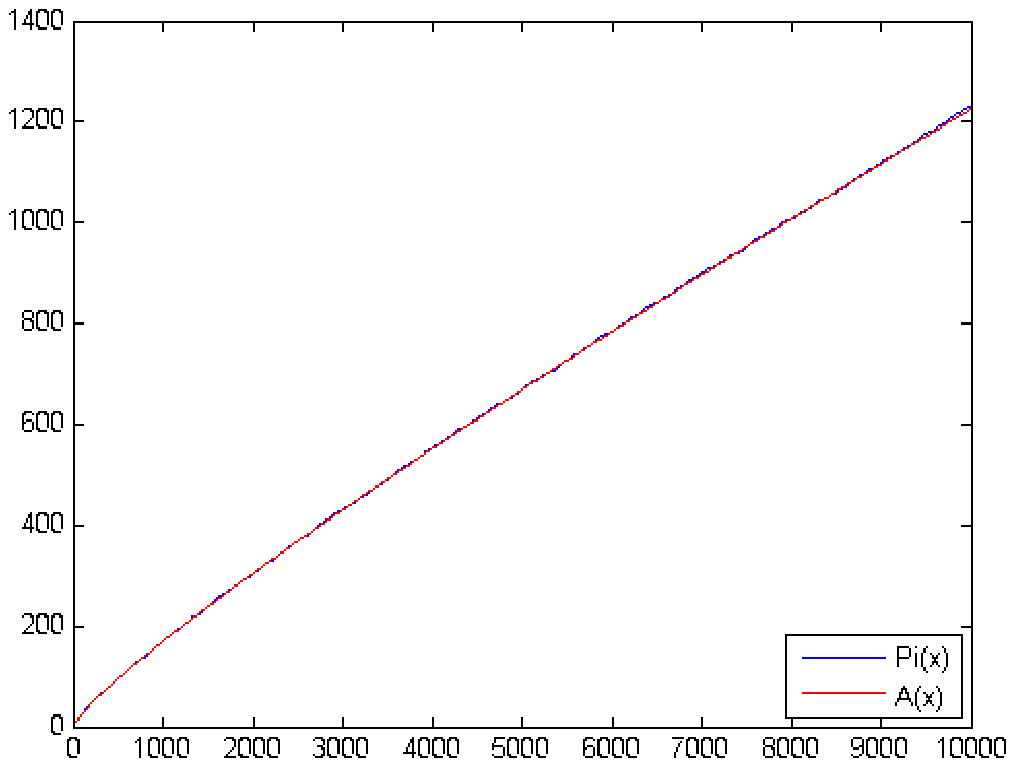


Figure 5: Plot of $\pi(x)$ (blue) and $A(x)$ (red) for $x = 4, ..., 100000$.

