THE GOLDBACH CONJECTURE

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ABSTRACT

This paper is a revision/expansion of the author's earlier published papers. It presents insights on the conjecture gained over a period of many years.

MSC: 11-XX (Number Theory)

Keywords: elimination; induction; contradiction; density of distribution; partitions; defining characteristic; extrapolation

INTRODUCTION

The expected mode of solving the Goldbach conjecture appears to be the utilization of advanced calculus or analysis, e.g., by the summation, or, integration, of the reciprocals involving directly or indirectly the primes to see whether they converge or diverge, in order to get a "feel" of the pattern of the distribution of the primes. But, such a method of solving the problem has evidently not succeeded so far. Some other approach or approaches could be more appropriate. This paper addresses the problem from several different angles, with reasoning backed by quantities that can be checked.

Theorem:- Every even number after 2 is the sum of 2 primes, as per the Goldbach conjecture.

Proof:-

Every even number after 2 is the sum of 2 odd numbers. Every odd number is either a prime which is odd or a composite - product of primes which are odd; notably, every prime with the exception of 2 is an odd number. Every even number after 2 is also a composite, but, a composite with at least 1 even prime factor, namely, 2, while the rest of its prime factors are odd, i.e., it is an even composite.

Therefore, every even number after 2 is the sum of 2 primes which are odd and/or the sum of 1 prime which is odd and 1 odd composite whose prime factors are odd and/or the sum of 2 odd composites whose prime factors are odd, besides being an even composite with at least 1 even prime factor, namely, 2, while the rest of its prime factors are odd.

Lemma:

By Euclid's proof, the primes are infinite; this implies that there would be an infinitude of sums of 2 primes as per the Goldbach conjecture. The even numbers, which are sums of 2 primes as per the conjecture, are also infinite. Thus, there are an infinite number of even numbers which are sums of 2 primes, both the even numbers and sums of 2 primes being infinite.

Corollary:
The odd numbers, which are either prime, every prime with the exception of 2 being an odd number, or composite (have prime factors which are odd), are infinite; this implies that there would be an infinite number of sums of 2 odd numbers, each of which is equal to an even number. Hence, as there is an infinitude of even numbers which are sums of 2 primes, as per the above lemma, and as all primes with the exception of 2 are odd numbers, there are an infinite number of even numbers which are sums of 2 odd numbers that are prime, all the even numbers, sums of 2 odd numbers and primes being infinite; i.e., every even number after 2 is also the sum of 2 odd numbers that are prime.

We thereby see the close interlink or relationship between the primes, even numbers and odd numbers, which are all infinite, which is significant.

The following are thus evident:

a) Every sum of 2 primes which are odd numbers is equal to an even number, as is below in consecutive order:

2 + 2 = 1 + 3 = **4**
3 + 3 = 1 + 5 = **6**
3 + 5 = 1 + 7 = **8**
5 + 5 = 3 + 7 = **10**
5 + 7 = 1 + 11 = **12**
7 + 7 = 3 + 11 = 1 + 13 = **14**
3 + 13 = 5 + 11 = **16**
7 + 11 = 5 + 13 = 1 + 17 = **18**
7 + 13 = 3 + 17 = 1 + 19 = **20**
11 + 11 = 3 + 19 = 5 + 17 = 11 + 11 = **22**
11 + 13 = 5 + 19 = 7 + 17 = 1 + 23 = **24**
13 + 13 = 3 + 23 = 7 + 19 = **26**
11 + 17 = 5 + 23 = **28**
13 + 17 = 11 + 19 = 7 + 23 = 1 + 29 = **30**
3 + 29 = 13 + 19 = 1 + 31 = **32**
17 + 17 = 3 + 31 = 5 + 29 = 11 + 23 = 17 + 17 = **34**
17 + 19 = 5 + 31 = 7 + 29 = 13 + 23 = **36**
19 + 19 = 7 + 31 = 1 + 37 = **38**
3 + 37 = 11 + 29 = 17 + 23 = **40**
19 + 23 = 5 + 37 = 11 + 31 = 13 + 29 = 1 + 41 = **42**
3 + 41 = 7 + 37 = 13 + 31 = 1 + 43 = **44**
23 + 23 = 3 + 43 = 5 + 41 = 17 + 29 = **46**
5 + 43 = 7 + 41 = 11 + 37 = 17 + 31 = 19 + 29 = 1 + 47 = **48**
3 + 47 = 7 + 43 = 13 + 37 = 19 + 31 = **50**
23 + 29 = 5 + 47 = 11 + 41 = **52**
7 + 47 = 11 + 43 = 13 + 41 = 17 + 37 = 23 + 31 = 1 + 53 = **54**
3 + 53 = 13 + 43 = 19 + 37 = **56**

29 + 29 = 5 + 53 = 11 + 47 = 17 + 41 = 29 + 29 = **58**
29 + 31 = 7 + 53 = 13 + 47 = 17 + 43 = 19 + 41 = 23 + 37 = 1 + 59 = **60**
31 + 31 = 3 + 59 = 19 + 43 = 1 + 61 = **62**
3 + 61 = 5 + 59 = 11 + 53 = 17 + 47 = 23 + 41 = **64**
5 + 61 = 7 + 59 = 13 + 53 = 19 + 47 = 23 + 43 = 29 + 37 = **66**
7 + 61 = 31 + 37 = 1 + 67 = **68**
3 + 67 = 11 + 59 = 17 + 53 = 23 + 47 = 29 + 41 = **70**
5 + 67 = 11 + 61 = 13 + 59 = 19 + 53 = 29 + 43 = 31 + 41 = 1 + 71 = **72**
37 + 37 = 3 + 71 = 7 + 67 = 13 + 61 = 31 + 43 = 37 + 37 = 1 + 73 = **74**
3 + 73 = 5 + 71 = 17 + 59 = 23 + 53 = 29 + 47 = **76**
37 + 41 = 5 + 73 = 7 + 71 = 11 + 67 = 31 + 47 = 37 + 41 = **78**
7 + 73 = 13 + 67 = 19 + 61 = 37 + 43 = 1 + 79 = **80**
41 + 41 = 3 + 79 = 11 + 71 = 23 + 59 = 29 + 53 = **82**
41 + 43 = 5 + 79 = 11 + 73 = 13 + 71 = 17 + 67 = 23 + 61 = 31 + 53 = 37 + 47 = 1+ 83 = **84**
43 + 43 = 3 + 83 = 7 + 79 = 13 + 73 = 19 + 67 = 43 + 43 = **86**
5 + 83 = 17 + 71 = 29 + 59 = 41 + 47 = **88**
7 + 83 = 11 + 79 = 17 + 73 = 19 + 71 = 23 + 67 = 29 + 61 = 31 + 59 = 37 + 53 = 43 + 47 = 1 + 89 = **90**
3 + 89 = 13 + 79 = 19 + 73 = 31 + 61 = 1 + 91 = **92**
47 + 47 = 5 + 89 = 11 + 83 = 23 + 71 = 41 + 53 = 47 + 47 = **94**
5 + 91 = 7 + 89 = 13 + 83 = 17 + 79 = 23 + 73 = 29 + 67 = 37 + 59 = 43 + 53 = **96**
7 + 91 = 19 + 79 = 31 + 67 = 37 + 61 = 1 + 97 = **98**
47 + 53 = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53 = **100**
5 + 97 = 11 + 91 = 13 + 89 = 19 + 83 = 23 + 79 = 29 + 73 = 31 + 71 = 41 + 61 = 43 + 59 = 1 + 101 = **102**

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b) Every sum of 1 prime which is an odd number & 1 odd composite which is the product of primes which are odd, is equal to the sum of 2 primes which are odd numbers, which are all each equal to an even number, as is below in consecutive order:

3 + **9** = 5 + 7 = 1 + 11 = **12**
5 + **9** = 3 + 11 = 7 + 7 = 1 + 13 = **14**
7 + **9** = 3 + 13 = 5 + 11 = **16**
3 + **15** = 7 + 11 = 5 + 13 = 1 + 17 = **18**
11 + **9** = 3 + 17 = 7 + 13 = 1 + 19 = **20**
13 + **9** = 3 + 19 = 5 + 17 = 11 + 11 = **22**
3 + **21** = 11 + 13 = 5 + 19 = 7 + 17 = 1 + 23 = **24**
17 + **9** = 3 + 23 = 7 + 19 = 13 + 13 = **26**
19 + **9** = 5 + 23 = 11 + 17 = **28**

5 + **25 =** 13 + 17 = 11 + 19 = 7 + 23 = 1 + 29 = **30**
23 + **9** = 3 + 29 = 13 + 19 = 1 + 31 = **32**
7 + **27 =** 17 + 17 = 3 + 31 = 5 + 29 = 11 + 23 = 17 + 17 = **34**
3 + **33 =** 17 + 19 = 5 + 31 = 7 + 29 = 13 + 23 = **36**
29 + **9** = 7 + 31 = 19 + 19 = 1 + 37 = **38**
31 + **9** = 3 + 37 = 11 + 29 = 17 + 23 = **40**
3 + **39** = 19 + 23 = 5 + 37 = 11 + 31 = 13 + 29 = 1 + 41 = **42**
5 + **39** = 3 + 41 = 7 + 37 = 13 + 31 = 1 + 43 = **44**
37 + **9** = 3 + 43 = 5 + 41 = 17 + 29 = 23 + 23 = **46**
3 + **45** = 5 + 43 = 7 + 41 = 11 + 37 = 17 + 31 = 19 + 29 = 1 + 47 = **48**
41 + **9** = 3 + 47 = 7 + 43 = 13 + 37 = 19 + 31 = **50**
43 + **9** = 5 + 47 = 11 + 41 =23 + 29 = **52**
5 + **49** = 7 + 47 = 11 + 43 = 13 + 41 = 17 + 37 = 23 + 31 = 1 + 53 = **54**
47 + **9** = 3 + 53 = 13 + 43 = 19 + 37 = **56**
3 + **55** = 29 + 29 = 5 + 53 = 11 + 47 = 17 + 41 = 29 + 29 = **58**
5 + **55** = 29 + 31 = 7 + 53 = 13 + 47 = 17 + 43 = 19 + 41 = 23 + 37 = 1 + 59 = **60**
53 + **9** = 3 + 59 = 19 + 43 = 31 + 31 = 1 + 61 = **62**
7 + **57** = 3 + 61 = 5 + 59 = 11 + 53 = 17 + 47 = 23 + 41 = **64**
11 + **55** = 5 + 61 = 7 + 59 = 13 + 53 = 19 + 47 = 23 + 43 = 29 + 37 = **66**
59 + **9** = 7 + 61 = 31 + 37 = 1 + 67 = **68**
61 + **9** = 3 + 67 = 11 + 59 = 17 + 53 = 23 + 47 = 29 + 41 = **70**
3 + **69** = 5 + 67 = 11 + 61 = 13 + 59 = 19 + 53 = 29 + 43 = 31 + 41 = 1 + 71 = **72**
5 + **69** = 37 + 37 = 3 + 71 = 7 + 67 = 13 + 61 = 31 + 43 = 37 + 37 = 1 + 73 = **74**
67 + **9** = 3 + 73 = 5 + 71 = 17 + 59 = 23 + 53 = 29 + 47 = **76**
3 + **75** = 37 + 41 = 5 + 73 = 7 + 71 = 11 + 67 = 31 + 47 = 37 + 41 = **78**
71 + **9** = 7 + 73 = 13 + 67 = 19 + 61 = 37 + 43 = 1 + 79 = **80**
73 + **9** = 3 + 79 = 11 + 71 = 23 + 59 = 29 + 53 = 41 + 41 = **82**
3 + **81** = 41 + 43 = 5 + 79 = 11 + 73 = 13 + 71 = 17 + 67 = 23 + 61 = 31 + 53 = 37 + 47 = 1+ 83 = **84**
5 + **81** = 43 + 43 = 3 + 83 = 7 + 79 = 13 + 73 = 19 + 67 = 43 + 43 = **86**
79 + **9** = 5 + 83 = 17 + 71 = 29 + 59 = 41 + 47 = **88**
3 + **87** = 7 + 83 = 11 + 79 = 17 + 73 = 19 + 71 = 23 + 67 = 29 + 61 = 31 + 59 = 37 + 53 = 43 + 47 = 1 + 89 = **90**
83 + **9** = 3 + 89 = 13 + 79 = 19 + 73 = 31 + 61 = 1 + 91 = **92**
7 + **87** = 47 + 47 = 5 + 89 = 11 + 83 = 23 + 71 = 41 + 53 = 47 + 47 = **94**
3 + **93** = 5 + 91 = 7 + 89 = 13 + 83 = 17 + 79 = 23 + 73 = 29 + 67 = 37 + 59 = 43 + 53 = **96**
89 + **9** = 7 + 91 = 19 + 79 = 31 + 67 = 37 + 61 = 1 + 97 = **98**
91 + **9** = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53 = **100**
3 + **99** = 5 + 97 = 11 + 91 = 13 + 89 = 19 + 83 = 23 + 79 = 29 + 73 = 31 + 71 = 41 + 61 = 43 + 59 = 1 + 101 = **102**

