ON THE VALIDITY OF THE RIEMANN HYPOTHESIS

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Abstract

In this paper, we have used the partial Euler product to examine the validity of the Riemann Hypothesis. The Dirichlet series with the Mobius function $M(s) = \sum_{n=1}^{\infty} 1/n^s$ has been modified and represented in terms of the partial Euler product by progressively eliminating the numbers that first have a prime factor 2, then 3, then 5, ..up to the prime number p_r to obtain the series $M(s, p_r)$. It is shown that the series $M(s)$ and the new series $M(s, p_r)$ have the same region of convergence for every p_r. Unlike the partial sum of $M(s)$ that has irregular behavior, the partial sum of the new series exhibits regular behavior as p_r approaches infinity. This has allowed the use of integration methods to compute the partial sum of the new series to determine its region of convergence and to provide an answer for the validity of the Riemann Hypothesis.

Keywords: Riemann zeta function, Mobius function, Riemann hypothesis, conditional convergence, Euler product.

Classification: Number Theory, 11M26

1 Introduction

The Riemann zeta function $\zeta(s)$ satisfies the following functional equation over the complex plain [1]

$$\zeta(1-s) = 2(2\pi)^2 \cos(0.5s\pi)\Gamma(s)\zeta(s), \tag{1}$$

where, $s = \sigma + it$ is a complex variable and $s \neq 1$.

For $\sigma > 1$ (or $\Re(s) > 1$), $\zeta(s)$ can be expressed by the following series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2}$$

or by the following product over the primes p_i's

$$\frac{1}{\zeta(s)} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right). \tag{3}$$

where, $p_1 = 2$, $\prod_{i=1}^{\infty}(1 - 1/{p_i}^s)$ is the Euler product and $\prod_{i=1}^{r}(1 - 1/{p_i}^s)$ is the partial Euler product. The above series and product representations of $\zeta(s)$ are absolutely convergent for $\sigma > 1$.

The region of the convergence for the sum in Equation (2) can be extended to $\Re(s) > 0$ by using the alternating series $\eta(s)$ where

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \tag{4}$$

and

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \eta(s). \tag{5}$$

One may notice that the term $1 - 2^{1-s}$ is zero at $s = 1$. This zero cancels the simple pole that $\zeta(s)$ has at $s = 1$ enabling the extension (or analog continuation) of the zeta function series representation over the critical strip $0 < \Re(s) < 1$.

It is well known that all of the non-trivial zeros of $\zeta(s)$ are located in the critical strip $0 < \Re(s) < 1$. Riemann stated that all non-trivial zeros were very probably located on the critical line $\Re(s) = 0.5$ [2]. There are many equivalent statements for the Riemann Hypothesis (RH) and one of them involves the Dirichlet series with the Mobius function.

The Mobius function $\mu(n)$ is defined as follows
$\mu(n) = 1$, if $n = 1$.
$\mu(n) = (-1)^k$, if $n = \prod_{i=1}^{k} p_i$, p_i's are distinct primes.
$\mu(n) = 0$, if $p^2 | n$ for some p.

The Dirichlet series $M(s)$ for the Mobius function is defined as

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(s)}{n^s}. \tag{6}$$

This series is absolutely convergent to $1/\zeta(s)$ for $\Re(s) > 1$ and conditionally convergent to $1/\zeta(s)$ for $\Re(s) = 1$. The Riemann hypothesis is equivalent to the statement that $M(s)$ is conditionally convergent to $1/\zeta(s)$ for $\Re(s) > 0.5$.

Gonek, Hughes and Keating [3] have done an extensive research into establishing a relationship between $\zeta(s)$ and its partial Euler product for $\Re(s) < 1$. Gonek stated "Analytic number theorists believe that an eventual proof of the Riemann Hypothesis must use both the Euler product and functional equation of the zeta-function. For there are functions with similar functional equations but no Euler product, and functions with an Euler product but no functional equation." In section 4, we will present a functional equation for $\zeta(s)$ using its partial Euler product. The method is based on writing the Euler product formula as follows

$$1/\zeta(s) = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right) = \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) \prod_{r+1}^{\infty} \left(1 - \frac{1}{p_i^s}\right).$$

The above equation is valid for $\sigma > 1$. To be able to represent $\zeta(s)$ in term of its partial Euler product for $\sigma \leq 1$, we have to replace the term $\prod_r^{\infty} (1 - 1/p_i^s)$ with an equivalent one that allows the analytic continuation for the representation of $\zeta(s)$ for $\sigma \leq 1$. Thus, the new term, that we need to introduce to replace $\prod_r^{\infty} (1 - 1/p_i^s)$, must have a zero that cancels the pole that $\zeta(s)$ has at $s = 1$. In the section 4, we will use the complex analysis to compute this new term and then represent $\zeta(s)$ in terms of its partial Euler product. In sections (2), (5) and (6), we have introduced an alternative method to compute $\zeta(s)$ in terms of its partial Euler product. This alternative method is based on modifying the Dirichlet series with the Mobius function. The results of these two methods were then analyzed and used to examine the validity of the Riemann Hypothesis

In this paper, we claim the the Riemann Hypothesis is invalid. We support our claim by proving that the series $M(\sigma)$ is divergent for $\sigma < 1$. We have achieved this result by introducing a method to represent the Dirichlet series $M(s)$ (defined by Equation (6)) in terms of the partial Euler product. This task is achieved by first eliminating the numbers that have the prime factor 2 to generate the series $M(s,2)$. For the series $M(s,2)$, we then eliminate the numbers with the prime factor 3 to generate the series $M(s,3)$, and so on, up to the prime number p_r. In essence, in sections 2, we have applied the sieving technique to modify the series $M(s)$ to include only the numbers with prime factors greater than p_r. In the literature, numbers with prime factors less than y are called y-smooth while numbers with prime factors greater than y are called y-rough. In essence, our approach is to compute the Dirichlet series over p_r-rough numbers. In section 3, we have shown that the series $M(s)$ and the new series $M(s, p_r)$ have the same region of convergence.

So far, the efforts to use the series $M(\sigma)$ to examine the validity of the Riemann Hypothesis have failed due to the irregular behavior of the partial sum of the series $M(\sigma)$. In sections 5 and 6, we have shown that the partial sum of the new series $M(\sigma, p_r)$ exhibits regular behavior as p_r approaches infinity. This has allowed the use of integration methods to compute the partial sum of the new series and consequently determine its region of convergence. With this analysis and using the zeta function representation in terms of its partial Euler product (section 4), we have been able to show in section 6 that the series $M(\sigma, p_r)$ and $M(\sigma)$ are divergent for $\sigma < 1$. Thus, non-trivial zeros can be found arbitrary close to the line $s = 1$.

2 Applying the Sieving Method to the Dirichlet Series $M(s)$.

The Dirichlet series $M(s)$ with the Mobius function is defined as

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(s)}{n^s},$$

where $\mu(n)$ is the Mobius function. Thus,

$$M(s) = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{1}{6^s} \cdots.$$

It should be pointed out that our definition of $M(s)$ is different from Mertins function $M(x)$ that is commonly found in the literature and defined as $M(x) = \sum_{1 \le n \le x} \mu(n)$.

Next, we introduce the series $M(s,2)$ by eliminating all the numbers that have a prime factor 2. Thus, $M(s,2)$ can be written as

$$M(s,2) = 1 - \frac{1}{3^s} - \frac{1}{5^s} - \frac{1}{7^s} + \frac{0}{9^s} - \frac{1}{11^s} - \frac{1}{13^s} + \frac{1}{15^s} \cdots.$$

Our analysis to test the conditional convergence of these series ($M(s)$ and $M(s,2)$ for $\sigma \leq 1$) is based on comparing correspondent terms of these two series. Therefore, rearrangement and permutation of the terms may have a significant impact on analyzing the region of convergence of both series. Thus, it essential to have the same index for both series $M(s)$ and $M(s,2)$ refer to the same term. Hence, we will represent $M(s,2)$ as follows

$$M(s,2) = 1 + \frac{0}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{0}{6^s} - \frac{1}{7^s} - \frac{0}{8^s}....,$$

or

$$M(s,2) = \sum_{n=1}^{\infty} \frac{\mu(n,2)}{n^s}, \tag{7}$$

where
$\mu(n,2) = \mu(n)$, if n is an odd number,
$\mu(n,2) = 0$, if n is an even number.

The above series $M(s,2)$ can be further modified by eliminating all the numbers that have a prime factor 3 to get the series $M(s,3)$ where

$$M(s,3) = 1 - \frac{1}{5^s} - \frac{1}{7^s} - \frac{1}{11^s} - \frac{1}{13^s} - \frac{1}{17^s} - \frac{1}{19^s} - \frac{1}{23^s} + \frac{0}{25^s}....,$$

or more conveniently

$$M(s,3) = 1 + \frac{0}{2^s} - \frac{0}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{0}{6^s} - \frac{1}{7^s} - \frac{0}{8^s}....,$$

and so on.

Let $I(p_r)$ represent, in ascending order, the integers with distinct prime factors that belong to the set $\{p_i : p_i > p_r\}$. Let $\{1, I(p_r)\}$ be the set of 1 and $I(p_r)$ (for example, $\{1, I(2)\}$ is the set of square-free odd numbers), then we define the series $M(s, p_r)$ as

$$M(s,p_r) = \sum_{n=1}^{\infty} \frac{\mu(n,p_r)}{n^s}, \tag{8}$$

where
$\mu(n,p_r) = \mu(n)$, if $n \in \{1, I(p_r)\}$,
otherwise, $\mu(n,p_r) = 0$.

It can be easily shown that, for every prime number p_r, the series $M(s,p_r)$ converges absolutely for $\Re(s) > 1$. Furthermore, it can be shown that, for $\Re(s) > 1$, $M(s,p_r)$ satisfies the following equation

$$M(s) = M(s,p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right). \tag{9}$$

Since

$$M(s) = \frac{1}{\zeta(s)} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right),$$

then we conclude that, for $\Re(s) > 1$, $M(s,p_r)$ approaches 1 as p_r approaches infinity.

3 The region of convergence for the series $M(s)$ and $M(s, p_r)$.

In this section, we will deal with the question of the relationship between the conditional convergence of the two series $M(s, p_r)$ and $M(s)$ over the strip $0.5 < \Re(s) \leq 1$. Theorem 1 establishes the the relationship between the conditional convergence of the series $M(s, p_r)$ and $M(s)$ along the real axis (or along the line $0.5 < \sigma \leq 1$) while Theorems 2 establishes the relationship between the conditional convergence of the two series $M(s)$ and $M(s, p_r)$ for $0.5 < \Re(s) \leq 1$.

Theorem 1 *For $s = \sigma + i0$, where $0.5 < \sigma \leq 1$ and for every prime number p_r, the series $M(\sigma)$ converges conditionally if and only if the series $M(\sigma, p_r)$ converges conditionally. Furthermore, $M(\sigma)$ and $M(\sigma, p_r)$ are related as follows*

$$M(\sigma) = M(\sigma, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{\sigma}}\right). \tag{10}$$

The proof of Theorem 1 is outlined in Appendix 1.