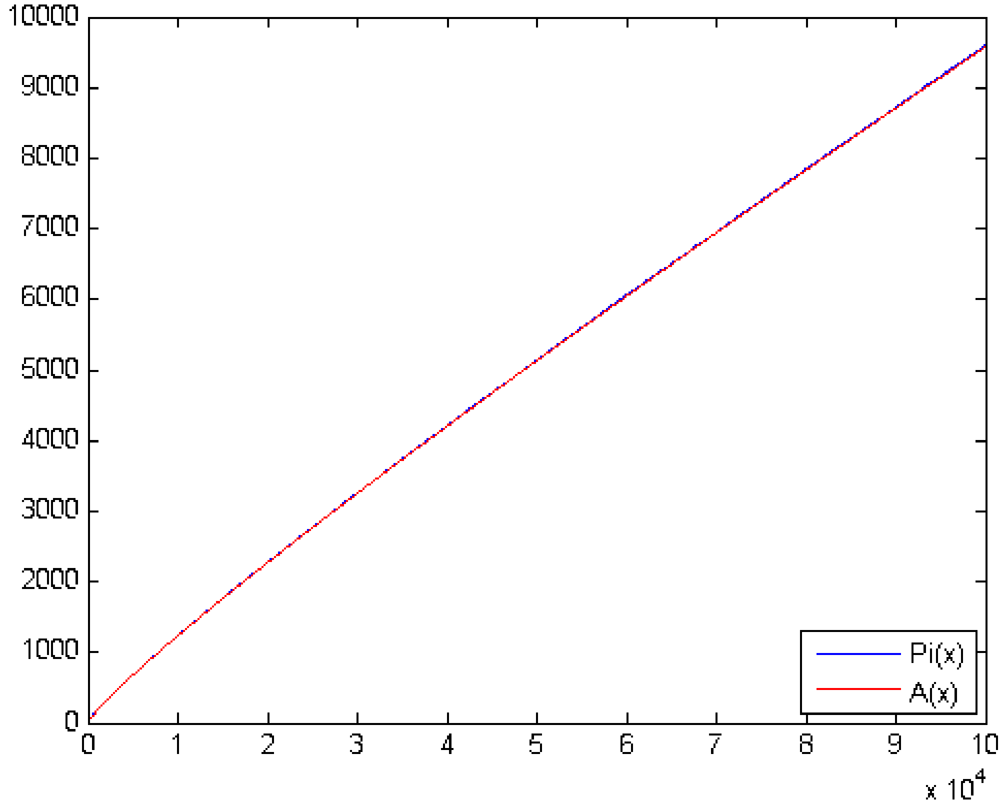


Figure 6: Comparison with $\int_2^x 1/\log(t)\ dt\ -\ \frac{1}{2}\ \int_2^{\sqrt{x}} 1/\log(t)\ dt$ for $x = 4, ..., 100$.

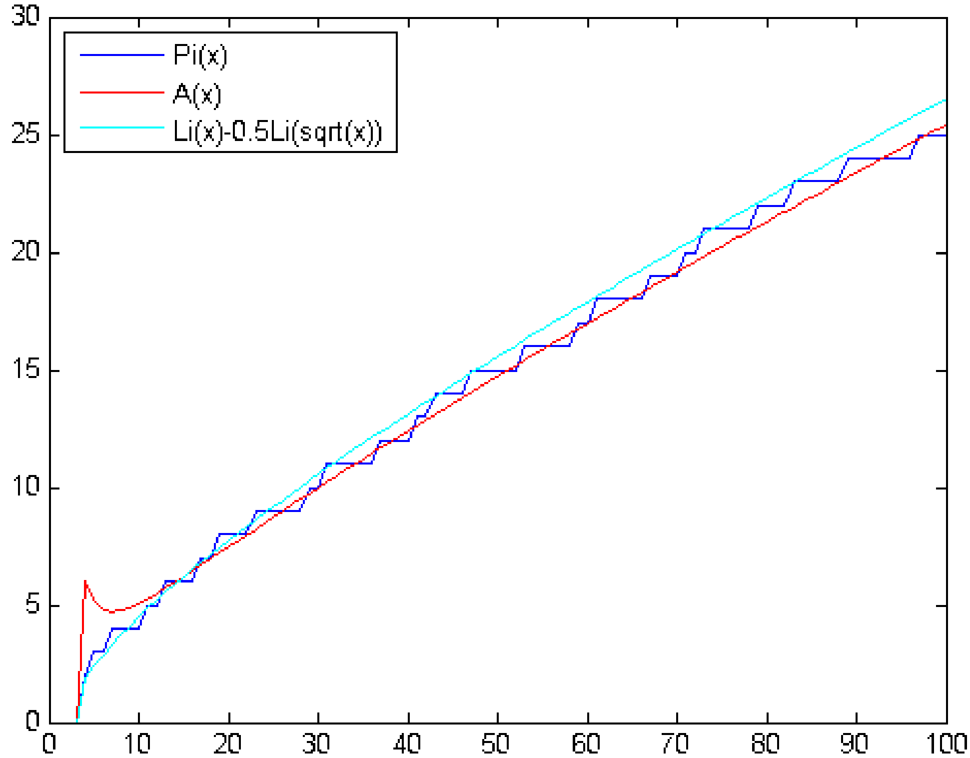


Figure 7: Comparison with $\int_2^x 1/\log(t)\ dt\ -\ \frac{1}{2}\ \int_2^{\sqrt{x}} 1/\log(t)\ dt$ for $x = 500, ..., 1000$.