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c) Every sum of 2 odd composites which are products of primes which are odd, is equal to the sum of 2 primes which are odd numbers, which are all each equal to an even number, as is below in consecutive order:

9 + **9** = 5 + 13 = 7 + 11 = 1 + 17 = **18**
9 + **15** = 5 + 19 = 7 + 17 = 11 + 13 = 1 + 23 = **24**
15 + **15** = 7 + 23 = 11 + 19 = 13 + 17 = 1 + 29 = **30**
9 + **25** = 7 + 27 = 17 + 17 = 3 + 31 = 5 + 29 = 11 + 23 = 17 + 17 = **34**
15 + **21** = 5 + 31 = 7 + 29 = 13 + 23 = 17 + 19 = **36**
15 + **25** = 3 + 37 = 11 + 29 = 17 + 23 = **40**
21 + **21** = 5 + 37 = 11 + 31 = 13 + 29 = 19 + 23 = 1 + 41 = **42**
9 + **35** = 3 + 41 = 7 + 37 = 13 + 31 = 1 + 43 = **44**
21 + **25** = 3 + 43 = 5 + 41 = 17 + 29 = 23 + 23 = **46**
9 + **39** = 5 + 43 = 7 + 41 = 11 + 37 = 17 + 31 = 19 + 29 = 1 + 47 = **48**
25 + **25** = 3 + 47 = 7 + 43 = 13 + 37 = 19 + 31 = **50**
25 + **27** = 5 + 47 = 11 + 41 =23 + 29 = **52**
27 + **27** = 7 + 47 = 11 + 43 = 13 + 41 = 17 + 37 = 23 + 31 = 1 + 53 = **54**
21 + **35** = 3 + 53 = 13 + 43 = 19 + 37 = **56**
9 + **49** = 29 + 29 = 5 + 53 = 11 + 47 = 17 + 41 = 29 + 29 = **58**
27 + **33** = 7 + 53 = 13 + 47 = 17 + 43 = 19 + 41 = 23 + 37 = 29 + 31 = 1 + 59 = **60**
27 + **35** = 31 + 31 = 3 + 59 = 19 + 43 = 1 + 61 = **62**
9 + **55** = 3 + 61 = 5 + 59 = 11 + 53 = 17 + 47 = 23 + 41 = **64**
33 + **33** = 5 + 61 = 7 + 59 = 13 + 53 = 19 + 47 = 23 + 43 = 29 + 37 = **66**
33 + **35** = 7 + 61 = 31 + 37 = 1 + 67 = **68**
35 + **35** = 3 + 67 = 11 + 59 = 17 + 53 = 23 + 47 = 29 + 41 = **70**
9 + **63** = 5 + 67 = 11 + 61 = 13 + 59 = 19 + 53 = 29 + 43 = 31 + 41 = 1 + 71 = **72**
35 + **39** = 3 + 71 = 7 + 67 = 13 + 61 = 31 + 43 = 37 + 37 = 1 + 73 = **74**
21 + **55** = 3 + 73 = 5 + 71 = 17 + 59 = 23 + 53 = 29 + 47 = **76**
39 + **39** = 5 + 73 = 7 + 71 = 11 + 67 = 31 + 47 = 37 + 41 = **78**
15 + **65** = 7 + 73 = 13 + 67 = 19 + 61 = 37 + 43 = 1 + 79 = **80**
25 + **57** = 41 + 41 = 3 + 79 = 11 + 71 = 23 + 59 = 29 + 53 = **82**
39 + **45** = 5 + 79 = 11 + 73 = 13 + 71 = 17 + 67 = 23 + 61 = 31 + 53 = 37 + 47 = 41 + 43 = 1 + 83 = **84**
9 + **77** = 43 + 43 = 3 + 83 = 7 + 79 = 13 + 73 = 19 + 67 = 43 + 43 = **86**
25 + **63** = 5 + 83 = 17 + 71 = 29 + 59 = 41 + 47 = **88**
45 + **45** = 7 + 83 = 11 + 79 = 17 + 73 = 19 + 71 = 23 + 67 = 29 + 61 = 31 + 59 = 37 + 53 = 43 + 47 = 1 + 89 = **90**
15 + **77** = 3 + 89 = 13 + 79 = 19 + 73 = 31 + 61 = 1 + 91 = **92**
45 + **49** = 5 + 89 = 11 + 83 = 23 + 71 = 41 + 53 = 47 + 47 = **94**
9 + **87** = 5 + 91 = 7 + 89 = 13 + 83 = 17 + 79 = 23 + 73 = 29 + 67 = 37 + 59 = 43 + 53 = **96**
49 + **49** = 7 + 91 = 19 + 79 = 31 + 67 = 37 + 61 = 1 + 97 = **98**
49 + **51** = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53 = **100**
51 + **51** = 5 + 97 = 11 + 91 = 13 + 89 = 19 + 83 = 23 + 79 = 29 + 73 = 31 + 71 = 41 + 61 = 43 + 59 = 1 + 101 = **102**

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d) From (a), (b) & (c) above, we have the even numbers from 4 to 102 … composed as follows:

1) **4** = 2 + 2 = 1 + 3 (sum of 2 primes only)
2) **6** = 3 + 3 = 1 + 5 (sum of 2 primes only)
3) **8** = 3 + 5 = 1 + 7 (sum of 2 primes only)
4) **10** = 5 + 5 = 3 + 7 (sum of 2 primes only)
5) **12** = 5 + 7 = 1 + 11 = **3** + **9** (sum of 1 prime & 1 odd composite)
6) **14** = 3 + 11 = 7 + 7 = 1 + 13 = **5** + **9** (sum of 1 prime & 1 odd composite)
7) **16** = 3 + 13 = 5 + 11 = **7** + **9** (sum of 1 prime & 1 odd composite)
8) **18** = 5 + 13 = 7 + 11 = 1 + 17 = **3** + **15** (sum of 1 prime & 1 odd composite) = **9 + 9** (sum of 2 odd composites)
9) **20** = 3 + 17 = 7 + 13 = 1 + 19 = **11** + **9** (sum of 1 prime & 1 odd composite)
10) **22** = 3 + 19 = 5 + 17 = 11 + 11 = **13** + **9** (sum of 1 prime & 1 odd composite)
11) **24** = 5 + 19 = 7 + 17 = 11 + 13 = 1 + 23 = **3** + **21** (sum of 1 prime & 1 odd composite) = **9** + **15** (sum of 2 odd composites)
12) **26** = 3 + 23 = 7 + 19 = 13 + 13 = **17** + **9** (sum of 1 prime & 1 odd composite)
13) **28** = 5 + 23 = 11 + 17 = **19** + **9** (sum of 1 prime & 1 odd composite)
14) **30** = 7 + 23 = 11 + 19 = 13 + 17 = 1 + 29 = **5** + **25** (sum of 1 prime & 1 odd composite) **=** **15** + **15** (sum of 2 odd composites)
15) **32** = 3 + 29 = 13 + 19 = 1 + 31 = **23** + **9** (sum of 1 prime & 1 odd composite)
16) **34** = 17 + 17 = 3 + 31 = 5 + 29 = 11 + 23 = 17 + 17 = **7** + **27** (sum of 1 prime & 1 odd composite) = **9** + **25** (sum of 2 odd composites)
17) **36** = 5 + 31 = 7 + 29 = 13 + 23 = 17 + 19 = **3** + **33** (sum of 1 prime & 1 odd composite) = **15** + **21** (sum of 2 odd composites)
18) **38** = 7 + 31 = 19 + 19 = 1 + 37 = **29** + **9** (sum of 1 prime & 1 odd composite)
19) **40** = 3 + 37 = 11 + 29 = 17 + 23 = **31** + **9** (sum of 1 prime & 1 odd composite) = **15** + **25** (sum of 2 odd composites)
20) **42** = 5 + 37 = 11 + 31 = 13 + 29 = 19 + 23 = 1 + 41 = **3** + **39** (sum of 1 prime & 1 odd composite) = **21** + **21** (sum of 2 odd composites)
21) **44** = 3 + 41 = 7 + 37 = 13 + 31 = 1 + 43 = **5** + **39** (sum of 1 prime & 1 odd composite) = **9** + **35** (sum of 2 odd composites)
22) **46** = 3 + 43 = 5 + 41 = 17 + 29 = 23 + 23 = **37** + **9** (sum of 1 prime & 1 odd composite) = **21** + **25** (sum of 2 odd composites)
23) **48** = 5 + 43 = 7 + 41 = 11 + 37 = 17 + 31 = 19 + 29 = 1 + 47 = **3** + **45** (sum of 1 prime & 1 odd composite) = **9** + **39** (sum of 2 odd composites)
24) **50** = 3 + 47 = 7 + 43 = 13 + 37 = 19 + 31 = **41** + **9** (sum of 1 prime & 1 odd composite) = **25** + **25** (sum of 2 odd composites)
25) **52** = 5 + 47 = 11 + 41 =23 + 29 = = **43** + **9** (sum of 1 prime & 1 odd composite) = **25** + **27** (sum of 2 odd composites)
26) **54** = 7 + 47 = 11 + 43 = 13 + 41 = 17 + 37 = 23 + 31 = 1 + 53 = **5** + **49** (sum of 1 prime & 1 odd composite) = **27** + **27** (sum of 2 odd composites)