Theorem 2 *For $s = \sigma + it$, where $0.5 < \sigma \leq 1$ and for every prime number p_r, the series $M(s)$ converges conditionally if and only if the series $M(s, p_r)$ converges conditionally. Furthermore, $M(s)$ and $M(s, p_r)$ are related as follows*

$$M(s) = M(s, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{s}}\right). \tag{11}$$

The proof of Theorem 2 follows from the fact that $M(s)$ and $M(s, p_r)$ are Dirichlet series. Consequently, the series $M(s)$ is conditionally convergent if and only if the series $M(\sigma)$ is conditionally convergent. Also, the series $M(s, p_r)$ is conditionally convergent if and only if the series $M(\sigma, p_r)$ is conditionally convergent. Using Theorem 1, we then conclude that the series $M(s)$ is conditionally convergent if and only if the series $M(s, p_r)$ is conditionally convergent.

The second part of the theorem can be also proved by first defining $M(s, p_r; N_1, N_2)$ as the partial sum

$$M(s, p_r; N_1, N_2) = \sum_{n=N_1}^{N_2} \frac{\mu(n, p_r)}{n^s}, \tag{12}$$

where $N_2 \geq p_r$. Then, we have

$$M(s, p_{r-1}; 1, Np_r) = M(s, p_r; 1, Np_r) - \frac{1}{p_r^s} M(s, p_r; 1, N). \tag{13}$$

Since the series $M(s, p_r)$ is conditionally convergent, then the partial sums $M(s, p_r; 1, Np_r)$ and $M(s, p_r; 1, N)$ are both convergent to $M(s, p_r)$ as N approaches infinity. Hence, as N approaches infinity, we obtain

$$M(s, p_{r-1}) = \lim_{N\to\infty} M(s, p_{r-1}; 1, Np_r) = M(s, p_r) \left(1 - \frac{1}{p_i^s}\right).$$

By repeating this process $r - 1$ times, we then obtain

$$M(s) = M(s, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right).$$

4 Functional representation of $\zeta(s)$ using its partial Euler product.

Theorems 1 and 2 of the previous section provide a relationship between $\zeta(s) = 1/M(s)$ and the partial Euler product $\prod_{i=1}^{r}(1 - 1/p_i^s)$. In this section, we will use the prime counting function to derive a functional representation for $\zeta(s)$ using its partial Euler product. This functional representation is then used to provide a second relationship between $\zeta(s) = 1/M(s)$ and the partial Euler product $\prod_{i=1}^{r}(1 - 1/p_i^s)$. These two relationships will then analyzed in sections (5) and (6) and later used to examine the validity of the Riemann Hypothesis.

We will start this task by first writing $\zeta(s)$ for $\sigma > 1$ as follows

$$1/\zeta(s) = \prod_{i=1}^{\infty}\left(1 - \frac{1}{p_i^s}\right) = \prod_{i=1}^{r}\left(1 - \frac{1}{p_i^s}\right)\prod_{r+1}^{\infty}\left(1 - \frac{1}{p_i^s}\right). \tag{14}$$

For $\sigma > 0.5$, we have

$$\log\prod_{i=r1}^{r2}\left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r1}^{r2}\log\left(1 - \frac{1}{{p_i}^s}\right),$$

or

$$\log\prod_{i=r1}^{r2}\left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r1}^{r2}\left(-\frac{1}{{p_i}^s} - \frac{1}{2{p_i}^{2s}} - \frac{1}{3{p_i}^{3s}} - ...\right).$$

Let δ be defined as the sum

$$\delta = \sum_{i=r1}^{r2}\left(-\frac{1}{2{p_i}^{2s}} - \frac{1}{3{p_i}^{3s}} - \frac{1}{4{p_i}^{4s}}...\right). \tag{15}$$

Thus,

$$\log\prod_{i=r1}^{r2}\left(1 - \frac{1}{p_i^s}\right) = -\sum_{i=r1}^{r2}\frac{1}{{p_i}^s} + \delta. \tag{16}$$

Since $|\delta| < \sum_{n=p_{r1}}^{\infty}\left(\frac{1}{2n^{2\sigma}} + \frac{1}{3n^{3s}} + \frac{1}{4n^{4s}}...\right)$, thus $\delta = O(p_{r1}^{1-2\sigma}/(2\sigma-1))$. Furthermore, if $2\sigma - 1$ is a fixed positive number, then $\delta = O(p_{r1}^{1-2\sigma})$.

Using the Prime Number Theorem (PNT) with a suitable constant $a > 0$, the number of primes less than x is given by [4, page 43]

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-a\sqrt{\log x}}\right), \tag{17}$$

or

$$\pi(x) = \mathrm{Li}(x) + O\left(x/(\log x)^k\right), \tag{18}$$

where $\mathrm{Li}(x)$ is the Logarithmic Integral of x and k is a number greater than zero.

Using Stieltjes integral [5], we may write the sum $\sum_{i=r1}^{r2}\frac{1}{{p_i}^\sigma}$ for $\sigma > 1$ as follows

$$\sum_{i=r1}^{r2}\frac{1}{{p_i}^\sigma} = \int_{x=p_{r1}}^{p_{r2}}\frac{d\pi(x)}{x^\sigma}. \tag{19}$$

Using Equation (18) for the representation of $\pi(x)$, we may then write the integral in Equation (19) as [5, Theorem 2, page 57]

$$\sum_{i=r1}^{r2} \frac{1}{p_i^\sigma} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} \frac{1}{\log x} dx + O\left(\frac{1}{(\log p_{r1})^k}\right), \tag{20}$$

where k is a number greater than zero. Therefore,

$$\sum_{i=r1}^{r2} \frac{1}{p_i^\sigma} = \int_{p_{r1}}^{\infty} \frac{1}{x^\sigma} \frac{1}{\log x} dx - \int_{p_{r2}}^{\infty} \frac{1}{x^\sigma} \frac{1}{\log x} dx + O\left(\frac{1}{(\log p_{r1})^k}\right). \tag{21}$$

Recalling that the Exponential Integral $E_1(r)$ is given by

$$E_1(r) = \int_r^\infty \frac{e^{-u}}{u} du,$$

and using the substitutions $u = (\sigma - 1)\log p_r$, $du = (\sigma - 1)dx/x$ and $x^\sigma/x = e^u$, then for $\sigma > 1$, we may write Equation (21) as

$$\sum_{i=r1}^{r2} \frac{1}{p_i^\sigma} = E_1\left((\sigma - 1)\log p_{r1}\right) - E_1\left((\sigma - 1)\log p_{r2}\right) + O\left(\frac{1}{(\log p_{r1})^k}\right). \tag{22}$$

Combining Equations (16) and ((22)) and noting that, for $\sigma > 1$, $E_1\left((\sigma - 1)\log p_{r2}\right)$ approaches zero as p_{r2} approaches infinity, we may write Equation (14) for $\sigma > 1$ as

$$-\log\zeta(\sigma) = \sum_{i=1}^{r} \log\left(1 - \frac{1}{{p_i}^\sigma}\right) - \sum_{i=r+1}^{\infty} \frac{1}{{p_i}^\sigma} + \delta,$$

or

$$\log\zeta(\sigma) + \sum_{i=1}^{r} \log\left(1 - \frac{1}{{p_i}^\sigma}\right) - E_1\left((\sigma - 1)\log p_{r+1}\right) = \epsilon,$$

where $\epsilon = O(1/(\log p_{r1})^k)$ is an arbitrarily small number attained by setting p_r sufficiently large. Therefore,

$$\zeta(\sigma) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^\sigma}\right) \exp\left(-E_1((\sigma - 1)\log p_{r+1})\right) = 1 + \epsilon. \tag{23}$$

As p_r approaches infinity, ϵ approaches zero. Hence, the right side of the above equation approaches 1 as p_r approaches infinity.

Similarly, for $\Re(s) > 1$, we can use the following expression for $E_1(s)$

$$E_1(s) = \int_1^\infty \frac{e^{-xs}}{x} dx,$$

to show that

$$\lim_{r\to\infty} \left\{\zeta(s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) \exp\left(-E_1((s - 1)\log p_{r+1})\right)\right\} = 1. \tag{24}$$

Let the function $G(s, p_r)$ be defined as

$$G(s, p_r) = \zeta(s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) \exp\left(-E_1((s-1)\log p_{r+1})\right) \tag{25}$$

where, $G(s, p_r)$ is a regular function for $\Re(s) > 1$. Referring to Equation (24), the function $G(s, p_r)$ approaches 1 as p_r approaches infinity. It should be noted that, for every p_r, the function $\exp\left(-E_1((s-1)\log p_{r+1})\right)$ is an entire function, the function $\zeta(s)$ is analytic everywhere except at $s = 1$ and the function $\prod_{i=1}^{r}(1 - 1/p_i^s)$ is analytic for $\Re(s) > 0$. Thus, for any $\sigma > 1$, the function $G(s, p_r)$ can be considered as a sequence of analytic functions. Furthermore, as p_r (or r) approaches infinity, this sequence is uniformly convergent over the half plane with $\sigma > 1 + \epsilon$ (where, ϵ is an arbitrary small number). Therefore, by the virtue of the Weiestrass theorem, the limit is also analytic function [6] (Weiestrass theorem states that if the function sequence f_n is analytic over the region Ω and f_n is uniformly convergent to a function f, then f is also analytic on Ω and ${f_n}'$ converges uniformly to f' on Ω). If we define this limit as $G(s)$, where

$$G(s) = \lim_{r\to\infty} G(s, p_r) \tag{26}$$

then, $G(s)$ is analytic over the half plane $\Re(s) > 1$ and it is equal to 1 by the virtue of Equation (24).

Next, we will extend the above results to the line $s = 1 + it$. We will then show that if RH is valid, then for the strip $s = \sigma + it$ where, $0.5 < \sigma < 1$, the above results will also be valid with the limit of $G(s, p_r)$ is 1 as p_r approaches infinity.