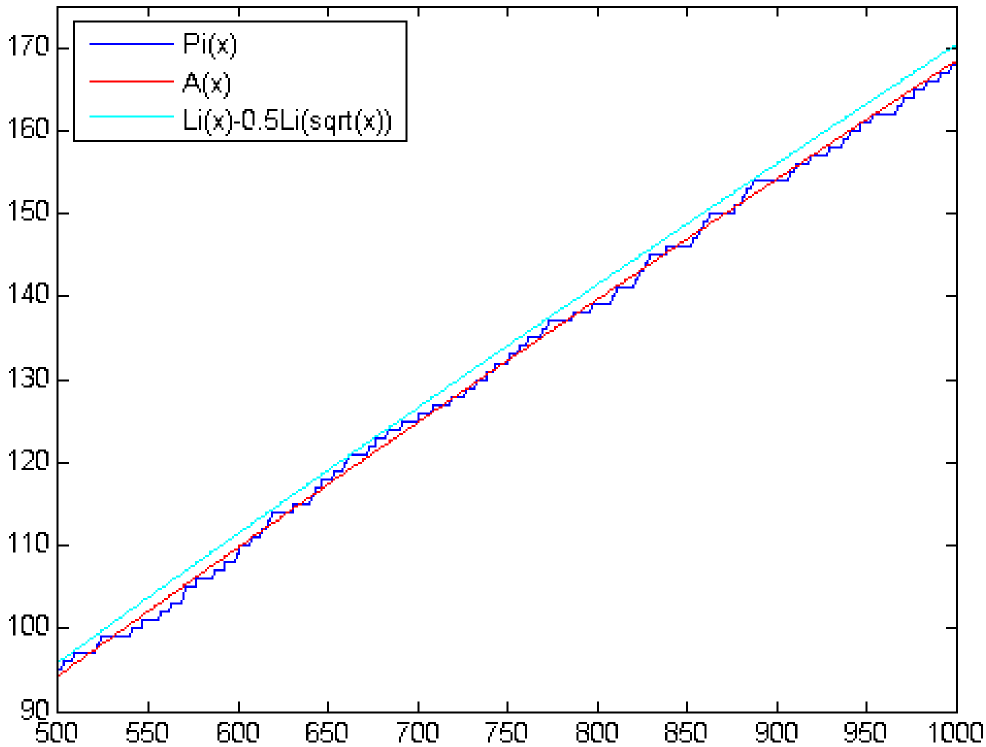


Figure 8: Comparison with $\int_2^x 1/\log(t)\ dt\ -\ \frac{1}{2}\ \int_2^{\sqrt{x}} 1/\log(t)\ dt$ for $x = 48000, ..., 50000$.

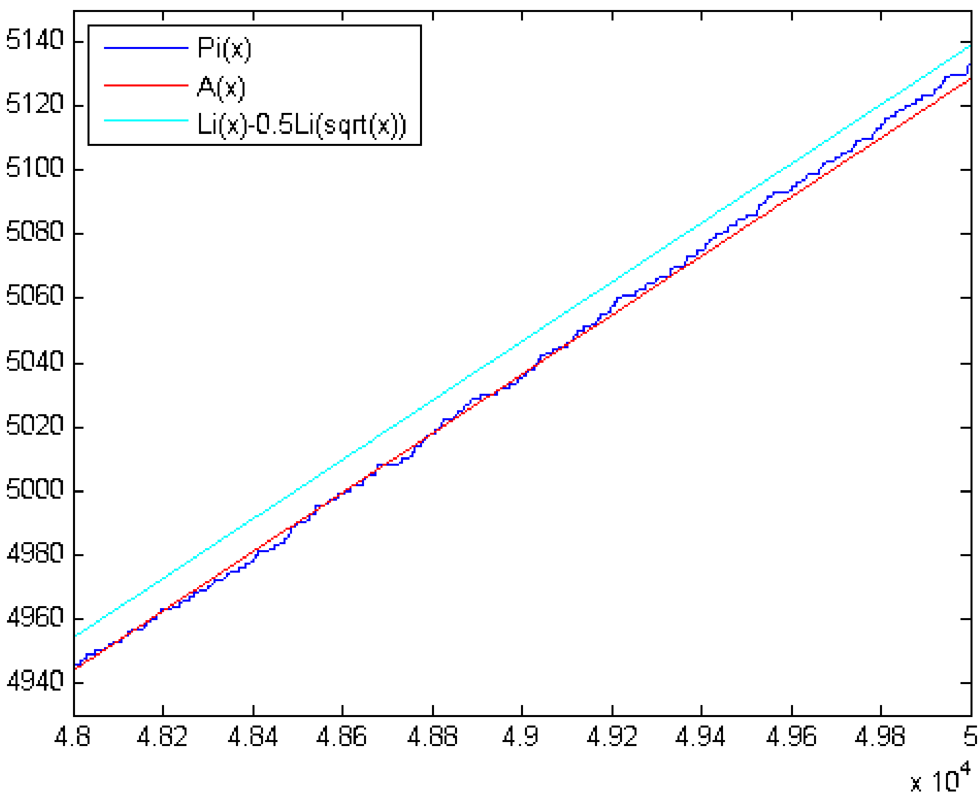


Figure 9: Comparison with $\int_2^x 1/\log(t)\,dt \;-\; \frac{1}{2}\int_2^{\sqrt{x}} 1/\log(t)\,dt$ for $x = 990000, \ldots, 1000000$.

