The continuum hypothesis

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In this paper we give a proof of the continuum hypothesis.

Theorem "generalized continuum hypothesis": For every transfinite cardinal number α, there is no cardinal number between α and 2^α.

Proof: Let **Card** be the class of transfinite cardinal numbers and let **Ord** be class of infinite initial ordinal numbers. We prove the equivalence of these two large categories showing that there exist a full and faithful functor $T\colon$ **Card**$\to$**Ord** such that each $\beta \in$ **Ord** is isomorphic to $T\alpha$ for some $\alpha \in$ **Card**.

Let $T\colon$ **Card**$\to$**Ord** be the functor which assigns to every transfinite cardinal number α the infinite initial ordinal number, $T\alpha$, of its equipotence class, and to every pair of transfinite cardinal numbers α and α' and to every arrow $f\colon \alpha \to \alpha'$ in **Card** the arrow $Tf\colon T\alpha \to T\alpha'$ in **Ord**. The functor T is well-defined by the definitions of initial ordinal number and of cardinal number, which assure that both categories, **Card** and **Ord**, are preorders. The functor T is order-preserving since **Card** and **Ord** are preorders, and by the definitions of initial ordinal number, of cardinal number and of T. The functor T is full because to every pair of transfinite cardinal numbers α and α' and to every arrow $g\colon T\alpha \to T\alpha'$ in **Ord** there is an arrow $f\colon \alpha \to \alpha'$ in **Card** such that $Tf = g$, because **Card** and **Ord** are linear orders and T is order-preserving. The functor T is faithful because to every pair of transfinite cardinal numbers α and α' and to every pair of arrows $f_1,\ f_2\colon \alpha \to \alpha'$ the equality $Tf_1 = Tf_2$ implies $f_1 = f_2$, for **Card** and **Ord** are preorders, and by definition of T. And, to each infinite initial ordinal number β, by definition of cardinal number, there is an order-preserving isomorphism of β onto the initial ordinal number, $T|\beta|$, of its transfinite cardinal number, $|\beta|$, that is, $\beta \cong T|\beta|$, thereby **Card** $\cong$ **Ord**.

Therefore, by the Gödel-Bernays-von Neumann axioms, by the fundamental theorem of cardinal arithmetic, because there is no initial ordinal number between ω and ω^ω, and because equivalent categories are equivalent theories, the equivalence between the categories **Card** and **Ord** proves that there is no cardinal number between the transfinite cardinal numbers $\aleph_0$ and $2^{\aleph_0}$, and that, in general, there is no cardinal number between any transfinite cardinal number α and 2^α. As a consequence, there exist no inaccessible cardinals. In fact, the class of transfinite cardinal numbers is isomorphic to ω, because the order-preserving function f of ω to **Card** which assigns to each finite ordinal number α the $\alpha-$th transfinite cardinal number is an isomorphism, which is unique by transfinite construction.

A theorem of universal algebra

Thus, not only does the theorem prove that the class of transfinite cardinal numbers is an infinite countable nondiscrete large category which is a closed complete and cocomplete semiring, with arrows, the polynomial maps and the exponential maps, that is an algebra by the action of the covariant exponential functor semiring e, itself, a functor algebra, but also, that the closed complete and cocomplete algebra of infinite initial ordinal numbers, **Ord**, is isomorphic to the closed complete and cocomplete algebra of transfinite cardinals, **Card**.

The theorem in categorical logic

In categorical logic, as all first order theories are infinite linear orders isomorphic to ω and have thereby cardinal number smaller than the cardinal of the continuum, the theorem proves that all higuer order theories are continuums, for they are partial orders isomorphic to countable products of first order theories.

The theorem in topos theory

Finally, in topos theory, this theorem not only proves that the category **Card** of cardinal numbers is a topos, but also that its topos of sheaves is the category $\mathbf{Sets}^{\mathbf{Card}^*}$, denoting the dual category of the category **Card** by $\mathbf{Card}^*$, of the set-valued contravariant functors on **Card** to **Sets** which assign to every cardinal number β its set of cardinal functions on β all of which turn out to be the cardinal continuous functions on the topology of the cardinal numbers.

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