We will start this task by showing that although both $\zeta(s)$ and $E_1((s-1)\log p_{r+1})$ have a singularity at $s = 1$, the product $G(s, p_r)$ has a removable singularity at $s = 1$ for every p_r. This can be shown by first expanding $\zeta(s)$ as a Laurent series about its singularity at $s = 1$

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \gamma_2\frac{(s-1)^2}{2!} - \gamma_3\frac{(s-1)^3}{3!} + ..., \tag{27}$$

where γ is the Euler-Mascheroni constant and γ_i's are the Stieltjes constants. For $s = 1 + \epsilon$, where $\epsilon = \epsilon_1 + i\epsilon_2$, ϵ_1 and ϵ_2 are arbitrary small numbers, the above equation can be written as

$$\zeta(s) = \frac{1}{\epsilon} + \gamma - \gamma_1\epsilon + \gamma_2\frac{\epsilon^2}{2!} - \gamma_3\frac{\epsilon^3}{3!} + ... \tag{28}$$

Furthermore, for $\sigma > 1$, using the definition of the Exponential Integral, we may write $E_1(s)$ as

$$E_1(s) = -\gamma - \log s + s - \frac{s^2}{2\,2!} + \frac{s^3}{3\,3!} - \frac{s^4}{4\,4!} + ... \tag{29}$$

Thus, for $s = 1 + \epsilon$, we have

$$\exp\left(-E_1((s-1)\log p_r)\right) = e^{\gamma}\epsilon\,\log p_r\,\exp\left(-\epsilon\log p_r + \frac{(\epsilon\log p_r)^2}{2\,2!} - \frac{(\epsilon\log p_r)^3}{3\,3!} + ...\right). \tag{30}$$

By taking the product $\zeta(s)\exp\left(-E_1((s-1)\log p_r)\right)$ and allowing ϵ to approach zero, we then obtain

$$\lim_{s\to 1}\left\{\zeta(s)\,\exp\left(-E_1((s-1)\log p_r)\right)\right\} = e^{\gamma}\,\log p_r. \tag{31}$$

However, it is well known that the partial Euler product at $s = 1$ can be written as [8]

$$\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) = \frac{e^{-\gamma}}{\log p_r} + O\left(\frac{1}{(\log p_r)^2}\right). \tag{32}$$

Multiplying Equations (31) and (32), we may conclude that at $s = 1$, $G(s, p_r)$ approaches 1 as p_r approaches infinity. Furthermore, for $s = 1 + it$ and $t \neq 1$, the value of $\exp(-E_1(it \log p_r))$ approaches 1 as p_r approaches infinity and since

$$\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{{p_i}^s}\right)\right\} = 1,$$

therefore, for $s = 1 + it$, we have the following

$$\lim_{r\to\infty} G(s, p_r) = \lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)\exp\left(-E_1((s-1)\log p_{r+1})\right)\right\} = 1.$$

So far, we have shown that the function $G(s, p_r)$ is uniformly convergent to 1 when $\Re(s) > 1$. We have also shown that $G(s, p_r)$ is convergent to 1 for $\Re(s) = 1$. In the following, we will show that, assuming the validity of the Riemann Hypothesis, the function $G(s, p_r)$ is uniformly convergent to 1 for every value of s with $\Re(s) > 0.5 + \epsilon$, where ϵ is an arbitrary small number. Toward this goal, we will first show that the function $G(s, p_r)$ is convergent for any value of s on the real axis with $\sigma > 0.5$. This can be achieved by first writing the expressions for $G(\sigma, p_{r1})$ and $G(\sigma, p_{r2})$ (where $r2$ is an arbitrary large number greater than $r1$)

$$G(\sigma, p_{r1}) = \zeta(\sigma)\exp\left(-E_1((\sigma-1)\log p_{r1+1})\right)\prod_{i=1}^{r1}\left(1-\frac{1}{p_i^\sigma}\right), \tag{33}$$

$$G(\sigma, p_{r2}) = \zeta(\sigma)\exp\left(-E_1((\sigma-1)\log p_{r2+1})\right)\prod_{i=1}^{r2}\left(1-\frac{1}{p_i^\sigma}\right). \tag{34}$$

Since the function $G(s, p_r)$ is analytic that is not equal to 0 for $\sigma > 0.5$, hence we can divide Equation (34) by Equation (33) and then take the logarithm to obtain

$$\log\left(\frac{G(\sigma, p_{r2})}{G(\sigma, p_{r1})}\right) = E_1\left((\sigma-1)\log p_{r1+1}\right) - E_1\left((\sigma-1)\log p_{r2+1}\right) + \log\left(\prod_{i=r1+1}^{r2}\left(1-\frac{1}{{p_i}^\sigma}\right)\right). \tag{35}$$

To compute the logarithm of the partial Euler product in Equation (35), we recall Equation (16)

$$\log\prod_{r1+1}^{r2}\left(1-\frac{1}{p_i^s}\right) = -\sum_{i=r1+1}^{r2}\frac{1}{{p_i}^s} + \delta,$$

where $\delta = O(p_{r1}^{1-2\sigma}/(2\sigma-1))$. Furthermore, on RH, we have

$$\pi(x) = \mathrm{Li}(x) + O\left(\sqrt{x}\,\log x\right), \tag{36}$$

where $\mathrm{Li}(x)$ is the Logarithmic Integral of x. Using Equation (36) for the representation of the prime counting function, we may then obtain (Appendix 2)

$$\sum_{i=r1+1}^{r2} \frac{1}{{p_i}^\sigma} = E_1((\sigma-1)\log p_{r1+1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon,$$

where $\varepsilon = O\left({p_{r1}}^{0.5-\sigma}\log p_{r1}/(\sigma-0.5)^2\right)$. Hence, Equation (35) can be written as

$$\log\left(\frac{G(\sigma,p_{r2})}{G(\sigma,p_{r1})}\right) = \varepsilon + \delta + E_1((\sigma-1)\log p_{r2}) - E_1((\sigma-1)\log p_{r2+1}).$$

Since $E_1((\sigma-1)\log p_{r2}) - E_1((\sigma-1)\log p_{r2+1})$ approaches zero as p_{r2} approaches zero, thus

$$\lim_{p_{r2}\to\infty} \log\left(\frac{G(\sigma,p_{r2})}{G(\sigma,p_{r1})}\right) = \varepsilon + \delta.$$

For the above equation, it should be pointed that we kept p_{r1} fixed while we allowed p_{r2} to approach infinity. Hence $G(\sigma,p_r)$ is bounded as p_r approaches infinity. Furthermore, for $\sigma > 0.5+\epsilon$, $\varepsilon+\delta$ can be made arbitrary small by choosing p_{r1} arbitrary large, thus the limit of $G(\sigma,p_r)$ exists as p_r approaches infinity and it is given by

$$G(\sigma) = \lim_{r\to\infty} G(\sigma,p_r) \tag{37}$$

This proves that, on RH, $G(\sigma,p_r)$ is convergent as p_r approaches infinity and thus $G(\sigma)$ exists for $\sigma > 0.5$. In Appendix 3, we have shown that, on RH and for $\Re(s) > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon, \tag{38}$$

where $\varepsilon = O\left(\frac{|s|+1}{(\sigma-0.5)^2}\,{p_{r1}}^{0.5-\sigma}\log p_{r1}\right)$. Thus, we can follow the same steps and show that $G(s,p_r)$ is convergent as p_r approaches infinity and thus $G(s)$ exists for $\Re(s) > 0.5$ (it should be pointed out, that the term ε in Equation (38) can be determined in terms of the non-trivial zero if the von Mangoldt function is used in deriving Equation (38) instead of using the prime counting function).

It should be noted that, while the function sequence $G(s,p_r)$ is not uniformly convergent when the region of convergence is extended all the way to the line $\sigma = 0.5$, it is however uniformly convergence for any rectangle extending from $-iT$ to iT (for any arbitrary large T) and with $\sigma > 0.5+\epsilon$, where ϵ is an arbitrary small number. This follows from the fact that, on RH, ε (or, the O term) is bounded for any $\sigma > 0.5+\epsilon$. Since $G(s,p_r)$ is analytic for $\Re(s) > 0$ and it is uniformly convergent for $\Re(s) > 0.5+\epsilon$, thus $G(s)$ is analytic for the half right complex plain with $\Re(s) > 0.5+\epsilon$ (Weiestrass theorem [6]). Since we have shown that $G(s) = 1$ for $\Re(s) \geq 1$, thus on RH, $G(s) = 1$ for $\Re(s) > 0.5+\epsilon$. Hence, we have the following theorem

Theorem 3 *For s = $\sigma + it$ and $\sigma > 0.5$, the following holds if RH is valid*

$$\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)\exp\left(-E_1((s-1)\log p_{r+1})\right)\right\} = 1. \tag{39}$$

$$\lim_{r\to\infty}\left\{M(s,p_r)\exp\left(E_1((s-1)\log p_{r+1})\right)\right\} = 1. \tag{40}$$

It should be also pointed out that Theorem 3 can be generalized to the case where there are no non-trivial zeros for values of s with $\Re(s) > a$ (where $a > 0.5$). For this case, Equation (39) is valid for every s with $\Re(s) > a$ and ε in Appendix 3 is given by $O\left(\frac{|s|+1}{(\sigma-a)^2}\, {p_{r1}}^{a-\sigma} \log p_{r1}\right)$.

Equation (39) of Theorem 3 can be written as follows

$$\log \zeta(s) + \log \prod_{i=1}^{r2} \left(1 - \frac{1}{p_i^s}\right) - E_1\left((s-1)\log p_{r2+1}\right) = 0,$$

where the equality of both sides is attained as r_2 (or p_{r2}) approaches infinity. It should be pointed out that both functions $\log \zeta(s)$ and $E_1((s-1)\log p_{r2+1})$ have a branch cut along the real axis where $0.5 \leq \sigma < 1$, while the difference (i.e. $\log \zeta(s) - E_1((s-1)\log p_{r2+1})$) does not have a branch cut. For $r < r2$, the above equation can be then written as

$$\log \zeta(s) = E_1\left((s-1)\log p_{r2+1}\right) - \sum_{i=1}^{r} \log\left(1 - \frac{1}{p_i^s}\right) - \sum_{i=r+1}^{r2} \log\left(1 - \frac{1}{p_i^s}\right).$$

Since, on RH and for $\Re(s) > 0.5$, (refer to Appendix 3)

$$-\sum_{i=r+1}^{r2} \log\left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r+1}^{r2} \frac{1}{{p_i}^s} + \delta = E_1\left((s-1)\log p_{r+1}\right) - E_1\left((s-1)\log p_{r2}\right) + \varepsilon$$

where $\varepsilon = O\left(\frac{t+1}{(\sigma-0.5)^2}\, {p_r}^{0.5-\sigma} \log p_r\right)$, therefore

$$\log \zeta(s) = -\sum_{i=1}^{r} \log\left(1 - \frac{1}{p_i^s}\right) + E_1\left((s-1)\log p_{r+1}\right) + O\left(\frac{t+1}{(\sigma-0.5)^2}\, {p_r}^{0.5-\sigma} \log p_r\right). \quad (41)$$

Taking the exponential of both side, we then obtain

$$\zeta(s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) = \exp\left(E_1\left((s-1)\log p_{r+1}\right)\right) \left(1 + O\left(\frac{t+1}{(\sigma-0.5)^2}\, {p_r}^{0.5-\sigma} \log p_r\right)\right). \quad (42)$$

or,

$$M(s, p_r) = \exp\left(-E_1\left((s-1)\log p_r\right)\right) \left(1 + O\left(\frac{t+1}{(\sigma-0.5)^2}\, {p_r}^{0.5-\sigma} \log p_r\right)\right). \quad (43)$$

In the following two sections, we will use Theorem 3 and Equation (38) in conjunction with Theorems 1 and 2 to show that the series $M(\sigma, p_r)$ and $M(\sigma)$ diverge for $\sigma < 1$

5 The series $M(\sigma, p_r)$ at $\sigma = 1$.

In this section, we will provide an estimate for the partial sum $M(1, p_r; 1, {p_r}^a)$ as a approaches infinity. This estimate will be computed by using Equation (38) and noting that $M(1, p_r)$ equals zero for every p_r. Therefore, for every p_r, $M(1, p_r; 1, {p_r}^a)$ approaches zero as a approaches infinity. In Appendix 4, we have first shown that the partial sum $M(1, p_r; 1, {p_r}^a)$ is bounded and that for every p_r and N, we have

$$|M(1,p_r;1,N)| = \left|\sum_{n=1}^{N}\frac{\mu(n,p_r)}{n}\right| \leq 2.$$

Before we present the details of our method, it is important to note that the partial sum $M(1,p_r;1,{p_r}^a)$ can be also generated using y-smooth numbers. The y-smooth numbers are the numbers that have only prime factors less than or equal to y. These numbers have been extensively analyzed in the literature [10]. In [10], Granville presented a clever method to generate the partial sum $M(1,p_r;1,{p_r}^a)$. With his method and using the inclusion-exclusion principle [10, page 248], one can then provide an estimate for the partial sum $M(1,p_r;1,{p_r}^a)$. In this section, we will provide a more general approach to compute $M(1,p_r;1,{p_r}^a)$. The main advantage of our approach is the ability to extend it to compute the partial sum for values of s other than 1. We will present our method in the following two steps.