27) **56** = 3 + 53 = 13 + 43 = 19 + 37 = **47** + **9** (sum of 1 prime & 1 odd composite) = **21** + **35** (sum of 2 odd composites)
28) **58** = 29 + 29 = 5 + 53 = 11 + 47 = 17 + 41 = 29 + 29 = **3** + **55** (sum of 1 prime & 1 odd composite) = **9** + **49** (sum of 2 odd composites)
29) **60** = 7 + 53 = 13 + 47 = 17 + 43 = 19 + 41 = 23 + 37 = 29 + 31 = 1 + 59 = **5** + **55** (sum of 1 prime & 1 odd composite) = **27** + **33** (sum of 2 odd composites)
30) **62** = 3 + 59 = 19 + 43 = 31 + 31 = 1 + 61 = **53** + **9** (sum of 1 prime & 1 odd composite) = **27** + **35** (sum of 2 odd composites)
31) **64** = 3 + 61 = 5 + 59 = 11 + 53 = 17 + 47 = 23 + 41 = **7** + **57** (sum of 1 prime & 1 odd composite) = **9** + **55** (sum of 2 odd composites)
32) **66** = 5 + 61 = 7 + 59 = 13 + 53 = 19 + 47 = 23 + 43 = 29 + 37 = **11** + **55** (sum of 1 prime & 1 odd composite) = **33** + **33** (sum of 2 odd composites)
33) **68** = 7 + 61 = 31 + 37 = 1 + 67 = **59** + **9** (sum of 1 prime & 1 odd composite) = **33** + **35** (sum of 2 odd composites)
34) **70** = 3 + 67 = 11 + 59 = 17 + 53 = 23 + 47 = 29 + 41 = **61** + **9** (sum of 1 prime & 1 odd composite) = **35** + **35** (sum of 2 odd composites)
35) **72** = 5 + 67 = 11 + 61 = 13 + 59 = 19 + 53 = 29 + 43 = 31 + 41 = 1 + 71 = **3** + **69** (sum of 1 prime & 1 odd composite) = **9 + 63** (sum of 2 odd composites)
36) **74** = 3 + 71 = 7 + 67 = 13 + 61 = 31 + 43 = 37 + 37 = 1 + 73 = **5** + **69** (sum of 1 prime & 1 odd composite) = **35** + **39** (sum of 2 odd composites)
37) **76** = 3 + 73 = 5 + 71 = 17 + 59 = 23 + 53 = 29 + 47 = **67** + **9** (sum of 1 prime & 1 odd composite) = **21** + **55** (sum of 2 odd composites)
38) **78** = 5 + 73 = 7 + 71 = 11 + 67 = 31 + 47 = 37 + 41 = **3** + **75** (sum of 1 prime & 1 odd composite) = **39** + **39** (sum of 2 odd composites)
39) **80** = 7 + 73 = 13 + 67 = 19 + 61 = 37 + 43 = 1 + 79 = **71** + **9** (sum of 1 prime & 1 odd composite) = **15** + **65** (sum of 2 odd composites)
40) **82** = 3 + 79 = 11 + 71 = 23 + 59 = 29 + 53 = 41 + 41 = **73** + **9** (sum of 1 prime & 1 odd composite) = **25** + **57** (sum of 2 odd composites)
41) **84** = 5 + 79 = 11 + 73 = 13 + 71 = 17 + 67 = 23 + 61 = 31 + 53 = 37 + 47 = 41 + 43 = 1 + 83 = **3** + **81** (sum of 1 prime & 1 odd composite) = **39** + **45** (sum of 2 odd composites)
42) **86** = 43 + 43 = 3 + 83 = 7 + 79 = 13 + 73 = 19 + 67 = 43 + 43 = **5** + **81** (sum of 1 prime & 1 odd composite) = **9** + **77** (sum of 2 odd composites)
43) **88** = 5 + 83 = 17 + 71 = 29 + 59 = 41 + 47 = **79** + **9** (sum of 1 prime & 1 odd composite) = **25** + **63** (sum of 2 odd composites)
44) **90** = 7 + 83 = 11 + 79 = 17 + 73 = 19 + 71 = 23 + 67 = 29 + 61 = 31 + 59 = 37 + 53 = 43 + 47 = 1 + 89 = **3** + **87** (sum of 1 prime & 1 odd composite) = **45** + **45** (sum of 2 odd composites)
45) **92** = 3 + 89 = 13 + 79 = 19 + 73 = 31 + 61 = 1 + 91 = **83** + **9** (sum of 1 prime & 1 odd composite) = **15** + **77** (sum of 2 odd composites)
46) **94** = 5 + 89 = 11 + 83 = 23 + 71 = 41 + 53 = 47 + 47 = **7** + **87** (sum of 1 prime & 1 odd composite) = **45 + 49** (sum of 2 odd composites)
47) **96** = 5 + 91 = 7 + 89 = 13 + 83 = 17 + 79 = 23 + 73 = 29 + 67 = 37 + 59 = 43 + 53 = **3** + **93** (sum of 1 prime & 1 odd composite) = **9** + **87** (sum of 2 odd composites)
48) **98** = 7 + 91 = 19 + 79 = 31 + 67 = 37 + 61 = 1 + 97 = **89** + **9** (sum of 1 prime

& 1 odd composite) = **49** + **49** (sum of 2 odd composites)

49) **100** = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53 = **91** + **9** (sum of 1 prime & 1 odd composite) = **49** + **51** (sum of 2 odd composites)

50) **102** = 5 + 97 = 11 + 91 = 13 + 89 = 19 + 83 = 23 + 79 = 29 + 73 = 31 + 71 = 41 + 61 = 43 + 59 = 1 + 101 = **3** + **99** (sum of 1 prime & 1 odd composite) = **51** + **51** (sum of 2 odd composites)

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(The above is only a partial or incomplete listing of sums of 1 prime & 1 odd composite, and, sums of 2 odd composites, each of which is equal to the sum of 2 primes as well as an even number. For example, in the list of compositions for the even numbers 4 to 102 … above, in Item (48), we could also have other "combinations" such as: **98** = 7 + 91 = 19 + 79 = 31 + 67 = 37 + 61 = 1 + 97 = <u>**25 + 73** (sum of 1 prime & 1 odd composite)</u> = <u>**21 + 77** (sum of 2 odd composites), et al.</u>, in Item (49), we could also have other "combinations" such as: **100** = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53 = <u>**31 + 69** (sum of 1 prime & 1 odd composite)</u> = <u>**45 + 55** (sum of 2 odd composites), et al.</u>, and, in Item (50), we could also have other "combinations" such as: **102** = 5 + 97 = 11 + 91 = 13 + 89 = 19 + 83 = 23 + 79 = 29 + 73 = 31 + 71 = 41 + 61 = 43 + 59 = 1 + 101 = <u>**17 + 85** (sum of 1 prime & 1 odd composite)</u> = <u>**21 + 81** (sum of 2 odd composites), et al.</u>. That is, there are more "combinations" than those shown in the above listing.)

In (d) above, in the list of compositions for the 50 consecutive even numbers 4 to 102 …, the even numbers 4, 6, 8 and 10 are only formed through the summing of 2 primes and not at all through the summing of 1 prime and 1 odd composite, or, the summing of 2 odd composites, which are impossibilities here. These sums of 2 primes are present (always present) throughout the whole list of compositions, from 4 right through to 102, while this is not the case for the sums of 1 prime and 1 odd composite, and, the sums of 2 odd composites.

We reason here by the process of elimination, through analyzing the information in (d) above which pertains to the compositions of the 50 consecutive even numbers 4 to 102 … taken from the infinite list of even numbers. We stated at the beginning the following about the even numbers after 2:-

<u>Firstly, every even number after 2 is:</u>

A) The sum of 2 odd numbers.
 (Every odd number is either a prime which is odd or a composite - product of primes which are odd.
 Notably, every prime with the exception of 2 is an odd number.)

<u>Secondly, every even number after 2 is also (the below-mentioned is the logical consequence of (A) above):</u>

1) The sum of 2 primes which are odd.
2) And/or the sum of 1 prime which is odd and 1 odd composite whose prime factors

are odd.
3) And/or the sum of 2 odd composites whose prime factors are odd.

Evidently, at least 1 of (1), (2) & (3) above has to be the "atom" or building-block of the even numbers. In (d) above, we observe the following:-

i) All the 50 consecutive even numbers 4 to 102 … in (d) above taken from the infinite list of even numbers are sums of 2 primes.
ii) It is impossible for each of the even numbers 4, 6, 8 & 10 in (d) above to be the sum of 1 prime which is odd and 1 odd composite whose prime factors are odd.
iii) It is impossible for each of the even numbers 4, 6, 8, 10, 12, 14, 16, 20, 22, 26, 28, 32 & 38 in (d) above to be the sum of 2 odd composites whose prime factors are odd.

It is evident from (i), (ii) & (iii) above that neither (2) nor (3) can be the "atom" or building-block of the even numbers since they are "incomplete". As (1) - the sum of 2 primes which are odd - is "complete", i.e., always present in the 50 consecutive even numbers 4 to 102 … in (d) above, unlike (2) & (3), it evidently is the "atom" or building-block of the even numbers. That is, every even number after 2 is evidently the sum of 2 primes which are odd. In fact, a distributed computer search completed in 2008 at the University of Aveiro, Portugal, had verified this for all even numbers up to 12×10^{17}, which is not a small list (it is in fact a long, impressive list, obtainable only with the help of modern computer technology). Definitely, due respectively to (ii) & (iii) above, we cannot say that every even number after 2 is the sum of 1 prime which is odd and 1 odd composite whose prime factors are odd, or, every even number after 2 is the sum of 2 odd composites whose prime factors are odd.

By the above lemma and corollary, the infinitudes of the primes, even numbers and odd numbers indeed imply that there are an infinite number of sums of 2 primes which are odd numbers, which are each equal to an even number. As the sums of 2 primes which are odd numbers are evidently the "atoms" or building-blocks of the even numbers, it also implies that they are infinite, since the even numbers are infinite.

Hypothetically, if on the other hand just 1 of the 3 items stated above, primes, even numbers and odd numbers, were finite, the above-said sums of 2 primes which are odd numbers, each of which is equal to an even number, would be finite. The primes, even numbers and odd numbers are evidently intricately linked, with the primes playing the part of building-blocks of both the even and odd numbers through various "combinations" as is described below. However, as the primes, even numbers and odd numbers are intricately linked, the finiteness (or, infinity) of any 1 of them implies the finiteness (or, infinity) of the other 2, and vice versa. These 3 items are evidently "close comrades-in-arm" working together to give special meaning to the integers. As these 3 are all infinite, it indeed implies that there is an infinitude of even numbers which are infinitely the sums of 2 primes that are odd and infinite.

The prime numbers are evidently the atoms or building-blocks of the integers. The integers are either primes (not divisible by other integers except 1) or composites (divisible by other integers, e.g., the prime numbers), and, even (the sums of 2 primes as conjectured by Goldbach) or odd (primes, or, composites whereby they are divisible by prime factors). Therefore, to determine whether the conjecture that every even number (except the number 2) is the sum of 2 primes is true, it would be appropriate to analyze the evident atoms or building-blocks of the even numbers, viz., the prime numbers. For the solution to this conjecture we note that the primes (vide Euclid's proof) and the even numbers are infinite, which implies that this conjecture should be true.

We here analyze the "behavior" of the first 2,400 consecutive prime numbers (divided into 12 batches of consecutive primes, each subsequent batch with an increment of 200 primes), leaving out 2 (because it is an even prime) and commencing with 3, which is the $2^{nd.}$ consecutive prime, the latter to be the first prime in our list of 2,400 consecutive primes (3 to 21,391), as follows:-

(1) <u>200 Consecutive Primes From 3 To 1,229</u>
- (a) Even numbers (obtained by summing of 2 primes) = 6 to 2,458
- (b) No. of even numbers = 1,227
- (c) No. of primes = 200
- (d) Average no. of even numbers "generated" by each of these 200 consecutive primes = 1,227 ÷ 200 = **6.14**
- (e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, …… et al.) for these 200 primes = 200 x 200 = 40,000
- (f) Average no. of summings of 2 primes/permutations for each of the 1,227 even numbers = 40,000 ÷ 1,227 = **32.60**

(2) <u>400 Consecutive Primes From 3 To 2,749</u>
- (a) Even numbers (obtained by summing of 2 primes) = 6 to 5,498
- (b) No. of even numbers = 2,747
- (c) No. of primes = 400
- (d) Average no. of even numbers "generated" by each of these 400 consecutive primes = 2,747 ÷ 400 = **6.87**
- (e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, …… et al.) for these 400 primes = 400 x 400 = 160,000
- (f) Average no. of summings of 2 primes/permutations for each of the 2,747 even numbers = 160,000 ÷ 2,747 = **58.25**

(3) <u>600 Consecutive Primes From 3 To 4,421</u>
- (a) Even numbers (obtained by summing of 2 primes) = 6 to 8,842
- (b) No. of even numbers = 4,419
- (c) No. of primes = 600
- (d) Average no. of even numbers "generated" by each of these 600 consecutive primes = 4,419 ÷ 600 = **7.37**
- (e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, ……

et al.) for these 600 primes = 600 x 600 = 360,000

(f) Average no. of summings of 2 primes/permutations for each of the 4,419 even numbers = 360,000 ÷ 4,419 = **81.47**

(4) <u>800 Consecutive Primes From 3 To 6,143</u>

(a) Even numbers (obtained by summing of 2 primes) = 6 to 12,286

(b) No. of even numbers = 6,141

(c) No. of primes = 800

(d) Average no. of even numbers "generated" by each of these 800 consecutive primes = 6,141 ÷ 800 = **7.68**

(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, …… et al.) for these 800 primes = 800 x 800 = 640,000