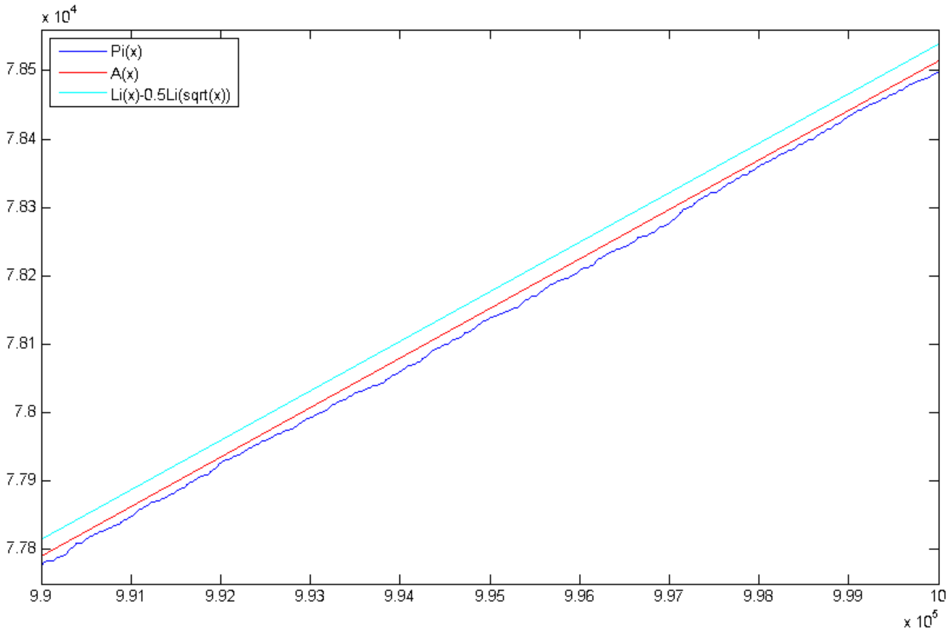
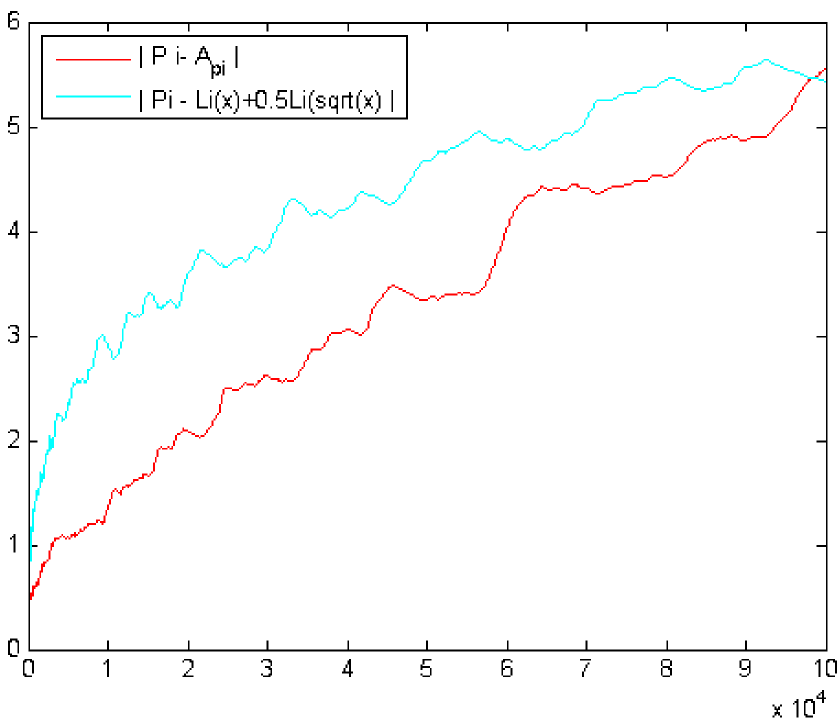


Figure 10: Average distance of $|\pi(x) - A(x)|$ and $|\pi(x) - Li(x) + 0.5Li(\sqrt{x})|$



Some more information on this approximation heuristic is available in [1].

References

[1] Schlueter, M.: *Some formulas and pattern.* Preprint (2013), available at http://vixra.org/abs/1307.0078