- In the first step of our approach, we will show that, for every a and as p_r approaches infinity, the partial sum $M(1,p_r;1,{p_r}^a)$ is a function of only a (independent of p_r).

Toward this end, we define the function $f(a,p_r)$ as

$$f(a,p_r) = M(1,p_r;1,{p_r}^a) = \sum_{n=1}^{{p_r}^a}\frac{\mu(n,p_r)}{n}.$$

We will then show that, for every a and as p_r approaches infinity, the function $f(a,p_r)$ approaches a deterministic function $\rho(a)$. In other words; if we plot $M(1,p_r;1,N)$ (where $N = {p_r}^a$) as a function of $a = \log N/\log p_r$, then for each value of a and as p_r approaches infinity, $f(a,p_r)$ approaches a unique value $\rho(a)$. This is equivalent to the statement

$$\rho(a) = \lim_{p_r\to\infty} f(a,p_r) = \lim_{p_r\to\infty} M(1,p_r;1,{p_r}^a).$$

This result can be achieved by first noting that the partial sum $M(1,p_r;1,{p_r}^a)$ for $1 < a < 2$ is given by

$$M(1,p_r;1,{p_r}^a) = 1 - \sum_{p_r\leq p_i<{p_r}^a}\frac{1}{p_i}.$$

If we define $M_1(1,p_r;1,{p_r}^a)$ as

$$M_1(1,p_r;p_r,{p_r}^a) = \sum_{p_r\leq p_i<{p_r}^a}\frac{1}{p_i},$$

then, using Stieltjes integral, we obtain

$$M(1,p_r;1,{p_r}^a) = 1 - M_1(1,p_r;p_r,{p_r}^a) = 1 - \int_{p_r}^{{p_r}^a}\frac{d\pi(x)}{x} = 1 - \int_1^a\frac{d\pi({p_r}^y)}{{p_r}^y}.$$

On RH, we have

$$d\pi({p_r}^y) = d\mathrm{Li}({p_r}^y) + dO(\sqrt{{p_r}^y}\log({p_r}^y)),$$

or

$$d\pi(p_r{}^y) = \frac{1}{\log(p_r{}^y)} dp_r{}^y + dO(\sqrt{p_r{}^y}\log(p_r{}^y)) = \frac{p_r^y}{y} dy + dO(\sqrt{p_r{}^y}\log(p_r{}^y)).$$

Hence, for $1 < a < 2$, we have

$$M(1, p_r; 1, p_r{}^a) = 1 - \int_1^a \frac{dy}{y} + \int_1^a \frac{dO(\sqrt{p_r{}^y}\log(p_r{}^y))}{p_r{}^y} = 1 - \log(a) + O(g_1(p_r, a)),$$

where

$$O(g_1(p_r, a)) = \int_1^a \frac{dO(\sqrt{p_r{}^y}\log(p_r{}^y))}{p_r{}^y}.$$

As p_r approaches infinity, $O(g_1(p_r, a)$ approaches zero. Consequently,

$$\lim_{p_r \to \infty} M(1, p_r; 1, p_r{}^a) = 1 - \log a.$$

The terms of the partial sum $M(1, p_r; 1, p_r{}^a)$ in the range $p_r \leq x < p_r{}^3$ are either a reciprocal of a prime or a reciprocal of the product of two primes. Therefore, for $1 < a < 3$, we have

$$M(1, p_r; 1, p_r{}^a) = 1 - \sum_{p_r \leq p_i < p_r{}^a} \frac{1}{p_i} + \sum_{p_r \leq p_{i1} < p_{i2} < p_{i1}p_{i2} < p_r{}^a} \frac{1}{p_{i1}p_{i2}},$$

where p_{i1} and p_{i2} are two distinct primes that are greater than or equal to p_r. Let $M_2(1, p_r; 1, p_r{}^a)$ be defined as

$$M_2(1, p_r; 1, p_r{}^a) = \sum_{p_r \leq p_{i1} < p_{i2} < p_{i1}p_{i2} < p_r{}^a} \frac{1}{p_{i1}p_{i2}} = \frac{1}{2} \sum_{p_r \leq p_i < p_r{}^{a-1}} \frac{1}{p_i} M_1(1, p_r; p_r, p_r^a/p_i) + r_2,$$

where the factor of half was added since each term of the form $1/(p_{i1}p_{i2})$ is repeated twice. It should be also noted that the second sum of the above equation includes non square-free terms (notice that, there is no repetition in any of the non square-free terms). The term r_2 was added to offset the contribution by these non square-free terms. We will show later that the contribution by these terms (or r_2) approaches zero as p_r approaches infinity. Using Stieltjes integral, we then have

$$M_2(1, p_r; 1, p_r{}^a) = \frac{1}{2} \int_1^{a-1} \frac{d\pi(p_r{}^y)}{p_r{}^y} \left(\log(a - y) + O(g_1(p_r, a - y))\right) + r_2.$$

Hence

$$M(1, p_r; 1, p_r{}^a) = 1 - \log(a) + O(g_1(p_r, a)) + \frac{1}{2} \int_1^{a-1} \frac{\log(a - y)}{y} dy + O(g_2(p_r, a - 1)),$$

where

$$O(g_2(p_r, a)) = \int_1^{a-1} \frac{O(g_1(p_r, a - y))}{y} dy + \int_1^{a-1} \log(a - y) \frac{dO(\sqrt{p_r{}^y}\log(p_r{}^y))}{p_r^y} +$$

$$\int_1^{a-1} O(g_1(p_r, a - y)) \frac{dO(\sqrt{p_r{}^y}\log(p_r{}^y))}{p_r^y} + r_2.$$

It can be easily shown that, for any fixed value of a, the three integrals on the right side of the above equation approach zero as p_r approaches infinity. We will also show later that r_2 approaches zero as p_r approaches infinity. Thus, for $1 \leq a < 3$, we have

$$\lim_{p_r \to \infty} M(1, p_r; 1, {p_r}^a) = 1 - \log a + \int_1^{a-1} \frac{\log(a-y)}{y} dy$$

Therefore, as p_r approaches infinity, $M(1, p_r; 1, {p_r}^a)$ is only dependent on a.

Repeating the previous process $\lfloor a \rfloor$ times (where $\lfloor x \rfloor$ is the integer value of x) and by using the induction method, we can show that, as p_r approaches infinity, the partial sum $M(1, p_r; 1, {p_r}^a)$ is dependent on only a. Specifically, we first write the partial sum $M(1, p_r; 1, {p_r}^a)$ as follows

$$M(1, p_r; 1, {p_r}^a) = 1 - M_1(1, p_r; 1, {p_r}^a) + M_2(1, p_r; 1, {p_r}^a) - ... + (-1)^j M_j(1, p_r; 1, {p_r}^a) + ... +$$
$$(-1)^{\lfloor a \rfloor - 1} M_{\lfloor a \rfloor - 1}(1, p_r; 1, {p_r}^a) + (-1)^{\lfloor a \rfloor} M_{\lfloor a \rfloor}(1, p_r; 1, {p_r}^a),$$

where

$$M_j(1, p_r; 1, {p_r}^a) = \sum_{p_r \leq p_{i1} < p_{i2} < .. < p_{ij} < p_{i1} p_{i2} .. p_{ij} < {p_r}^a} \frac{1}{p_{i1} p_{i2} ... p_{ij}}.$$

and $p_{i1}, p_{i2}, ..., p_{ij}$ are j distinct prime numbers greater than or equal to p_r. If we assume that $M_{j-1}(1, p_r; 1, {p_r}^a)$ is given by

$$M_{j-1}(1, p_r; 1, {p_r}^a) = h_{j-1}(a) + O(g_{j-1}(p_r, a))$$

where $h_{j-1}(a)$ is a function of a and $O(g_{j-1}(p_r, a))$ approaches zero as p_r approaches infinity, then

$$M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j} \sum_{p_r \leq p_i < {p_r}^{a-1}} \frac{1}{p_i} M_{j-1}(1, p_r; p_r, p_r^a / p_i) + r_j,$$

where the factor of $1/j$ was added since each term of the form $1/(p_{i1} p_{i2} ... p_{ij})$ is repeated j times. It should be also noted that the sum of the above equation includes non square-free terms. The term r_j was added to offset the contribution by these non square-free terms. We will show later that the contribution by these terms (or r_j) approaches zero as p_r approaches infinity. Using Stieltjes integral, we then have

$$M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j} \int_1^{a-1} \frac{d\pi({p_r}^y)}{{p_r}^y} \left(h_{j-1}(a-y) + O(g_{j-1}(p_r, a-y))\right) + r_j.$$

Hence

$$M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j} \int_1^{a-1} \frac{h_{j-1}(a-y)}{y} dy + O(g_j(p_r, a)),$$

where the first term is a definite integral with only one variable a. Thus, the definite integral is a function of a. We define this function as $h_j(a)$. The second term is given by

$$O(g_j(p_r, a)) = \int_1^{a-1} \frac{O(g_{j-1}(p_r, a-y))}{y} dy + \int_1^{a-1} h_{j-1}(a-y) \frac{dO(\sqrt{{p_r}^y} \log({p_r}^y))}{p_r^y} +$$

$$\int_1^{a-1} O(g_{j-1}(p_r, a-y))\frac{dO(\sqrt{p_r{}^y}\log(p_r{}^y))}{p_r^y} + r_j.$$

It can be easily shown that, for a fixed value of a, the three integrals on the right side of the above equation approach zero as p_r approaches infinity. We will also show later that r_j approaches zero as p_r approaches infinity. Hence, as p_r approaches infinity, we have

$$\lim_{p_r\to\infty} M_j(1, p_r; 1, p_r{}^a) = \frac{1}{j}\int_1^{a-1}\frac{h_{j-1}(a-y)}{y}dy = h_j(a)$$

where $h_1(a) = \log(a)$. Hence, for every a and as p_r approaches infinity, we have

$$\lim_{p_r\to\infty} M(1, p_r; 1, p_r{}^a) = 1 - h_1(a) + h_2(a) - h_3(a) + ... + (-1)^{\lfloor a\rfloor} h_{\lfloor a\rfloor}(a) = \rho(a). \tag{44}$$

It should be pointed out that the above equation implies that the partial sums $M(1, p_r; 1, p_r{}^a)$ and $M(1, p_r^y; 1, p_r{}^{ay})$ (where, p_r^y is a prime number) have the same limit as p_r approaches infinity. Hence,

$$\lim_{p_r\to\infty} M(1, p_r; 1, p_r{}^a) = \lim_{p_r\to\infty} M(1, p_r^y; 1, p_r{}^{ay}) = \rho(a). \tag{45}$$

Equation (45) will be used in the next step to estimate the asymptotic behavior of the function $\rho(a)$ as a approaches infinity.