(f) Average no. of summings of 2 primes/permutations for each of the 6,141 even numbers = 640,000 ÷ 6,141 = **104.22**

(5) <u>1,000 Consecutive Primes From 3 To 7,927</u>

(a) Even numbers (obtained by summing of 2 primes) = 6 to 15,854

(b) No. of even numbers = 7,925

(c) No. of primes = 1,000

(d) Average no. of even numbers "generated" by each of these 1,000 consecutive primes = 7,925 ÷ 1,000 = **7.93**

(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, …… et al.) for these 1,000 primes = 1,000 x 1,000 = 1,000,000

(f) Average no. of summings of 2 primes/permutations for each of the 7,925 even numbers = 1,000,000 ÷ 7,925 = **126.18**

(6) <u>1,200 Consecutive Primes From 3 To 9,739</u>

(a) Even numbers (obtained by summing of 2 primes) = 6 to 19,478

(b) No. of even numbers = 9,737

(c) No. of primes = 1,200

(d) Average no. of even numbers "generated" by each of these 1,200 consecutive primes = 9,737 ÷ 1,200 = **8.11**

(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, …… et al.) for these 1,200 primes = 1,200 x 1,200 = 1,440,000

(f) Average no. of summings of 2 primes/permutations for each of the 9,737 even numbers = 1,440,000 ÷ 9,737 = **147.89**

(7) <u>1,400 Consecutive Primes From 3 To 11,677</u>

(a) Even numbers (obtained by summing of 2 primes) = 6 to 23,354

(b) No. of even numbers = 11,675

(c) No. of primes = 1,400
(d) Average no. of even numbers "generated" by each of these 1,400 consecutive primes = 11,675 ÷ 1,400 = **8.34**
(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, …… et al.) for these 1,400 primes = 1,400 x 1,400 = 1,960,000
(f) Average no. of summings of 2 primes/permutations for each of the 11,675 even numbers = 1,960,000 ÷ 11,675 = **167.88**

(8) <u>1,600 Consecutive Primes From 3 To 13,513</u>
(a) Even numbers (obtained by summing of 2 primes) = 6 to 27,026
(b) No. of even numbers = 13,511
(c) No. of primes = 1,600
(d) Average no. of even numbers "generated" by each of these 1,600 consecutive primes = 13,511 ÷ 1,600 = **8.44**
(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, …… et al.) for these 1,600 primes = 1,600 x 1,600 = 2,560,000
(f) Average no. of summings of 2 primes/permutations for each of the 13,511 even numbers = 2,560,000 ÷ 13,511 = **189.48**

(9) <u>1,800 Consecutive Primes From 3 To 15,413</u>
(a) Even numbers (obtained by summing of 2 primes) = 6 to 30,826
(b) No. of even numbers = 15,411
(c) No. of primes = 1,800
(d) Average no. of even numbers "generated" by each of these 1,800 consecutive primes = 15,411 ÷ 1,800 = **8.56**
(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, …… et al.) for these 1,800 primes = 1,800 x 1,800 = 3,240,000
(f) Average no. of summings of 2 primes/permutations for each of the 15,411 even numbers = 3,240,000 ÷ 15,411 = **210.24**

(10) <u>2,000 Consecutive Primes From 3 To 17,393</u>
(a) Even numbers (obtained by summing of 2 primes) = 6 to 34,786
(b) No. of even numbers = 17,391
(c) No. of primes = 2,000
(d) Average no. of even numbers "generated" by each of these 2,000 consecutive primes = 17,391 ÷ 2,000 = **8.70**
(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, …… et al.) for these 2,000 primes = 2,000 x 2,000 = 4,000,000
(f) Average no. of summings of 2 primes/permutations for each of the 17,391 even numbers = 4,000,000 ÷ 17,391 = **230.00**

(11) <u>2,200 Consecutive Primes From 3 To 19,427</u>

(a) Even numbers (obtained by summing of 2 primes) = 6 to 38,854
(b) No. of even numbers = 19,425
(c) No. of primes = 2,200
(d) Average no. of even numbers "generated" by each of these 2,200 consecutive primes = 19,425 ÷ 2,200 = **8.83**
(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, …… et al.) for these 2,200 primes = 2,200 x 2,200 = 4,840,000
(f) Average no. of summings of 2 primes/permutations for each of the 19,425 even numbers = 4,840,000 ÷ 19,425 = **249.16**

(12) <u>2,400 Consecutive Primes From 3 To 21,391</u>

(a) Even numbers (obtained by summing of 2 primes) = 6 to 42,782
(b) No. of even numbers = 21,389
(c) No. of primes = 2,400
(d) Average no. of even numbers "generated" by each of these 2,400 consecutive primes = 21,389 ÷ 2,400 = **8.91**
(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, …… et al.) for these 2,400 primes = 2,400 x 2,400 = 5,760,000
(f) Average no. of summings of 2 primes/permutations for each of the 21,389 even numbers = 5,760,000 ÷ 21,389 = **269.30**

There would evidently be more and more profuse repetitions and overlaps of the even numbers "generated" by the primes the higher up the infinite list of prime numbers we go, which is significant. (For a better insight of this, refer to Appendix 1 and Appendix 2.)

We compare all the (d)s and (f)s in (1) to (12) above, which is as follows:-

(d) Average no. of even numbers "generated" by each of the consecutive primes in (1) to (12) above, as follows according to the listings (1) to (12):

(1) **6.14**
(2) **6.87**
(3) **7.37**
(4) **7.68**
(5) **7.93**
(6) **8.11**
(7) **8.34**
(8) **8.44**

(9) **8.56**
(10) **8.70**
(11) **8.83**
(12) **8.91**

(f) Average no. of summings of 2 primes/permutations for each of the even numbers in (1) to (12) above, as follows according to the listings (1) to (12):

(1) **32.60**
(2) **58.25**
(3) **81.47**
(4) **104.22**
(5) **126.18**
(6) **147.89**
(7) **167.88**
(8) **189.48**
(9) **210.24**
(10) **230.00**
(11) **249.16**
(12) **269.30**

The following is evident from the above information:-

(A): (d) Average no. of even numbers "generated" by each of the consecutive primes in the above 12 listings increases continually all the way from the list: (1) 200 Consecutive Primes From 3 To 1,229 to the list: (12) 2,400 Consecutive Primes From 3 To 21,391, from **6.14** even numbers per prime number in List (1) to **8.91** even numbers per prime number in List (12).

(B): (f) Average no. of summings of 2 primes/permutations for each of the even numbers in the above 12 listings increases continually all the way from the list: (1) 200 Consecutive Primes From 3 To 1,229 to the list: (12) 2,400 Consecutive Primes From 3 To 21,391, from **32.60** number of summings of 2 primes/permutations per even number in List (1) to **269.30** number of summings of 2 primes/permutations per even number in List (12).

Lemma:
According to the principle of complete induction in set theory, if a set of natural numbers contains 1 and, for each *n*, it contains *n* + 1 whenever it contains all numbers less than *n* + 1, then it must contain every natural number, e.g., complete induction proves that every natural number is a product of primes.

By induction, we now deduce the following:

The larger the list of consecutive primes becomes, the greater would be the average number of even numbers "generated" by each of the primes in the list of consecutive primes (inferred from (A) above).

The larger the list of consecutive primes becomes, the greater would be the average number of summings of 2 primes/permutations for each of the even numbers in the infinite list of even numbers (inferred from (B) above).

Furthermore, the Goldbach conjecture had been tested and found to be correct for every even number up to 12×10^{17}, which is not a small list, by a distributed computer search carried out at the University of Aveiro, Portugal, in 2008.

As the primes and the even numbers are infinite, by the above lemma and all the above deductions and information, it could be inferred that the increases stated in (A) and (B) above, with the even numbers each being the sum of 2 primes, continue to infinity, *i.e., the Goldbach conjecture becomes stronger and stronger the higher up the infinite list of prime numbers/even numbers we go - all the way to infinity*.

Next, we resort to the proof by contradiction. The above deduction would be reversed if, e.g., the following takes place (which is the reversal of the above-mentioned information):

(A): (d) Average no. of even numbers "generated" by each of the consecutive primes in the above 12 listings decreases continually all the way from the list: (1) 200 Consecutive Primes From 3 To 1,229 to the list: (12) 2,400 Consecutive Primes From 3 To 21,391, from **8.91** even numbers per prime number in List (1) to **6.14** even numbers per prime number in List (12).

(B): (f) Average no. of summings of 2 primes/permutations for each of the even numbers in the above 12 listings decreases continually all the way from the list: (1) 200 Consecutive Primes From 3 To 1,229 to the list: (12) 2,400 Consecutive Primes From 3 To 21,391, from **269.30** number of summings of 2 primes/permutations per even number in List (1) to **32.60** number of summings of 2 primes/permutations per even number in List (12).

If this reversed state happens, the implication is that there would reach a point when there are no more batches of 2 prime numbers summing together to form even numbers, in which case the Goldbach conjecture would be false. Evidently this would happen when the prime numbers are finite. As the prime numbers are infinite (as Euclid had proved long ago) this would never happen.

Since the above information indicate otherwise, and, the prime numbers are infinite, we accept the above induction/deduction and infer that the Goldbach conjecture could not be false, i.e., the Goldbach conjecture is true, and, every even number (except 2) is indeed the sum of 2 prime numbers. This concludes the proof by contradiction.

We take a look at the following example to see how effectively the primes "generate" new even numbers in accordance with the Goldbach conjecture:-

<u>Density Of New Even Numbers "Generated" (*See Appendix 1 For Example Of Computation Method*)</u>

(a) Set Of Integers, 51 To 100, With 10 Primes Within It = 5 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 50. No. Of Primes = 10.)

(b) Set Of Integers, 101 To 150, With 10 Primes Within It = 5.2 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 52. No. Of Primes = 10.)

(c) Set Of Integers, 151 To 200, With 11 Primes Within It = **4.55 New Even Nos. Per Prime No.**
(No. Of New Even Nos. "Generated" = 50. No. Of Primes = 11.)

(d) Set Of Integers, 201 To 250, With 7 Primes Within It = 6 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 42. No. Of Primes = 7.)

(e) Set Of Integers, 251 To 300, With 9 Primes Within It = 5.78 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 52. No. Of Primes = 9.)

(f) Set Of Integers, 301 To 350, With 8 Primes Within It = 7 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 56. No. Of Primes = 8.)

(g) Set Of Integers, 351 To 400, With 8 Primes Within It = 6 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 48. No. Of Primes = 8.)

(h) Set Of Integers, 401 To 450, With 9 Primes Within It = 5.78 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 52. No. Of Primes = 9.)

(i) Set Of Integers, 451 To 500, With 8 Primes Within It = 6.25 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 50. No. Of Primes = 8.)

(j) Set Of Integers, 501 To 550, With 6 Primes Within It = 8 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 48. No. Of Primes = 6.)

(k) Set Of Integers, 551 To 600, With 8 Primes Within It = 6.5 New Even Nos. Per Prime No.

(No. Of New Even Nos. "Generated" = 52. No. Of Primes = 8.)

(l) Set Of Integers, 601 To 650, With 9 Primes Within It = 5.33 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 48. No. Of Primes = 9.)

(m) Set Of Integers, 651 To 700, With 7 Primes Within It = 6.29 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 44. No. Of Primes = 7.)

(n) Set Of Integers, 701 To 750, With 7 Primes Within It = 7.43 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 52. No. Of Primes = 7.)

(o) Set Of Integers, 751 To 800, With 7 Primes Within It = 7.71 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 54. No. Of Primes = 7.)

(p) Set Of Integers, 801 To 850, With 7 Primes Within It = 6 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 42. No. Of Primes = 7.)

(q) Set Of Integers, 851 To 900, With 8 Primes Within It = 6 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 48. No. Of Primes = 8.)

(r) Set Of Integers, 901 To 950, With 7 Primes Within It = 8.57 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 60. No. Of Primes = 7.)

(s) Set Of Integers, 951 To 1,000, With 7 Primes Within It = 7.14 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 50. No. Of Primes = 7.)