As it mentioned earlier, the partial sum $M(1, p_r; 1, p_r{}^a)$ constructed by this process included non square-free terms (i.e r_i's). In the following, we will show that, for every a and as p_r approaches infinity, the total contribution by these non square-free terms approaches zero as well. Toward this end, let S_0 be the sum of the terms with the factor $1/p_r^2$. Let S_1 be the sum of the remaining terms with the factor $1/(p_{r+1})^2$, S_2 be the sum of the remaining terms with the factor $1/(p_{r+2})^2$, and so on. Let S be sum of all the terms associated with non square-free terms. Thus, S is given by

$$S = \frac{1}{p_r{}^2}S_0 + \frac{1}{p_{r+1}{}^2}S_1 + ... + \frac{1}{p_{r+l}{}^2}S_L,$$

where p_{r+l} is the largest prime where its square is less than $p_r{}^a$. However,

$$|S_0|, |S_1|, ..., |S_l| < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{p_r{}^a}.$$

Thus,

$$|S_0|, |S_1|, ..., |S_l| = O(a\log p_r).$$

Therefore

$$S = \left(\frac{1}{p_r{}^2} + \frac{1}{p_{r+1}{}^2} + ... + \frac{1}{p_{r+l}{}^2}\right) O(a\log p_r).$$

Hence, the contribution by the non square-free terms R is given by,

$$S = O(a\log p_r/p_r).$$

Consequently, for every a and as p_r approaches infinity, S (or the contribution by the non square-free terms) approaches zero.

- In the second step, we write the partial sum $M(1, p_r; 1, {p_r}^a)$ as the sum of two components. The first one is the deterministic or regular component and it is given by $\rho(a)$. The second one is the irregular component $R(1, p_r; 1, {p_r}^a)$ given by $M(1, p_r; 1, {p_r}^a) - \rho(a)$. We will then show that the function $\rho(a)$ is the Dickman function that has been extensively used to analyze the properties of y-smotht numbers.

Toward this end, we write the partial sum $M(1, p_r; 1, {p_r}^a)$ as the following sum

$$M(1, p_r; 1, {p_r}^a) = 1 - \sum_{p_r \le p_i < {p_r}^{a/2}} \frac{1}{p_i} M(1, p_i; 1, {p_r}^a/p_i) - \sum_{{p_r}^{a/2} \le p_i < {p_r}^a} \frac{1}{p_i}. \tag{46}$$

Notice that the above equation is justified by the virtue that $M(1, p_i; 1, {p_r}^a/p_i)$ is comprised of 1 and the terms of the form $1/n$ where $p_i < n \le p^a/p_i$, Furthermore, every factor of n is greater than p_i. The second sum was added since the first sum is void of the terms $1/p_i$'s for ${p_i}^{a/2} \le p_i \le p_r^a$. Using Stieltjes integral, we can write the above equation as follows

$$M(1, p_r; 1, {p_r}^a) = 1 - \int_1^{a/2} \frac{d\pi({p_r}^y)}{p_r^y} M(1, {p_r}^y; 1, p_r^a/p_r^y) - \int_{a/2}^{a} \frac{d\pi({p_r}^y)}{p_r^y}, \tag{47}$$

where, on RH, $d\pi({p_r}^y)$ is given by $d\text{Li}({p_r}^y) + dO(\sqrt{{p_r}^y}\log({p_r}^y))$. It should pointed out that while Equations (46) and (47) provide the value of the partial sum $M(s, p_r; 1, p_r^a)$ at $s = 1$, they can be easily modified to compute the partial sum for any values of s to the right of the line $\sigma = 0.5$. This task will be achieved in the next section and it will be the key step to examine the validity of the Riemann Hypothesis

As p_r approaches infinity, $M(1, {p_r}^y; 1, p_r^{a-y})$ approaches $\rho(a/y - 1)$ (refer to Equation (45)). Therefore, as p_r approaches infinity, we have

$$\rho(a) = 1 - \int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy - \int_{a/2}^{a} \frac{dy}{y}. \tag{48}$$

It is shown in Appendix 4 that $|M(1, p_r; 1, {p_r}^a)| \le 2$ for every p_r and a. Hence, $|\rho(a)| \le 2$. Consequently, $\rho(a)$ approaches zero as a approaches infinity (this follows from the fact that if $\rho(a)$ does not converge to zero, then the first integral of the above equation diverges as a approaches infinity which then leads to the divergence of $\rho(a)$. This contradicts our earlier statement that $|\rho(a)| \le 2$). Thus, as a approaches infinity, we have

$$\int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy = 1 - \log 2. \tag{49}$$

A key step in our method to examine the validity of the the Riemann Hypothesis is the computation of the rate at which $\rho(a)$ decays to zero. This task will be achieved by using Equation (48) to compute the difference $\rho(a + \Delta a) - \rho(a)$ (where, Δa is an arbitrary small number) to obtain

$$\rho(a+\Delta a) - \rho(a) = -\int_1^{(a+\Delta a)/2} \frac{\rho\left(\frac{a+\Delta a}{y} - 1\right)}{y} dy + \int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy - \int_{(a+\Delta a)/2}^{(a+\Delta a)} \frac{dy}{y} + \int_{a/2}^{a} \frac{dy}{y}.$$

Since the third integral of the above equation is equal to the fourth integral, therefore

$$\rho(a + \Delta a) - \rho(a) = -\int_1^{(a+\Delta a)/2} \frac{\rho\left(\frac{a+\Delta a}{y} - 1\right)}{y} dy + \int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy.$$

If we define $y = (1 + \Delta a/a)z$, then we have

$$\rho(a + \Delta a) - \rho(a) = -\int_{1/(1+\Delta a/a)}^{(a/2)(1+\Delta a/a)/(1+a/\Delta a)} \frac{\rho\left(\frac{a}{z} - 1\right)}{z} dz + \int_{1}^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy.$$

Thus,

$$\rho(a + \Delta a) - \rho(a) = -\int_{1/(1+\Delta a/a)}^{1} \frac{\rho\left(\frac{a}{z} - 1\right)}{z} dz.$$

Dividing both sides of the above equation by Δa and letting Δa approach zero, we then obtain

$$\frac{d\rho(a)}{da} = -\frac{\rho(a-1)}{a}, \tag{50}$$

where $\rho(a) = 1 - \log(a)$ for $1 \le a \le 2$. Equation (50) is a first order non-linear delay differential equation that has been extensively analyzed in the literature [10]. The function $\rho(a)$ is known as the Dickman function. As a approaches infinity, $\rho(a)$ can be given by the following estimate [10]

$$\rho(a) = \left(\frac{e + o(1)}{a \log a}\right)^{a}. \tag{51}$$

For sufficiently large values of a ($a > 20$), we have $\rho(a) < a^{-a}$.

To compute the irregular component of $M(1, p_r; 1, p_r^a)$, we notice that $R(1, p_r; 1, {p_r}^a)$ is given by

$$R(1, p_r; 1, {p_r}^a) = M(1, p_r; 1, p_r^a) - \rho(a).$$

Thus $R(1, p_r; 1, {p_r}^a)$ can be computed by subtracting Equation (48) from Equation (47) to obtain

$$R(1, p_r; 1, {p_r}^a) = \int_{1}^{a/2} \frac{M(1, {p_r}^y; 1, p_r^{a-y})}{p_r^y} dO(\sqrt{{p_r}^y} \log({p_r}^y)) - \int_{1}^{a/2} \frac{R(1, {p_r}^y; 1, p_r^{a-y})}{y} dy - \int_{a/2}^{a} \frac{dO(\sqrt{{p_r}^y} \log({p_r}^y))}{p_r^y}, \tag{52}$$

or

$$R(1, p_r; 1, {p_r}^a) = \int_{1}^{a/2} \frac{\rho\left(a/y - 1\right) + R(1, {p_r}^y; 1, p_r^{a-y})}{p_r^y} dO(\sqrt{{p_r}^y} \log({p_r}^y)) - \int_{1}^{a/2} \frac{R(1, {p_r}^y; 1, p_r^{a-y})}{y} dy - \int_{a/2}^{a} \frac{dO(\sqrt{{p_r}^y} \log({p_r}^y))}{p_r^y}. \tag{53}$$

For sufficiently large a and due to the exponential decay of $\rho(a/y - 1)$ and ${p_r}^{-y}$, the contribution by first integral of the above equation is negligible. Thus, for sufficiently large a, the above equation can be written as follows

$$R(1, p_r; 1, {p_r}^a) = -\int_{1}^{a/2} \frac{R(1, {p_r}^y; 1, p_r^{a-y})}{y} dy - \int_{a/2}^{a} \frac{dO(\sqrt{{p_r}^y} \log({p_r}^y))}{p_r^y}. \tag{54}$$

So far, we have shown that the regular component of $M(1, p_r; 1, p_r^a)$ is given by $\rho(a)$. Since $\int_1^a d\rho(x) = \rho(a) - \rho(1) = \rho(a) - 1$, therefore the regular component of $M(1, p_r; 1, p_r^a)$ can be also written as

$$\rho(a) = \int_1^a d\rho(x) + 1 = 1 + \int_0^a \rho'(x)dx.$$

Similarly, for values of $s \neq 1$, we can consider that $M(s, p_r; 1, p_r^a)$ is comprised of two components. The first component is the regular component defined as $F(a, \alpha)$ (where $\alpha = (s - 1)\log p_r$) and is given by

$$F(a, \alpha) = 1 + \int_1^a {p_r}^{(1-s)x} d\rho(x) = 1 + \int_0^a {p_r}^{(1-s)x} \rho'(x)dx,$$

or,

$$F(a, \alpha) = 1 + \int_0^a e^{-\alpha x} \rho'(x)dx, \tag{55}$$

while the irregular component is given by $M(s, p_r; 1, p_r^a) - F(a, \alpha)$. Notice that for $s = 1$, we have $\alpha = 0$ and $F(a, 0) = \rho(a)$. We now define $F(\alpha)$ as

$$F(\alpha) = \lim_{a \to \infty} F(a, \alpha) = 1 + \int_0^\infty e^{-\alpha x} \rho'(x)dx. \tag{56}$$

Thus, for $\Re(s) \geq 1$, α is a complex variable in the complex plane to the right of the line $\sigma = 1$. Hence, the integral $\int_0^\infty e^{-\alpha x} \rho'(x)dx$ is the Laplace transform of the function $\rho^{'}(x)$ and is given by $F(\alpha) - 1$ (where $F(\alpha)$ is the regular component of the series $M(s, p_r)$, i.e. $M(s, p_r; 1, \infty)$). Since the Laplace transform of $\rho(x)$ is given by $e^{-E_1(s)}/s$ [11, page 569], therefore the Laplace transform of $\rho^{'}(x)$ is then given by $s\mathcal{L}(\rho(x)) - \rho(0)$. Hence

$$F(\alpha) = e^{-E_1(\alpha)}$$

Remarkably, these results agree with what we have obtained in Theorem 3. In Theorem 3, we have shown that

$$\lim_{r \to \infty} \{M(s, p_r) \exp\left(E_1((s-1)\log p_{r+1})\right)\} = 1,$$

or referring to Equation (43), we have

$$M(s, p_r) = e^{-E_1(\alpha)} \left(1 + \epsilon(p_r, s)\right), \tag{57}$$

where $\epsilon(p_r, s) = O\left(\frac{1+t}{(\sigma-0.5)^2} p_r^{1/2-\sigma} \log p_r\right)$. Consequently, for $\Re(s) \geq 1$, we then obtain

$$M(s, p_r) = F(\alpha) \left(1 + \epsilon(p_r, s)\right). \tag{58}$$

where $F(\alpha)$ is the regular component of the series $M(s, p_r)$ and $F(\alpha)\epsilon(p_r, s)$ is the irregular component of the series $M(s, p_r)$. It should be emphasized here that the regular component $F(\alpha)$ is the value of $M(s, p_r)$ due to the $\mathrm{Li}(x)$ component of the prime counting function $\pi(x)$. It is also important to note that the irregular component is not the same as the difference between the partial sum $M(s, p_r; 1, {p_r}^a)$ and the series $M(s, p_r)$.Therefore, except for $s = 1$ (where the irregular component $\epsilon(p_r, s)$ is zero for every p_r), $\epsilon(p_r, s)$ is not zero although it approaches zero as p_r approaches infinity

Notice that on RH, the previous analysis should also hold for $\Re(s) > 0.5$. This analysis and its application to examine the validity of the Riemann Hypothesis will be presented in the next section.

6 The series $M(\sigma, p_r)$ for $\sigma < 1$ and the Riemann Hypothesis.

In this section, we will examine the convergence of the series $M(s, p_r)$ for $0.5 < \Re(s) < 1$ by first computing the partial sum $M(s, p_r; 1, {p_r}^a)$. In the previous section, Equation (46) was used to compute $M(1, p_r; 1, {p_r}^a)$. In this section, we will modify this equation to compute $M(s, p_r; 1, {p_r}^a)$ for $s \neq 1$ as follows

$$M(s, p_r; 1, {p_r}^a) = 1 - \sum_{p_r \leq p_i < {p_r}^{a/2}} \frac{1}{p_i^s} M(s, p_i; 1, {p_r}^a/p_i) - \sum_{{p_r}^{a/2} \leq p_i < {p_r}^a} \frac{1}{p_i^s}. \tag{59}$$

Using Stieltjes integral, we can write the above equation as

$$M(s, p_r; 1, {p_r}^a) = 1 - \int_1^{a/2} \frac{d\pi({p_r}^y)}{p_r^{sy}} M(s, {p_r}^y; 1, p_r^a/p_r^y) - \int_{a/2}^a \frac{d\pi({p_r}^y)}{p_r^{sy}}. \tag{60}$$

On the real axis (i.e. $s = \sigma$), we then have

$$M(\sigma, p_r; 1, {p_r}^a) = 1 - \int_1^{a/2} \frac{d\pi({p_r}^y)}{p_r^{\sigma y}} M(\sigma, {p_r}^y; 1, p_r^{a-y}) - \int_{a/2}^a \frac{d\pi({p_r}^y)}{p_r^{\sigma y}}. \tag{61}$$

Using Theorem 3, on RH, the partial sum $M(\sigma, p_r; 1, {p_r}^a)$ is convergent as a approaches infinity and its value is given by

$$\lim_{a \to \infty} M(\sigma, p_r; 1, {p_r}^a) = M(\sigma, p_r) = \exp\left(-E_1(-\beta)\right)\left(1 + \epsilon(p_r, s)\right), \tag{62}$$

where $\beta = -\alpha = (1 - \sigma) \log p_r$ (note that $\beta > 0$ for $\sigma < 1$).

Furthermore, referring to Appendix 2, the second integral on the right side of Equation (61) is given by

$$\int_{a/2}^a \frac{d\pi({p_r}^y)}{p_r^{\sigma y}} = -E_1(-a\beta) + E_1(-a\beta/2) + \varepsilon,$$

or

$$\int_{a/2}^a \frac{d\pi({p_r}^y)}{p_r^{\sigma y}} < |E_1(-a\beta)| \approx e^{a\beta}/a\beta, \tag{63}$$

where, for $x >> 1$, $-E_1(-x) \approx e^x/x + i\pi$.

Since $d\pi(x) \geq 0$ and $M(\sigma, {p_r}^y; 1, p_r^{a-y}) < 0$ for $1 \leq y < a/2 - \epsilon$ (where ϵ can be made arbitrary small by choosing p_r sufficiently large), therefore

$$\left| \int_1^{a/2} \frac{d\pi({p_r}^y)}{p_r^{\sigma y}} M(\sigma, {p_r}^y; 1, p_r^{a-y}) \right| \geq \left| \int_1^{J} \frac{d\pi({p_r}^y)}{p_r^{\sigma y}} M(\sigma, {p_r}^y; 1, p_r^{a-y}) \right|$$

where $1 \leq J \leq a/3$. Furthermore, the integral on the right side of of the above equation can be decomposed into the following two components

$$\int_1^{J} \frac{d\pi({p_r}^y)}{p_r^{\sigma y}} M(\sigma, {p_r}^y; 1, p_r^{a-y}) = \int_1^{J} \frac{d\mathrm{Li}({p_r}^y)}{p_r^{\sigma y}} M(\sigma, {p_r}^y; 1, p_r^{a-y}) + \int_1^{J} \frac{dO(\sqrt{{p_r}^y}\log({p_r}^y))}{p_r^{\sigma y}} M(\sigma, {p_r}^y; 1, p_r^{a-y}).$$

For sufficiently large p_r, the second integral on the right side of the above equation is negligible compared with the first one, therefore we will be interested in computing only the first integral where

$$\int_1^{J} \frac{d\mathrm{Li}({p_r}^y)}{p_r^{\sigma y}} M(\sigma, {p_r}^y; 1, p_r^{a-y}) = \int_1^{J} \frac{e^{\beta y}}{y} M(\sigma, {p_r}^y; 1, p_r^{a-y}) dy.$$

In Appendix 5, we have shown that for $y < \log a$, we can substitute the value of $\exp(-E_1(-\beta y))$ (or $M(\sigma, {p_r}^y)$) for the partial sum $M(\sigma, {p_r}^y; 1, p_r^{a-y})$

$$\int_1^{\log a} \frac{d\mathrm{Li}({p_r}^y)}{p_r^{\sigma y}} M(\sigma, {p_r}^y; 1, p_r^{a-y}) \approx \int_1^{\log a} \frac{e^{\beta y}}{y} \exp(-E_1(-\beta y)) dy.$$

Hence,

$$\left| \int_1^{a/2} \frac{d\pi({p_r}^y)}{p_r^{\sigma y}} M(\sigma, {p_r}^y; 1, p_r^{a-y}) \right| > |\exp(-E_1(-\beta \log a)| \approx e^{a^\beta / \beta \log a} \tag{64}$$

Assuming the validity of the Riemann Hypothesis, then the left side of Equation (61) should converge as a approaches infinity. However, by the virtue of Equations (63) and (64), the right side of Equation (61) diverges to infinity as a approaches infinity. Therefore, the series $M(\sigma, p_r)$ and $M(\sigma)$ diverge for $\sigma < 1$. This implies that the Riemann Hypothesis is invalid and the zeros can be found arbitrary close to line $\Re(s) = 1$.

Appendix 1

To prove the first part of Theorem 1 (i.e. for $s = \sigma + i0$ and $0.5 < \sigma \leq 1$, the series $M(\sigma, p_r)$ converges conditionally if $M(\sigma)$ converges conditionally), we first start with proving that $M(\sigma, 2)$ is conditionally convergent if $M(\sigma)$ is convergent. Since $M(\sigma)$ is convergent, then for any arbitrary small number δ, there exists an integer N_0 such that for every integer $N > N_0$

$$|M(\sigma; N, \infty)| = \left| \sum_{n=N}^{\infty} \frac{\mu(n)}{n^\sigma} \right| < \delta \tag{65}$$

Let the sums $M(\sigma;1,N)$, $M(\sigma;N+1,2N)$, $M(\sigma;2N+1,2^2N)$, $M(\sigma;2^2N+1,2^3N)$, ..., $M(\sigma;2^{L-1}N+1,2^LN)$ be defined as

$$M(\sigma;1,N)=\sum_{n=1}^{N}\frac{\mu(n)}{n^\sigma}=A_1,$$

$$M(\sigma;N+1,2N)=\sum_{n=N+1}^{2N}\frac{\mu(n)}{n^\sigma}=\delta_1,$$

$$M(\sigma;2N+1,2^2N)=\sum_{n=2N+1}^{2^2N}\frac{\mu(n)}{n^\sigma}=\delta_2,$$

$$M(\sigma;2^2N+1,2^3N)=\sum_{n=2^2N+1}^{2^3N}\frac{\mu(n)}{n^\sigma}=\delta_3,$$

$$M(\sigma;2^{L-1}N+1,2^LN)=\sum_{n=2^{L-1}N+1}^{2^LN}\frac{\mu(n)}{n^\sigma}=\delta_L.$$

Throughout the analysis in this appendix, N will be a fixed number (that is larger than N_0) while the test for the convergence will be achieved by letting L approach infinity.