(t) Set Of Integers, 1,001 To 1,050, With 8 Primes Within It = 6.5 New Even Nos. Prime No.
(No. Of New Even Nos. "Generated" = 52. No. Of Primes = 8.)

(u) Set Of Integers, 1,051 To 1,100, With 8 Primes Within It = 6 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 48. No. Of Primes = 8.)

(v) Set Of Integers, 1,101 To 1,150, With 5 Primes Within It = 6.4 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 32. No. Of Primes = 5.)

(w) Set Of Integers, 1,151 To 1,200, With 7 Primes Within It = **9.14 New Even Nos. Per Prime No.**
(No. Of New Even Nos. "Generated" = 64. No. Of Primes = 7.)

(x) Set Of Integers, 1,201 To 1,250, With 8 Primes Within It = 7 New Even Nos. Per Prime No.
(No. Of New Even Nos. "Generated" = 56. No. Of Primes = 8.)

Average Density For The Above 24 Items ((a) To (x)) = 155.54 ÷ 24 = 6.48 New Even Nos. Per Prime No.

Maximum Density = 9.14 New Even Nos. Per Prime No. (No. Of New Even Nos. "Generated" = 64. No. Of Primes = 7.)

Minimum Density = 4.55 New Even Nos. Per Prime No. (No. Of New Even Nos. "Generated" = 50. No. Of Primes = 11.)

Such a "profuse generation" of "regular batches" of even numbers by the prime numbers is significant and lends further support to the validity of the Goldbach conjecture.

There is further proof which is obtainable by analyzing a number of even numbers, e.g., we could split a group of 240 even consecutive numbers, from 4 to 482, into 8 equal batches (30 even numbers per batch) and analyze the batches; this would corroborate the fact that the infinite quantity of primes would “generate” a regular, continuous (*without breaks or gaps*) and infinite list of even numbers. The density of distribution or prime additions/combinations per even number evidently become greater and greater the higher up the infinite list of the even numbers we go, i.e., *the Goldbach conjecture evidently becomes stronger and stronger the higher up the infinite list of the even numbers we go*. This pattern is significant and is discernable in the following example:-

(1) <u>1 st. Batch Of 30 Even Numbers (4 To 62)</u> (*See Appendix 2 For Example Of Computation Method*)

a) Maximum No. Of Prime Additions/Combinations Per Even Number = **5**
b) Minimum No. Of Prime Additions/Combinations Per Even Number = **1**
c) Density Of Distribution = Average Prime Additions/Combinations Per Even Number = **2.77** Prime Additions/Combinations Per Even Number

(2) <u>2 nd. Batch Of 30 Even Numbers (64 To 122)</u>

a) Maximum No. Of Prime Additions/Combinations Per Even Number = 14
b) Minimum No. Of Prime Additions/Combinations Per Even Number = 2
c) Density Of Distribution = Average Prime Additions/Combinations Per Even Number = **6.1** Prime Additions/Combinations Per Even Number
d) Percentage Increase In Density Of Distribution = (6.1 - 2.77) ÷ 2.77 x 100% = 120.22%

(3) <u>3 rd. Batch Of 30 Even Numbers (124 To 182)</u>

a) Maximum No. Of Prime Additions/Combinations Per Even Number = 16
b) Minimum No. Of Prime Additions/Combinations Per Even Number = 4
c) Density Of Distribution = Average Prime Additions/Combinations Per Even Number = **9.07** Prime Additions/Combinations Per Even Number
d) Percentage Increase In Density Of Distribution = (9.07 - 6.1) ÷ 6.1 x 100% = 48.69%

(4) <u>4 th. Batch Of 30 Even Numbers (184 To 242)</u>

a) Maximum No. Of Prime Additions/Combinations Per Even Number = 22
b) Minimum No. Of Prime Additions/Combinations Per Even Number = 5
c) Density Of Distribution = Average Prime Additions/Combinations Per Even Number = **10.53** Prime Additions/Combinations Per Even Number
d) Percentage Increase In Density Of Distribution = (10.53 - 9.07) ÷ 9.07 x 100% = 16.1%

(5) <u>5 th. Batch Of 30 Even Numbers (244 To 302)</u>

a) Maximum No. Of Prime Additions/Combinations Per Even Number = 21
b) Minimum No. Of Prime Additions/Combinations Per Even Number = 7
c) Density Of Distribution = Average Prime Additions/Combinations Per Even Number = **12.37** Prime Additions/Combinations Per Even Number
d) Percentage Increase In Density Of Distribution = (12.37 - 10.53) ÷ 10.53 x 100% = 17.47%

(6) <u>6 th. Batch Of 30 Even Numbers (304 To 362)</u>

a) Maximum No. Of Prime Additions/Combinations Per Even Number = 27
b) Minimum No. Of Prime Additions/Combinations Per Even Number = 7
c) Density Of Distribution = Average Prime Additions/Combinations Per Even Number = **13.77** Prime Additions/Combinations Per Even Number
d) Percentage Increase In Density Of Distribution = (13.77 - 12.37) ÷ 12.37 x 100%

= 11.32%

(7) 7 th. Batch Of 30 Even Numbers (364 To 422)

a) Maximum No. Of Prime Additions/Combinations Per Even Number = 30
b) Minimum No. Of Prime Additions/Combinations Per Even Number = 7
c) Density Of Distribution = Average Prime Additions/Combinations Per Even Number = **15.23** Prime Additions/Combinations Per Even Number
d) Percentage Increase In Density Of Distribution = (15.23 - 13.77) ÷ 13.77 x 100% = 10.6%

(8) 8 th. Batch Of 30 Even Numbers (424 To 482)

a) Maximum No. Of Prime Additions/Combinations Per Even Number = 30
b) Minimum No. Of Prime Additions/Combinations Per Even Number = 9
c) Density Of Distribution = Average Prime Additions/Combinations Per Even Number = **16.93** Prime Additions/Combinations Per Even Number
d) Percentage Increase In Density Of Distribution = (16.93 - 15.23) ÷ 15.23 x 100% = 11.16%

The Density Of Distribution is expected to increase to infinity, though the Percentage Increase In Density Of Distribution is expected to thin out towards infinity - it could be seen above to increase from 2.77 prime additions/combinations per even number for batch of even numbers, 4 to 62, all the way up to 16.93 prime additions/combinations per even number for batch of even numbers, 424 to 482. This is nevertheless significant evidence that lends support to the validity of the Goldbach conjecture. Also, the Maximum No. Of Prime Additions/Combinations Per Even Number and the Minimum No. Of Prime Additions/Combinations Per Even Number could be seen to range from 5 and 1 respectively for batch of even numbers, 4 to 62, to 30 and 9 respectively for batch of even numbers, 424 to 482. This trend of "upward increase" of the (maximum and minimum) numbers of prime additions/combinations for each even number implies that at some points toward infinity the numbers of prime additions/combinations for each even number could be thousands, millions, billions, trillions, and more, if only we have the computing power to compute/check such prime additions/combinations (this again indicates that *the Goldbach conjecture becomes evidently stronger and stronger the higher up the infinite list of the even numbers we go*). This is significant too and is also evidence that lends support to the validity of the Goldbach conjecture. By the infinitude of the primes (vide Euclid's proof) and even numbers, these "patterns", as described here, would be there all the way to infinity, which would be in accordance with the Goldbach conjecture.

The following evidence would further affirm the validity of the Goldbach conjecture:-

1) 10 consecutive primes, commencing from the odd prime 3, would give rise to 10 x 10, or, 100 sums of 2 primes/partitions/permutations, but less than 100 different even numbers, with many repetitions/overlaps (e.g., for these first 10 consecutive primes 3, 5, 7, 11, 13, 17, 19, 23, 29 & 31, 10 = 3 + 7 = 5 + 5 (2 partitions/permutations), 22 = 3 + 19 = 5 + 17 = 11 + 11 (3 partitions/permutations), &, 34 = 3 + 31 = 5 + 29 = 11 + 23 = 17 + 17 (4 partitions/permutations)).
2) 20 consecutive primes, commencing from the odd prime 3, (increase of **100%** in no. of

consecutive primes compared to (1) above) would give rise to 20 x 20, or, 400 sums of 2 primes/partitions/permutations (increase of **300%** in no. of sums of 2 primes/partitions/permutations compared to (1) above), but less than 400 different even numbers, with many repetitions/overlaps.

3) 30 consecutive primes, commencing from the odd prime 3, (increase of **200%** in no. of consecutive primes compared to (1) above) would give rise to 30 x 30, or, 900 sums of 2 primes/partitions/permutations (increase of 8**00%** in no. of sums of 2 primes/partitions/permutations compared to (1) above), but less than 900 different even numbers, with many repetitions/overlaps.
4) 40 consecutive primes, commencing from the odd prime 3, (increase of **300%** in no. of consecutive primes compared to (1) above) would give rise to 40 x 40, or, 1,600 sums of 2 primes/partitions/permutations (increase of **1,500%** in no. of sums of 2 primes/partitions/permutations compared to (1) above), but less than 1,600 different even numbers, with many repetitions/overlaps.
5) 50 consecutive primes, commencing from the odd prime 3, (increase of **400%** in no. of consecutive primes compared to (1) above) would give rise to 50 x 50, or, 2,500 sums of 2 primes/partitions/permutations (increase of **2,400%** in no. of sums of 2 primes/partitions/permutations compared to (1) above), but less than 2,500 different even numbers, with many repetitions/overlaps.
6) 60 consecutive primes, commencing from the odd prime 3, (increase of **500%** in no. of consecutive primes compared to (1) above) would give rise to 60 x 60, or, 3,600 sums of 2 primes/partitions/permutations (increase of **3,500%** in no. of sums of 2 primes/partitions/permutations compared to (1) above), but less than 3,600 different even numbers, with many repetitions/overlaps.
7) 70 consecutive primes, commencing from the odd prime 3, (increase of **600%** in no. of consecutive primes compared to (1) above) would give rise to 70 x 70, or, 4,900 sums of 2 primes/partitions/permutations (increase of **4,800%** in no. of sums of 2 primes/partitions/permutations compared to (1) above), but less than 4,900 different even numbers, with many repetitions/overlaps.
8) 80 consecutive primes, commencing from the odd prime 3, (increase of **700%** in no. of consecutive primes compared to (1) above) would give rise to 80 x 80, or, 6,400 sums of 2 primes/partitions/permutations (increase of **6,300%** in no. of sums of 2 primes/partitions/permutations compared to (1) above), but less than 6,400 different even numbers, with many repetitions/overlaps.
9) 90 consecutive primes, commencing from the odd prime 3, (increase of **800%** in no. of consecutive primes compared to (1) above) would give rise to 90 x 90, or, 8,100 sums of 2 primes/partitions/permutations (increase of **8,000%** in no. of sums of 2 primes/partitions/permutations compared to (1) above), but less than 8,100 different even numbers, with many repetitions/overlaps.
10) 100 consecutive primes, commencing from the odd prime 3, (increase of **900%** in no. of

consecutive primes compared to (1) above) would give rise to 100 x 100, or, 10,000 sums of 2 primes/partitions/permutations (increase of **9,900%** in no. of sums of 2 primes/partitions/permutations compared to (1) above), but less than 10,000 different even numbers, with many repetitions/overlaps.