Let $\delta(l)$ be defined as the maximum of $|\delta_l|,|\delta_{l+1}|,|\delta_{l+2}|,...,|\delta_L|,|\delta_l+\delta_{l+1}|,|\delta_l+\delta_{l+1}+\delta_{l+2}|,...,|\delta_l+\delta_{l+1}+...+\delta_L|$, then by the virtue of the convergence of $M(\sigma)$,

$$|\delta_1|,|\delta_2|,|\delta_3|,...,|\delta_L|,|\delta_1+\delta_2|,|\delta_1+\delta_2+\delta_3|,...,|\delta_1+\delta_2+\delta_3+...+\delta_L|\leq\delta(1)\leq 2\delta.$$

We also have

$$|\delta_l|,|\delta_{l+1}|,|\delta_{l+2}|,...,|\delta_L|,|\delta_l+\delta_{l+1}|,|\delta_l+\delta_{l+1}+\delta_{l+2}|,...,|\delta_l+\delta_{l+1}+...+\delta_L|\leq\delta(l),$$

where by the virtue of the convergence of $M(\sigma)$, $\delta(l)$ can be set arbitrary close to zero (since δ, defined in Equation 65, can be set arbitrary close to zero by setting N_0 arbitrary large).

Furthermore, let the sums $M(\sigma,2;1,N)$, $M(\sigma,2;N+1,2N)$, $M(\sigma,2;2N+1,2^2N)$, $M(\sigma,2;2^2N+1,2^3N)$, ..., $M(\sigma,2;2^{L-1}N+1,2^LN)$ be defined as

$$M(\sigma,2;1,N)=\sum_{n=1}^{N}\frac{\mu(n,2)}{n^\sigma}=B_1,$$

$$M(\sigma,2;N+1,2N)=\sum_{n=N+1}^{2N}\frac{\mu(n,2)}{n^\sigma}=\epsilon_1,$$

$$M(\sigma,2;2N+1,2^2N)=\sum_{n=2N+1}^{2^2N}\frac{\mu(n,2)}{n^\sigma}=\epsilon_2,$$

$$M(\sigma,2;2^2N+1,2^3N)=\sum_{n=2^2N+1}^{2^3N}\frac{\mu(n,2)}{n^\sigma}=\epsilon_3,$$

$$M(\sigma,2;2^{L-1}N+1,2^LN)=\sum_{n=2^{L-1}N+1}^{2^LN}\frac{\mu(n,2)}{n^\sigma}=\epsilon_L,$$

Since

$$\sum_{n=1}^{2N}\frac{\mu(n)}{n^\sigma}=\sum_{n=1}^{2N}\frac{\mu(n,2)}{n^\sigma}-\sum_{n=1}^{N}\frac{\mu(n,2)}{(2n)^\sigma},$$

thus

$$M(\sigma;1,2N)=M(\sigma,2;1,2N)-\frac{1}{2^\sigma}M(\sigma,2;1,N).$$

Similarly, since

$$\sum_{n=2^lN+1}^{2^{l+1}N}\frac{\mu(n)}{n^\sigma}=\sum_{n=2^lN+1}^{2^{l+1}N}\frac{\mu(n,2)}{n^\sigma}-\sum_{n=2^{l-1}N+1}^{2^lN}\frac{\mu(n,2)}{(2n)^\sigma},$$

thus

$$M(\sigma;2^lN+1,2^{l+1}N)=M(\sigma,2;2^lN+1,2^{l+1}N)-\frac{1}{2^\sigma}M(\sigma,2;2^{l-1}N+1,2^lN).$$

Rearranging the previous equations, we then have

$$\begin{aligned}A_1+\delta_1&=B_1+\epsilon_1-\frac{1}{2^\sigma}B_1,\\ \delta_2&=\epsilon_2-\frac{1}{2^\sigma}\epsilon_1,\\ \delta_3&=\epsilon_3-\frac{1}{2^\sigma}\epsilon_2,\\ \\ \delta_L&=\epsilon_L-\frac{1}{2^\sigma}\epsilon_{L-1},\end{aligned}\tag{66}$$

where $|\delta_1|,|\delta_2|,|\delta_3|,...,|\delta_L|,|\delta_1+\delta_2|,|\delta_1+\delta_2+\delta_3|,|\delta_1+\delta_2+\delta_3+...+\delta_L|\leq\delta(1)\leq 2\delta$ and δ can be set arbitrary close to zero. Hence

$$\epsilon_2=\frac{1}{2^\sigma}\epsilon_1+\delta_2,$$

$$\epsilon_3=\frac{1}{2^\sigma}\epsilon_2+\delta_3=\frac{1}{2^{2\sigma}}\epsilon_1+\frac{1}{2^\sigma}\delta_2+\delta_3,$$

$$\epsilon_4=\frac{1}{2^\sigma}\epsilon_3+\delta_4=\frac{1}{2^{3\sigma}}\epsilon_1+\frac{1}{2^{2\sigma}}\delta_2+\frac{1}{2^\sigma}\delta_3+\delta_4,$$

$$\epsilon_L=\frac{1}{2^\sigma}\epsilon_{L-1}+\delta_L=\frac{1}{2^{(L-1)\sigma}}\epsilon_1+\frac{1}{2^{(L-2)\sigma}}\delta_2+\frac{1}{2^{(L-3)\sigma}}\delta_3+...+\delta_L.$$

Therefore,

$$\epsilon_1+\epsilon_2+\epsilon_2+...+\epsilon_L=\left(1+\frac{1}{2^{\sigma}}+\frac{1}{2^{2\sigma}}+...+\frac{1}{2^{(L-1)\sigma}}\right)\epsilon_1+(\delta_2+\delta_3+...+\delta_L)+$$
$$\frac{1}{2^{\sigma}}(\delta_2+\delta_3+...+\delta_{L-1})+\frac{1}{2^{2\sigma}}(\delta_2+\delta_3+...+\delta_{L-2})+...+\frac{1}{2^{(L-2)\sigma}}\delta_2.$$

Since $|\delta_2|\leq\delta(1)$, $|\,|\delta_2+\delta_3|\leq\delta(1)$, ..., $|\delta_1+\delta_2+\delta_3+...+\delta_L|\leq\delta(1)$, hence

$$|\delta_2+\delta_3+...+\delta_L|+\frac{1}{2^{\sigma}}|\delta_2+\delta_3+...+\delta_{L-1}|+...+\frac{1}{2^{(L-2)\sigma}}|\delta_2|\leq\left|\delta(1)+\frac{1}{2^{\sigma}}\delta(1)+...+\frac{1}{2^{(L-2)\sigma}}\delta(1)\right|,$$

or

$$|\delta_2+\delta_3+...+\delta_L|+\frac{1}{2^{\sigma}}|\delta_2+\delta_3+...+\delta_{L-1}|+...+\frac{1}{2^{(L-2)\sigma}}|\delta_2|\leq\frac{2^{\sigma}}{2^{\sigma}-1}|\delta(1)|.$$

Therefore

$$\epsilon_1+\epsilon_2+\epsilon_3+...+\epsilon_L=\left(1+\frac{1}{2^{\sigma}}+\frac{1}{2^{2\sigma}}+...+\frac{1}{2^{L\sigma}}\right)\epsilon_1+\gamma_1,$$

where γ_1 is of the same order as that of $\delta(1)$ (where $\delta(1)$ can be set arbitrary close to zero by setting δ, defined in Equation 65, arbitrary close to zero).

As L approaches infinity, we then obtain

$$\sum_{i=1}^{\infty}\epsilon_i=\frac{2^{\sigma}}{2^{\sigma}-1}\epsilon_1+\gamma_1.$$

Therefore, if the series $M(\sigma)$ is convergent, then the sum $M(\sigma,2;N+1,\infty)$ (which is equal to $\epsilon_1+\epsilon_2+\epsilon_3+...$) is bounded by the sum $M(\sigma,2;N+1,2N)$ (which is equal to ϵ_1).

The previous process can be repeated with the substitution of A_1 and B_1 in Equation (66) with A_2 and B_2, where $A_2=M(\sigma;1,2N)$ and $B_2=M(\sigma,2;1,2N)$, to obtain

$$A_2+\delta_2=B_2+\epsilon_2-\frac{1}{2^{\sigma}}B_2.$$

Thus,

$$A_2=B_2-\frac{1}{2^{\sigma}}B_2+\frac{1}{2^{\sigma}}\epsilon_1.$$

Following the same process, we can show that the sum $M(\sigma,2;2N+1,\infty)$ is given by

$$\sum_{i=2}^{\infty}\epsilon_i=\frac{1}{2^{\sigma}-1}\epsilon_1+\gamma_2.$$

where γ_2 is of the same order as that of $\delta(2)$ (where $\delta(2)$ can be set arbitrary close to zero by setting δ, defined in Equation 65, arbitrary close to zero).

If we repeat the process l times, we obtain

$$A_l=B_l-\frac{1}{2^{\sigma}}B_l+\frac{1}{2^{(l-1)\sigma}}\epsilon_1,$$

where $A_l = M(\sigma; 1, 2^l N)$ and $B_l = M(\sigma, 2; 1, 2^l N)$ and the sum $M(\sigma, 2; 2^l N + 1, \infty)$ is given by

$$\sum_{i=l}^{\infty} \epsilon_i = \frac{1}{2^{(l-2)\sigma}} \frac{1}{2^\sigma - 1} \epsilon_1 + \gamma_l.$$

where γ_l is of the same order as that of $\delta(l)$. Since by the virtue of the convergence of $M(\sigma)$, $\delta(l)$ tends to zero as l approaches infinity, therefore γ_l and the above sum approach zero as l approaches infinity.