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The following is evident from the above:-

1) The 1st. marginal increase of **100%** in no. of consecutive primes (increase of 200% - increase of 100%) results in marginal increase of **500%** in no. of sums of 2 primes/partitions/permutations (increase of 800% - increase of 300%).
2) The 2nd. marginal increase of **100%** in no. of consecutive primes (increase of 300% - increase of 200%) results in marginal increase of **700%** in no. of sums of 2 primes/partitions/permutations (increase of 1,500% - increase of 800%).
3) The 3rd. marginal increase of **100%** in no. of consecutive primes (increase of 400% - increase of 300%) results in marginal increase of **900%** in no. of sums of 2 primes/partitions/permutations (increase of 2,400% - increase of 1,500%).
4) The 4th. marginal increase of **100%** in no. of consecutive primes (increase of 500% - increase of 400%) results in marginal increase of **1,100%** in no. of sums of 2 primes/partitions/permutations (increase of 3,500% - increase of 2,400%).
5) The 5th. marginal increase of **100%** in no. of consecutive primes (increase of 600% - increase of 500%) results in marginal increase of **1,300%** in no. of sums of 2 primes/partitions/permutations (increase of 4,800% - increase of 3,500%).
6) The 6th. marginal increase of **100%** in no. of consecutive primes (increase of 700% - increase of 600%) results in marginal increase of **1,500%** in no. of sums of 2 primes/partitions/permutations (increase of 6,300% - increase of 4,800%).
7) The 7th. marginal increase of **100%** in no. of consecutive primes (increase of 800% - increase of 700%) results in marginal increase of **1,700%** in no. of sums of 2 primes/partitions/permutations (increase of 8,000% - increase of 6,300%).
8) The 8th. marginal increase of **100%** in no. of consecutive primes (increase of 900% - increase of 800%) results in marginal increase of **1,900%** in no. of sums of 2 primes/partitions/permutations (increase of 9,900% - increase of 8,000%).

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(1) to (8) above show that while the marginal increase in no. of consecutive primes remains constant at 100% from (1) to (8), the marginal increase in no. of sums of 2 primes/partitions/permutations goes up progressively from 500% in (1) to 1,900% in (8). It is evident here that the higher up the infinite list of primes we go, the more "overwhelming" or dense the (one-to-one) combinations of primes (i.e., sums of 2 primes, in the formation of even numbers) would become, the number of permutations of the combinations of primes tending towards infinity (with the infinity of the prime numbers). *In other words, the Goldbach conjecture becomes stronger and stronger the higher up the infinite list of prime numbers/even numbers we go*. The infinitude of the prime numbers (vide Euclid's proof) and even numbers would hence imply the validity of the Goldbach conjecture.

The prime number theorem, which had been proven, states that the limit of the quotient of the 2 functions $\pi(n)$ and $n/\log n$ as n approaches infinity is 1, which is expressed by the formula:

$$\lim_{n\to\infty} \pi(n)/(n/\log n) = 1$$

where $\pi(n)$ is approximately equal to $(n/\log n)$

The function $\pi(n)$ represents the number of primes less than or equal to the number n. This function measures the distribution of the prime numbers. With it, we compute the ratio $n/\pi(n)$ which says what fraction of the numbers up to a given point are primes. (It is actually the reciprocal of this fraction.) The following is the result of a computation:-

n	***π(n)***	***n/π(n)***
10	4 (a)	2.5
100	25 (b)	4.0
1,000	168 (c)	6.0
10,000	1,229 (d)	8.1
100,000	9,592 (e)	10.4
1,000,000	78,498 (f)	12.7
10,000,000	664,579 (g)	15.0
100,000,000	5,761,455 (h)	17.4
1,000,000,000	50,847,534 (i)	19.7
10,000,000,000	455,052,512 (j)	22.0

It is noticeable that as one moves from 1 power of 10 to the next, the ratio $n/\pi(n)$ increases by about 2.3, e.g., 22.0 - 19.7 = 2.3. As $\log_e 10 = 2.30258 \ldots$, we may thus regard $\pi(n)$ as approximately equal to $n/\log n$.

We have the following partitions with the primes described in the “$\pi(n)$” column above:-

1) With (a) above, we have the following “prime + prime = even number” combinations:

   a) prime a + prime a: 4 x 4 “prime + prime” combinations
   b) prime a + prime b: 4 x 25 “prime + prime” combinations
   c) prime a + prime c: 4 x 168 “prime + prime” combinations
   d) prime a + prime d: 4 x 1,229 “prime + prime” combinations
   e) prime a + prime e: 4 x 9,592 “prime + prime” combinations
   f) prime a + prime f: 4 x 78,498 “prime + prime” combinations
   g) prime a + prime g: 4 x 664,579 “prime + prime” combinations
   h) prime a + prime h: 4 x 5,761,455 “prime + prime” combinations
   i) prime a + prime i: 4 x 50,847,534 “prime + prime” combinations
   j) prime a + prime j: 4 x 455,052,512 “prime + prime” combinations

   For example, for (j) above, a prime described in (a) in the “$\pi(n)$” column above plus a prime described in (j) in the “$\pi(n)$” column above give an even number, and there are 4 x 455,052,512 such “prime + prime = even number” combinations.

2) With (b) above, we have the following “prime + prime = even number” combinations:

   a) prime b + prime a: 25 x 4 “prime + prime” combinations
   b) prime b + prime b: 25 x 25 “prime + prime” combinations
   c) prime b + prime c: 25 x 168 “prime + prime” combinations
   d) prime b + prime d: 25 x 1,229 “prime + prime” combinations
   e) prime b + prime e: 25 x 9,592 “prime + prime” combinations
   f) prime b + prime f: 25 x 78,498 “prime + prime” combinations
   g) prime b + prime g: 25 x 664,579 “prime + prime” combinations
   h) prime b + prime h: 25 x 5,761,455 “prime + prime” combinations
   i) prime b + prime i: 25 x 50,847,534 “prime + prime” combinations
   j) prime b + prime j: 25 x 455,052,512 “prime + prime” combinations

3) With (c) above, we have the following “prime + prime = even number” combinations:

   a) prime c + prime a: 168 x 4 “prime + prime” combinations
   b) prime c + prime b: 168 x 25 “prime + prime” combinations

c) prime c + prime c: 168 x 168 “prime + prime” combinations
d) prime c + prime d: 168 x 1,229 “prime + prime” combinations
e) prime c + prime e: 168 x 9,592 “prime + prime” combinations
f) prime c + prime f: 168 x 78,498 “prime + prime” combinations
g) prime c + prime g: 168 x 664,579 “prime + prime” combinations
h) prime c + prime h: 168 x 5,761,455 “prime + prime” combinations
i) prime c + prime i: 168 x 50,847,534 “prime + prime” combinations
j) prime c + prime j: 168 x 455,052,512 “prime + prime” combinations

4) With (d) above, we have the following “prime + prime = even number” combinations:

a) prime d + prime a: 1,229 x 4 “prime + prime” combinations
b) prime d + prime b: 1,229 x 25 “prime + prime” combinations
c) prime d + prime c: 1,229 x 168 “prime + prime” combinations
d) prime d + prime d: 1,229 x 1,229 “prime + prime” combinations
e) prime d + prime e: 1,229 x 9,592 “prime + prime” combinations
f) prime d + prime f: 1,229 x 78,498 “prime + prime” combinations
g) prime d + prime g: 1,229 x 664,579 “prime + prime” combinations
h) prime d + prime h: 1,229 x 5,761,455 “prime + prime” combinations
i) prime d + prime i: 1,229 x 50,847,534 “prime + prime” combinations
j) prime d + prime j: 1,229 x 455,052,512 “prime + prime” combinations

5) With (e) above, we have the following “prime + prime = even number” combinations:

a) prime e + prime a: 9,592 x 4 “prime + prime” combinations
b) prime e + prime b: 9,592 x 25 “prime + prime” combinations
c) prime e + prime c: 9,592 x 168 “prime + prime” combinations
d) prime e + prime d: 9,592 x 1,229 “prime + prime” combinations
e) prime e + prime e: 9,592 x 9,592 “prime + prime” combinations
f) prime e + prime f: 9,592 x 78,498 “prime + prime” combinations
g) prime e + prime g: 9,592 x 664,579 “prime + prime” combinations
h) prime e + prime h: 9,592 x 5,761,455 “prime + prime” combinations
i) prime e + prime i: 9,592 x 50,847,534 “prime + prime” combinations
j) prime e + prime j: 9,592 x 455,052,512 “prime + prime” combinations

6) With (f) above, we have the following “prime + prime = even number” combinations:

a) prime f + prime a: 78,498 x 4 "prime + prime" combinations
b) prime f + prime b: 78,498 x 25 "prime + prime" combinations
c) prime f + prime c: 78,498 x 168 "prime + prime" combinations
d) prime f + prime d: 78,498 x 1,229 "prime + prime" combinations
e) prime f + prime e: 78,498 x 9,592 "prime + prime" combinations
f) prime f + prime f: 78,498 x 78,498 "prime + prime" combinations
g) prime f + prime g: 78,498 x 664,579 "prime + prime" combinations
h) prime f + prime h: 78,498 x 5,761,455 "prime + prime" combinations
i) prime f + prime i: 78,498 x 50,847,534 "prime + prime" combinations
j) prime f + prime j: 78,498 x 455,052,512 "prime + prime" combinations

7) With (g) above, we have the following "prime + prime = even number" combinations:

a) prime g + prime a: 664,579 x 4 "prime + prime" combinations
b) prime g + prime b: 664,579 x 25 "prime + prime" combinations
c) prime g + prime c: 664,579 x 168 "prime + prime" combinations
d) prime g + prime d: 664,579 x 1,229 "prime + prime" combinations
e) prime g + prime e: 664,579 x 9,592 "prime + prime" combinations
f) prime g + prime f: 664,579 x 78,498 "prime + prime" combinations
g) prime g + prime g: 664,579 x 664,579 "prime + prime" combinations
h) prime g + prime h: 664,579 x 5,761,455 "prime + prime" combinations
i) prime g + prime i: 664,579 x 50,847,534 "prime + prime" combinations
j) prime g + prime j: 664,579 x 455,052,512 "prime + prime" combinations

8) With (h) above, we have the following "prime + prime = even number" combinations:

a) prime h + prime a: 5,761,455 x 4 "prime + prime" combinations
b) prime h + prime b: 5,761,455 x 25 "prime + prime" combinations
c) prime h + prime c: 5,761,455 x 168 "prime + prime" combinations
d) prime h + prime d: 5,761,455 x 1,229 "prime + prime" combinations
e) prime h + prime e: 5,761,455 x 9,592 "prime + prime" combinations
f) prime h + prime f: 5,761,455 x 78,498 "prime + prime" combinations
g) prime h + prime g: 5,761,455 x 664,579 "prime + prime" combinations
h) prime h + prime h: 5,761,455 x 5,761,455 "prime + prime" combinations
i) prime h + prime i: 5,761,455 x 50,847,534 "prime + prime" combinations
j) prime h + prime j: 5,761,455 x 455,052,512 "prime + prime" combinations

9) With (i) above, we have the following "prime + prime = even number"

combinations:

a) prime i + prime a: 50,847,534 x 4 “prime + prime” combinations
b) prime i + prime b: 50,847,534 x 25 “prime + prime” combinations
c) prime i + prime c: 50,847,534 x 168 “prime + prime” combinations
d) prime i + prime d: 50,847,534 x 1,229 “prime + prime” combinations
e) prime i + prime e: 50,847,534 x 9,592 “prime + prime” combinations
f) prime i + prime f: 50,847,534 x 78,498 “prime + prime” combinations
g) prime i + prime g: 50,847,534 x 664,579 “prime + prime” combinations
h) prime i + prime h: 50,847,534 x 5,761,455 “prime + prime” combinations
i) prime i + prime i: 50,847,534 x 50,847,534 “prime + prime” combinations
j) prime i + prime j: 50,847,534 x 455,052,512 “prime + prime” combinations

10) With (j) above, we have the following “prime + prime = even number” combinations:

a) prime j + prime a: 455,052,512 x 4 “prime + prime” combinations
b) prime j + prime b: 455,052,512 x 25 “prime + prime” combinations
c) prime j + prime c: 455,052,512 x 168 “prime + prime” combinations
d) prime j + prime d: 455,052,512 x 1,229 “prime + prime” combinations
e) prime j + prime e: 455,052,512 x 9,592 “prime + prime” combinations
f) prime j + prime f: 455,052,512 x 78,498 “prime + prime” combinations
g) prime j + prime g: 455,052,512 x 664,579 “prime + prime” combinations
h) prime j + prime h: 455,052,512 x 5,761,455 “prime + prime” combinations
i) prime j + prime i: 455,052,512 x 50,847,534 “prime + prime” combinations
j) prime j + prime j: 455,052,512 x 455,052,512 “prime + prime” combinations

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The above partitions/“prime + prime = even number” combinations are evidently progressively more “overwhelming”, dense (refer to Figure 1 below), and repetitive (overlapping). *That is, the Goldbach conjecture becomes evidently progressively stronger and stronger towards infinity, which corroborates the earlier observation/induction*. It is not surprising that computer searches completed in 2000 had verified that all even numbers up to 400 trillion (4 x 10^{14}), which is not a small list, are sums of 2 primes, while in 2008, a distributed computer search ran by Tomas Oliveira e Silva, a researcher at the University of Aveiro, Portugal, had further verified the Goldbach conjecture up to 12 x 10^{17}, which is a long, impressive list.

Though the distribution of primes evidently becomes progressively less and less dense, e.g., ranging from 40% of primes within the first 10 integers to 4.55% of primes within the first 10,000,000,000 integers, the density of partitions/"prime + prime = even number" combinations evidently becomes progressively greater and greater as is shown below:-

1) For the 1st. **10-fold** increase in no. of integers (100 integers ÷ 10 integers), the no. of partitions/"prime + prime = even number" combinations increases **39.06 times** ([25 x 25 partitions] ÷ [4 x 4 partitions]).
2) For the 2nd. **10-fold** increase in no. of integers (1,000 integers ÷ 100 integers), the no. of partitions/"prime + prime = even number" combinations increases **45.16 times** ([168 x 168 partitions] ÷ [25 x 25 partitions]).
3) For the 3rd. **10-fold** increase in no. of integers (10,000 integers ÷ 1,000 integers), the no. of partitions/"prime + prime = even number" combinations increases **53.52 times** ([1,229 x 1,229 partitions] ÷ [168 x 168 partitions]).
4) For the 4th. **10-fold** increase in no. of integers (100,000 integers ÷ 10,000 integers), the no. of partitions/"prime + prime = even number" combinations increases **60.91 times** ([9,592 x 9,592 partitions] ÷ [1,229 x 1,229 partitions]).
5) For the 5th. **10-fold** increase in no. of integers (1,000,000 integers ÷ 100,000 integers), the no. of partitions/"prime + prime = even number" combinations increases **66.97 times** ([78,498 x 78,498 partitions] ÷ [9,592 x 9,592 partitions]).
6) For the 6th. **10-fold** increase in no. of integers (10,000,000 integers ÷ 1,000,000 integers), the no. of partitions/"prime + prime = even number" combinations increases **71.68 times** ([664,579 x 664,579 partitions] ÷ [78,498 x 78,498 partitions]).
7) For the 7th. **10-fold** increase in no. of integers (100,000,000 integers ÷ 10,000,000 integers), the no. of partitions/"prime + prime = even number" combinations increases **75.16 times** ([5,761,455 x 5,761,455 partitions] ÷ [664,579 x 664,579 partitions]).
8) For the 8th. **10-fold** increase in no. of integers (1,000,000,000 integers ÷ 100,000,000 integers), the no. of partitions/"prime + prime = even number" combinations increases **77.89 times** ([50,847,534 x 50,847,534 partitions] ÷ [5,761,455 x 5,761,455 partitions]).
9) For the 9th. **10-fold** increase in no. of integers (10,000,000,000 integers ÷ 1,000,000,000 integers), the no. of partitions/"prime + prime = even number" combinations increases **80.09 times** ([455,052,512 x 455,052,512 partitions] ÷ [50,847,534 x 50,847,534 partitions]).

Figure 1

The infinitude of the primes, as per Euclid's proof, together with the infinitude of the even numbers, however imply that the above partitions/"prime + prime = even number" combinations would become increasingly more "overwhelming", dense, and repetitive (overlapping) towards infinity (the Goldbach conjecture becoming evidently stronger and stronger the higher up the infinite list of prime numbers/even numbers we go), hence "ensuring" the continuity (without any breaks or gaps) of the even numbers generated, and would be so all the way to infinity, thus proving that every even number after 2 is the sum of 2 primes. (For a better insight of how the above partitions/"prime + prime = even number" combinations would become increasingly more "overwhelming", dense, and repetitive (overlapping) towards infinity, refer to Appendix 1 and Appendix 2.)

The partitions/"prime + prime = even number" combinations, as had been conjectured by Goldbach, are evidently effusive, or in great abundance, in their occurrences, as is shown above and in the appendices below. This has important consequence. For instance, in Appendix 2, the number of partitions/"prime + prime = even number" combinations for each of the 30 even numbers (424 to 482) ranges from the minimum 9 (for the even numbers 428 and 458) to the maximum 30 (for the even numbers 462 and 480), giving an average of 16.93 partitions/"prime + prime = even number" combinations per even number. This is significant and is in stark contrast to the results of the Fundamental Theorem of Arithmetic or Unique Factorization Theorem, which states that there is only 1 possible combination of primes which will multiply together to produce any particular number, e.g., the only combination of primes which will produce the number 2,079 is as follows:-

3 x 3 x 3 x 7 x 11 (only)

In the same manner, the following numbers are also uniquely factorized:-

63 = 3 x 3 x 7 (only)
153 = 3 x 3 x 17 (only)
1,021,020 = 2 x 2 x 3 x 5 x 7 x 11 x 13 x 17 (only)

In other words, every positive whole number can be broken up into prime factors, and, this can happen in only 1 way. In contrast, every even number is the sum of 2 primes in more than 1 way, e.g., 30 ways (i.e., 30 possible partitions) in the cases of the even numbers 462 and 480 as is described above. As is stated above, this is significant. This effusiveness or abundance of partitions/"prime + prime = even number" combinations somehow implies that the continuity (without any breaks or gaps) of the even numbers as sums of 2 primes (which are "generated" through the various additions of 2 primes) is "ensured", i.e., the possible breaks in the continuity of the even numbers as sums of 2 primes (wherein some even numbers in-between can never be sums of 2 primes, as are shown in the example in Figure 2 below where there are 4 breaks in the continuity of the even numbers x_1 to x_{12}, where the 4 even numbers x_4, x_5, x_8 & x_{10} can never be sums of 2 primes), which implies the falsity of the Goldbach conjecture, are somehow "prevented from happening" by this effusiveness or abundance:-

x below represents, say, an extremely large even number. p below represents a prime. c below represents a composite number or non-prime whence the Goldbach conjecture would be false (i.e., not every even number is the sum of 2 primes as the composite numbers or non-primes would be the exceptions).

$$\begin{array}{l} \vdots \\ x_1 = p_1 + p_2 \\ x_2 = p_3 + p_4 \\ x_3 = p_5 + p_6 \\ x_4 = c_1 + c_2 \text{ (break)} \\ x_5 = p_7 + c_3 \text{ (break)} \\ x_6 = p_8 + p_9 \\ x_7 = p_{10} + p_{11} \\ x_8 = c_4 + p_{12} \text{ (break)} \\ x_9 = p_{13} + p_{14} \\ x_{10} = c_5 + c_6 \text{ (break)} \\ x_{11} = p_{15} + p_{16} \\ x_{12} = p_{17} + p_{18} \\ \vdots \end{array}$$

Figure 2

There appears to be some deep meaning in the ease and effusiveness with which the partitions/"prime + prime = even number" combinations or sums of 2 primes show up, as is shown in this paper. If every even number which is the sum of 2 primes is the sum of 2 primes in only 1 way (a la the results of the Fundamental Theorem of Arithmetic or Unique Factorization Theorem described above), there could be possible breaks in the continuity of the even numbers as sums of 2 primes (as are shown in Figure 2 above), in other words, there could be some reason to doubt the validity of the Goldbach conjecture. But, on the contrary, the sums of 2 primes are evidently much effusive; they are evidently a defining characteristic of the even numbers. Under such a circumstance, it would be difficult to doubt the validity of the Goldbach conjecture.

In elaborating further on the above point, we take a look at the following:-

<u>No. Of Old/Repeated (Also Appeared Earlier) Even Numbers/Overlaps "Generated" (By The Additions/Combinations Of Two Primes), For Integers 1 To 1,250 (*See Appendix 1 For Example Of Computation Method*)</u>

(a) Set Of Integers, 1 To 50, With 14 Primes Within It = Not Applicable
(aa) Percentage Increase In Repetition = Not Applicable

(b) Set Of Integers, 51 To 100, With 10 Primes Within It = **20** Repeated Even Nos.
(bb) Percentage Increase In Repetition = Not Applicable

(c) Set Of Integers, 101 To 150, With 10 Primes Within It = 46 Repeated Even Nos.
(cc) Percentage Increase In Repetition = (46 - 20) ÷ 20 x 100% = **130%**

(d) Set Of Integers, 151 To 200, With 11 Primes Within It = 73 Repeated Even Nos.
(dd) Percentage Increase In Repetition = (73 - 46) ÷ 46 x 100% = 58.7%

(e) Set Of Integers, 201 To 250, With 7 Primes Within It = 93 Repeated Even Nos.
(ee) Percentage Increase In Repetition = (93 - 73) ÷ 73 x 100% = 27.4%

(f) Set Of Integers, 251 To 300, With 9 Primes Within It = 115 Repeated Even Nos.
(ff) Percentage Increase In Repetition = (115 - 93) ÷ 93 x 100% = 23.66%

(g) Set Of Integers, 301 To 350, With 8 Primes Within It = 139 Repeated Even Nos.
(gg) Percentage Increase In Repetition = (139 - 115) ÷ 115 x 100% = 20.87%

(h) Set Of Integers, 351 To 400, With 8 Primes Within It = 172 Repeated Even Nos.
(hh) Percentage Increase In Repetition = (172 - 139) ÷ 139 x 100% = 23.74%

(i) Set Of Integers, 401 To 450, With 9 Primes Within It = 196 Repeated Even Nos.
(ii) Percentage Increase In Repetition = (196 - 172) ÷ 172 x 100% = 13.95%

(j) Set Of Integers, 451 To 500, With 8 Primes Within It = 220 Repeated Even Nos.
(jj) Percentage Increase In Repetition = (220 - 196) ÷ 196 x 100% = 12.24%

(k) Set Of Integers, 501 To 550, With 6 Primes Within It = 247 Repeated Even Nos.
(kk) Percentage Increase In Repetition = (247 - 220) ÷ 220 x 100% = 12.27%

(l) Set Of Integers, 551 To 600, With 8 Primes Within It = 268 Repeated Even Nos.
(ll) Percentage Increase In Repetition = (268 - 247) ÷ 247 x 100% = 8.5%

(m) Set Of Integers, 601 To 650, With 9 Primes Within It = 298 Repeated Even Nos.
(mm) Percentage Increase In Repetition = (298 - 268) ÷ 268 x 100% = 11.19%

(n) Set Of Integers, 651 To 700, With 7 Primes Within It = 320 Repeated Even Nos.
(nn) Percentage Increase In Repetition = (320 - 298) ÷ 298 x 100% = 7.38%

(o) Set Of Integers, 701 To 750, With 7 Primes Within It = 340 Repeated Even Nos.
(oo) Percentage Increase In Repetition = (340 - 320) ÷ 320 x 100% = 6.25%

(p) Set Of Integers, 751 To 800, With 7 Primes Within It = 367 Repeated Even Nos.
(pp) Percentage Increase In Repetition = (367 - 340) ÷ 340 x 100% = 7.94%

(q) Set Of Integers, 801 To 850, With 7 Primes Within It = 392 Repeated Even Nos.
(qq) Percentage Increase In Repetition = (392 - 367) ÷ 367 x 100% = 6.81%

(r) Set Of Integers, 851 To 900, With 8 Primes Within It = 412 Repeated Even Nos.
(rr) Percentage Increase In Repetition = (412 - 392) ÷ 392 x 100% = 5.1%

(s) Set Of Integers, 901 To 950, With 7 Primes Within It = 433 Repeated Even Nos.
(ss) Percentage Increase In Repetition = (433 - 412) ÷ 412 x 100% = 5.1%

(t) Set Of Integers, 951 To 1,000, With 7 Primes Within It = 470 Repeated Even Nos.
(tt) Percentage Increase In Repetition = (470 - 433) ÷ 433 x 100% = 8.55%

(u) Set Of Integers, 1,001 To 1,050, With 8 Primes Within It = 492 Repeated Even Nos.
(uu) Percentage Increase In Repetition = (492 - 470) ÷ 470 x 100% = 4.68%

(v) Set Of Integers, 1,051 To 1,100, With 8 Primes Within It = 523 Repeated Even Nos.
(vv) Percentage Increase In Repetition = (523 - 492) ÷ 492 x 100% = 6.3%

(w) Set Of Integers, 1,101 To 1,150, With 5 Primes Within It = 545 Repeated Even Nos.
(ww) Percentage Increase In Repetition = (545 - 523) ÷ 523 x 100% = 4.21%

(x) Set Of Integers, 1,151 To 1,200, With 7 Primes Within It = 553 Repeated Even Nos.
(xx) Percentage Increase In Repetition = (553 - 545) ÷ 545 x 100% = **1.47%**

(y) Set Of Integers, 1,201 To 1,250, With 8 Primes Within It = **592** Repeated Even Nos.
(yy) Percentage Increase In Repetition = (592 - 553) ÷ 553 x 100% = **7.05%**

It could be seen above that on the whole the No. Of Old/Repeated (Also Appeared Earlier) Even Numbers/Overlaps "Generated" (By The Additions/Combinations Of Two Primes) increases progressively from 20 in (b) to 592 in (y), while it could be seen that the Percentage Increase In Repetition on the whole thins out from 130% in (cc) to 7.05% in (yy), with the lowest percentage increase of 1.47% found in (xx). This statistical trend or feature is not surprising and represents significant evidence that lends support to the validity of the Goldbach conjecture - the infinitude of both the primes and the even numbers implies that the above overlaps increase progressively to infinity.

It is evident here that the higher up the primes we go the more “overwhelmingly” the even numbers “generated” would repeat themselves and overlap. This is significant. Though the infinitude of the prime numbers would ensure that there would always be new even numbers being “generated”, there is also the “fear” that there might be gaps, breaks or lack of continuity in the even numbers thus "generated" wherein some of the even numbers in-between can never be sums of 2 primes (as are shown in the example in Figure 2 above), thereby disproving the Goldbach conjecture. But, it is evident that these more and more profuse repetitions and overlaps of the even numbers thus “generated” by the primes the higher up the infinite list of prime numbers we go “ensure” that such gaps or breaks would not appear between the even numbers “generated” - they “ensure” that the even numbers thus “generated” by the primes in the infinite list of primes would be regular, continuous, *without breaks or gaps*, and, in consecutive running order. This evident greater and greater effusiveness or exuberance of the repetitions and overlaps of the even numbers thus “generated” by the primes the higher up the infinite list of prime numbers we go can be likened to a “play-safe measure” wherein there is “safety derived from large numbers”. In other words, since an even number could be formed in so many ways by adding 2 primes, i.e., so easily formed by adding 2 primes, evidently more so the higher up the infinite list of prime numbers we go, as has been shown above, the sums of 2 primes thus becoming evidently a defining characteristic of the even numbers, we could expect every larger and larger even number to be the sum of 2 primes in more and more ways as has been shown above (and in the appendices below).

We note again that a long, impressive list of consecutive even numbers, from 4 to 12×10^{17}, had already been verified to be sums of 2 primes, and, these partitions/“prime + prime = even number” combinations would become increasingly more “overwhelming”, dense, and repetitive (overlapping) towards infinity (the Goldbach conjecture becoming evidently stronger and stronger the higher up the infinite list of prime numbers/even numbers we go), as is described above. The moot question now is, of course, whether after 12×10^{17} there would be an even number in the infinite list of even numbers which is the last, or, largest, even number that is the sum of 2 primes - this largest even number, if it exists (thereby proving the falsehood of the Goldbach conjecture), must (of necessity) be the sum of 2 primes that are each the largest existing prime. However, as the primes are infinite (vide Euclid’s proof), a largest existing prime is an impossibility. Therefore, there can never be a largest even number comprising of the summation of 2 largest existing primes which would disprove the Goldbach conjecture. As a matter of fact, the infinity of the primes implies that there would be an infinite number of double primes which sum up to an even number.

The Goldbach conjecture is thus valid.

CONCLUDING REMARKS

A number of methods have been adopted in this paper in proving the Goldbach conjecture.

The inductive method, which is a well-established proof, is one of the methods utilized. The following lends support to this inductive proof of the Goldbach conjecture: (a) The characteristic of a mountain or infinite volume of sand is reflected in the characteristic of some grains of sand found there so that studying the characteristic of some grains of sand found there is enough for deducing the characteristic of the mountain or infinite volume of sand, to ascertain the quality of a batch of products it is only necessary to inspect some carefully selected samples from that batch of products and not every one of the products and to carry out a population census, i.e., find out the characteristics of a population, it is only necessary to carry out a survey on some carefully selected respondents and not the whole population; in like manner, by the same principle, we just need to study a carefully selected list of even numbers, find out whether they are all sums of 2 primes and deduce by induction whether all even numbers after this list would also be sums of 2 primes - this act is rather like extrapolation. (For example, a distributed computer search completed in 2008 at the University of Aveiro, Portugal, had confirmed that every even number up to 12×10^{17}, which is no small list of numbers, is the sum of 2 primes. By the principle of induction in this case we could deduce that all the even numbers after 12×10^{17} would also be sums of 2 primes.) (b) Thus, in this way every even number after 2 could be reasonably proved to be the sum of 2 primes. In fact, induction plays an important part in the proof.

The other argument used to prove the conjecture is the indirect (reductio ad absurdum) method, which had been used by Euclid and other mathematicians after him. Logically, 1 or 2 examples of “contradiction” should be sufficient proof of infinity, for it does not make sense to have a need for an infinite number of cases of “contradiction”, as our proof would then have to be infinitely and impossibly long, an absurdity. This method of proof is “proof by implication” as a result of “contradiction” - which is a “short-cut” and smart way in proving infinity, instead of “proving infinity by counting to infinity”, which is ludicrous, and, impossible. Hence, 1 or 2 cases of “contradiction” should be sufficient for implying that there would be an infinitude of even numbers which are sums of 2 primes, which of course also tacitly implies that there would be an infinitude of the number of cases of such “contradiction”. (Euclid evidently had this logical point in mind when he formulated the indirect (reductio ad absurdum) proof of the infinity of the primes.) This method of proof had been cleverly used by a number of mathematicians, not the least by the great German mathematician, David Hilbert. For example, Hilbert had used an indirect method (the “reductio ad absurdum” proof) to prove Gordan’s Theorem without having to show an actual “construction”, a proof which had been accepted by his peers.

One important query here, which many might not have considered: What if the list of prime numbers is not infinite? Of course, if that is the case, the Goldbach conjecture would be false. It would then have been absurd for the Goldbach conjecture to have been conceived at all. However, the list of primes is infinite (vide Euclid’s proof). This gives credence to the Goldbach conjecture.

A very important related point must be highlighted here. If the Goldbach conjecture were indeed false, there must be an ultimate (largest) even number which is (and must necessarily be) the result of the summation of 2 primes that are each the largest existing prime. It must be noted that this is actually an impossibility, as there can never be a largest existing prime - by Euclid's proof, the primes are infinite (refer to argument just above). Hence, the Goldbach conjecture cannot be false, and, by both reductio ad absurdum (contradiction), and, induction (wherein all even numbers up to 12 x 10^{17}, a long, impressive list, had been confirmed to be sums of 2 primes), has to be true.

Another very important point is that the Goldbach conjecture becomes evidently stronger and stronger the higher up the infinite list of prime numbers/even numbers we go, as has been shown above. Thus, by implication, induction, extrapolation, it could be concluded that the Goldbach conjecture is valid - that every even number after 2 is the sum of 2 primes.

So far, there has been no indication or confirmation at all that the number of even numbers after the number 2 which are each the sum of 2 primes is finite and the largest existing even number which is the sum of 2 primes has not been found and confirmed. (This would of course be the case if the Goldbach conjecture is true.) Also, no counter-example (i.e., an even number which is never the sum of 2 primes) has been found so far. On the other hand, practically everyone could intuit that the list of even numbers after the number 2 which are each the sum of 2 primes is infinite. Besides, the evidence, as shown in this paper, is strongly in support of the infinity of this list.

APPENDIX 1

(20) Set Of Integers, 1,201 To 1,250, With 8 Primes Within It

(a) Primes: 1,201; 1,213; 1,217; 1,223; 1,229; 1,231; 1,237 and 1,249

(b) No. Of Primes: 8

(c) No. Of Even Numbers "Generated" (Including Repetitions) By The 8 Primes = 648 (1,204 [1,201 + 3] To 2,498 [1,249 + 1,249])

(d) No. Of New Even Numbers "Generated" = 56 (2,388 To 2,498)

(e) No. Of Old/Repeated (Also Appeared In (19) Above, With Some Also Having Appeared In (18), (17), (16), (15), (14), (13), (12), (11), (10), (9) And (8) Above) Even Numbers "Generated" (I.e., Repetitions/Overlaps) = 592 (1,204 To 2,386)

(f) Density Of New Even Numbers "Generated" = (d) ÷ 8 Primes = 56 ÷ 8 = 7 New Even Numbers Per Prime Number

APPENDIX 2

(8) 8 th. Batch Of 30 Even Numbers (424 To 482) - Partitions/"Prime + Prime = Even Number" Combinations

(a) 424: No. Of Above-mentioned Prime Additions/Combinations = 12

(b) 426: No. Of Above-mentioned Prime Additions/Combinations = 21
(c) 428: No. Of Above-mentioned Prime Additions/Combinations = **9**
(d) 430: No. Of Above-mentioned Prime Additions/Combinations = 14
(e) 432: No. Of Above-mentioned Prime Additions/Combinations = 19
(f) 434: No. Of Above-mentioned Prime Additions/Combinations = 14
(g) 436: No. Of Above-mentioned Prime Additions/Combinations = 11
(h) 438: No. Of Above-mentioned Prime Additions/Combinations = 22
(i) 440: No. Of Above-mentioned Prime Additions/Combinations = 15
(j) 442: No. Of Above-mentioned Prime Additions/Combinations = 13
(k) 444: No. Of Above-mentioned Prime Additions/Combinations = 22
(l) 446: No. Of Above-mentioned Prime Additions/Combinations = 12
(m) 448: No. Of Above-mentioned Prime Additions/Combinations = 13
(n) 450: No. Of Above-mentioned Prime Additions/Combinations = 29
(o) 452: No. Of Above-mentioned Prime Additions/Combinations = 14
(p) 454: No. Of Above-mentioned Prime Additions/Combinations = 12
(q) 456: No. Of Above-mentioned Prime Additions/Combinations = 26
(r) 458: No. Of Above-mentioned Prime Additions/Combinations = **9**
(s) 460: No. Of Above-mentioned Prime Additions/Combinations = 17
(t) 462: No. Of Above-mentioned Prime Additions/Combinations = **30**
(u) 464: No. Of Above-mentioned Prime Additions/Combinations = 13
(v) 466: No. Of Above-mentioned Prime Additions/Combinations = 14
(w) 468: No. Of Above-mentioned Prime Additions/Combinations = 26
(x) 470: No. Of Above-mentioned Prime Additions/Combinations = 16
(y) 472: No. Of Above-mentioned Prime Additions/Combinations = 14
(z) 474: No. Of Above-mentioned Prime Additions/Combinations = 24
(aa) 476: No. Of Above-mentioned Prime Additions/Combinations = 14
(bb) 478: No. Of Above-mentioned Prime Additions/Combinations = 12
(cc) 480: No. Of Above-mentioned Prime Additions/Combinations = **30**
(dd) 482: No. Of Above-mentioned Prime Additions/Combinations = 11
(i) Maximum No. Of Prime Additions/Combinations = 30
(ii) Minimum No. Of Prime Additions/Combinations = 9
(iii) Total No. Of Prime Additions/Combinations For (a) To (dd) = 508
(iv) Total No. Of Even Numbers = 30
(v) Density Of Distribution = Average Prime Additions/Combinations Per Even Number = (iii) ÷ (iv) = 508 ÷ 30 = 16.93 Prime Additions/Combinations Per Even Number

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