Thus, we conclude that $M(\sigma, 2; 2^l N + 1, \infty)$ (given by $\sum_{i=l}^{\infty} \epsilon_i$) approaches zero as l approaches infinity. Furthermore, as l approaches infinity, $B = \lim_{l \to \infty} B_l$ approaches its limit given by

$$\left(1 - \frac{1}{2^\sigma}\right) B = M(\sigma; 1, \infty).$$

Hence,

$$\left(1 - \frac{1}{2^\sigma}\right) M(\sigma, 2) = M(\sigma).$$

Similarly, following the same steps, we can show that

$$\left(1 - \frac{1}{3^\sigma}\right) M(\sigma, 3; 1, \infty) = M(\sigma, 2; 1, \infty).$$

or

$$\left(1 - \frac{1}{2^\sigma}\right) \left(1 - \frac{1}{3^\sigma}\right) M(\sigma, 3; 1, \infty) = M(\sigma; 1, \infty).$$

This task can be achieved by first defining

$$M(\sigma, 2; 1, N) = \sum_{n=1}^{N} \frac{\mu(n,2)}{n^\sigma} = A_1,$$

$$M(\sigma, 2; N+1, 3N) = \sum_{n=N+1}^{3N} \frac{\mu(n,2)}{n^\sigma} = \delta_1,$$

$$M(\sigma, 2; 3N+1, 3^2 N) = \sum_{n=3N+1}^{3^2 N} \frac{\mu(n,2)}{n^\sigma} = \delta_2,$$

$$M(\sigma, 2; 3^{L-1} N + 1, 3^L N) = \sum_{n=3^{L-1} N+1}^{3^L N} \frac{\mu(n,2)}{n^\sigma} = \delta_L,$$

and

$$M(\sigma, 3; 1, N) = \sum_{n=1}^{N} \frac{\mu(n,3)}{n^\sigma} = B_1,$$

$$M(\sigma, 3; N+1, 3N) = \sum_{n=N+1}^{3N} \frac{\mu(n,3)}{n^{\sigma}} = \epsilon_1,$$

$$M(\sigma, 3; 3N+1, 3^2N) = \sum_{n=3N+1}^{3^2N} \frac{\mu(n,3)}{n^{\sigma}} = \epsilon_2,$$

$$M(\sigma, 3; 3^{L-1}N+1, 3^LN) = \sum_{n=3^{L-1}N+1}^{3^LN} \frac{\mu(n,3)}{n^{\sigma}} = \epsilon_L,$$

Since

$$\sum_{n=1}^{3N} \frac{\mu(n,2)}{n^{\sigma}} = \sum_{n=1}^{3N} \frac{\mu(n,3)}{n^{\sigma}} - \sum_{n=1}^{N} \frac{\mu(n,3)}{(3n)^{\sigma}},$$

thus

$$M(\sigma, 2; 1, 3N) = M(\sigma, 3; 1, 3N) - \frac{1}{3^{\sigma}} M(\sigma, 3; 1, N)$$

Similarly,

$$M(\sigma, 2; 3^lN+1, 3^{l+1}N) = M(\sigma, 3; 3^lN+1, 3^{l+1}N) - \frac{1}{3^{\sigma}} M(\sigma, 3; 3^{l-1}N+1, 3^lN)$$

Following the same process, we can show that

$$\sum_{i=1}^{\infty} \epsilon_i = \frac{3^{\sigma}}{3^{\sigma}-1}\epsilon_1 + \gamma_1,$$

where γ_1 is of the same order as that of $\delta(1)$ ($\delta(l)$ is defined as the maximum of $|\delta_l|, |\delta_{l+1}|, |\delta_{l+2}|, ..., |\delta_L|, |\delta_l + \delta_{l+1}|, |\delta_l + \delta_{l+1} + \delta_{l+2}|, ..., |\delta_l + \delta_{l+1} + ... + \delta_L|$).

Similarly, if we define $A_2 = M(\sigma, 2; 1, 3N)$ and $B_2 = M(\sigma, 3; 1, 3N)$, then

$$A_2 = B_2 - \frac{1}{3^{\sigma}} B_2 + \frac{1}{3^{\sigma}} \epsilon_1.$$

Therefore

$$\sum_{i=2}^{\infty} \epsilon_i = \frac{1}{3^{\sigma}-1}\epsilon_1 + \gamma_2.$$

where γ_2 is of the same order as that of $\delta(2)$.

Repeating the steps l times, we then obtain

$$\sum_{i=l}^{\infty} \epsilon_i = \frac{1}{3^{(l-2)\sigma}} \frac{1}{3^{\sigma}-1}\epsilon_1 + \gamma_l.$$

where γ_l is of the same order as that of $\delta(l)$. Hence the above sum approaches zero as l approaches infinity

Thus, we conclude that $M(\sigma, 3; 3^l N + 1, \infty)$ (given by $\sum_{i=l}^{\infty} \epsilon_i$) approaches zero as l approaches infinity. Furthermore, as l approaches infinity, $B = \lim_{l \to \infty} B_l$ approaches its limit given by

$$\left(1 - \frac{1}{3^\sigma}\right) B = M(\sigma, 2; 1, \infty).$$

Hence,

$$\left(1 - \frac{1}{3^\sigma}\right) M(\sigma, 3) = M(\sigma, 2).$$

Repeating the process r times, we then conclude

$$M(\sigma) = M(\sigma, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{{p_i}^\sigma}\right).$$

The second part of the theorem can be proved by recalling

$$M(s, p_{r-1}; 1, Np_r) = M(s, p_r; 1, Np_r) - \frac{1}{p_r^s} M(s, p_r; 1, N).$$

If both series $M(s, p_{r-1})$ and $M(s, p_r)$ are convergent, then as N approaches infinity, we obtain

$$M(s, p_{r-1}) = M(s, p_r) \left(1 - \frac{1}{p_i^s}\right).$$

Repeating the process r times, we then conclude

$$M(\sigma) = M(\sigma, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{{p_i}^\sigma}\right).$$

Appendix 2

Assuming RH is valid and for $\sigma > 0.5$, to show that

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^\sigma} = E_1((\sigma - 1) \log p_{r1}) - E_1((\sigma - 1) \log p_{r2}) + \varepsilon$$

where, $\varepsilon = O\left(\frac{t}{(\sigma - 0.5)^2} {p_{r1}}^{1/2-\sigma} \log p_{r1}\right)$, we first recall that

$$\sum_{i=r1}^{r2} \frac{1}{p_i^\sigma} = \int_{p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^\sigma} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx + \int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dO\left(\sqrt{x} \log x\right).$$

We will first compute the integral with the O notation. This can be done by integration by parts to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dO\left(\sqrt{x}\log x\right) = \frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{{p_{r2}}^\sigma} - \frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{{p_{r1}}^\sigma} - \int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^\sigma}\right)$$

Since $x > 0$, thus

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dO\left(\sqrt{x}\log x\right) = \frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{{p_{r2}}^\sigma} - \frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{{p_{r1}}^\sigma} - O\left(\int_{p_{r1}}^{p_{r2}} \sqrt{x}\log x \, d\left(\frac{1}{x^\sigma}\right)\right)$$

With the substitution of variables $y = \log x$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \sqrt{x}\log x \, d\left(\frac{1}{x^\sigma}\right) = -\int_{p_{r1}}^{p_{r2}} \sigma y e^{(\frac{1}{2}-\sigma)y} dy.$$

Since

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax},$$

therefore

$$\int_{p_{r1}}^{p_{r2}} \sqrt{x}\log x \, d\left(\frac{1}{x^\sigma}\right) = -\sigma\left(\frac{\log p_{r2}}{0.5-\sigma} - \frac{1}{(0.5-\sigma)^2}\right) {p_{r2}}^{0.5-\sigma} + \sigma\left(\frac{\log p_{r1}}{0.5-\sigma} - \frac{1}{(0.5-\sigma)^2}\right) {p_{r1}}^{0.5-\sigma}.$$

Hence, for $\sigma > 0.5$, we have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dO\left(\sqrt{x}\log x\right) = O\left(\frac{{p_{r1}}^{0.5-\sigma}\log p_{r1}}{(\sigma-0.5)^2}\right) \tag{67}$$

For $\sigma \geq 1$, the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx$ can be computed directly from the definition of the Exponential Integral $E_1(r) = \int_r^\infty \frac{e^{-u}}{u} du$ (where $r \geq 0$) to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2})$$

It should be pointed out that although the functions $E_1((\sigma-1)\log p_{r1})$ and $E_1((\sigma-1)\log p_{r2})$ have a singularity at $\sigma = 1$, the difference has a removable singularity at $\sigma = 1$. This follows from the fact that as σ approaches 1, the difference can be written as

$$E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) = -\log\left((1-\sigma)\log p_{r1}\right) - \gamma + \log\left((1-\sigma)\log p_{r2}\right) + \gamma$$

or,

$$\lim_{\sigma\to 1} \int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \lim_{\sigma\to 1} E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) = -\log\log p_{r1} + \log\log p_{r2}$$

To compute the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx$ for $\sigma < 0$, we first use the substantiation $y = \log x$ to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy = \int_{\epsilon}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy - \int_{\epsilon}^{\log p_{r1}} \frac{e^{(1-\sigma)y}}{y} dy$$

where, ϵ is an arbitrary small positive number. With the variable substantiations $z_1 = y/\log p_{r1}$ and $z_2 = y/\log p_{r2}$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2} dz_2 - \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1} dz_1.$$

With the variable substantiations $w_1 = (1-\sigma)(\log p_{r1})z_1$ and $w_2 = (1-\sigma)(\log p_{r2})z_1$ and by adding and subtracting the terms $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}$, we then have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{e^{w_2}-1}{w_2} dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{e^{w_1}-1}{w_1} dw_1 +$$
$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}.$$

Using the following identity [9, page 230]

$$\int_0^a \frac{e^t-1}{t} dt = -E_1(-a) - \log(a) - \gamma$$

where $a > 0$, we then obtain for $\sigma < 1$,

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2})$$

Hence, for $\sigma > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^\sigma} = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon$$

It should be pointed out that in general, if there are no non-trivial zeros for values of s with $\Re(s) > a$, then by following the same steps, we may also show that for $\sigma > a$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^\sigma} = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon$$

where, $\varepsilon = O\left({p_{r1}}^{a-\sigma} \log p_{r1}/(\sigma-a)^2\right)$.

Appendix 3

Assuming RH is valid and for $\sigma > 0.5$, to show that

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon$$

where, $\varepsilon = O\left(\frac{|s|+1}{(\sigma-0.5)^2} {p_{r1}}^{1/2-\sigma} \log p_{r1}\right)$, we first recall that

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} = \int_{p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^s} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx + \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dO\left(\sqrt{x}\log x\right).$$

We will first compute the integral with the O notation. This can be done by integration by parts to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dO\left(\sqrt{x}\log x\right) = \frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{{p_{r2}}^s} - \frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{{p_{r1}}^s} - \int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^s}\right)$$

The integral on the right side of the above equation can be then written as

$$\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^s}\right) = -s\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) x^{-s-1}dx.$$

Hence,

$$\left|\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^s}\right)\right| \leq |s|\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) |x^{-s-1}|dx.$$

Consequently,

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dO\left(\sqrt{x}\log x\right) = O\left((|s|+1)\ \frac{{p_{r1}}^{0.5-\sigma}\log p_{r1}}{(\sigma-0.5)^2}\right).$$

For $\Re(s) \geq 1$, the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s\log x}dx$ can be computed directly from the definition of the Exponential Integral $E_1(z) = \int_1^{\infty} \frac{e^{-tz}}{t}dt$ (where $\Re(z) \geq 0$) to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s\log x}dx = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2})$$

To compute the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s\log x}dx$ for $\Re(z) < 1$, we first write the integral as follows

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s\log x}dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x}dx - i\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\sin(t\log x)}{\log x}dx.$$

The first integral on the right side $\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x}dx$ can be computed by using the substitution $y = \log x$ to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x}dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y}\cos(ty)}{y}dy,$$

or

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x}dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y}\cos(ty)}{y}dy + \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y}dy - \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y}dy.$$

Hence,

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x}dx = \int_{\epsilon}^{p_{r1}} \frac{e^{(1-\sigma)y}(1-\cos(ty))}{y}dy - \int_{\epsilon}^{p_{r2}} \frac{e^{(1-\sigma)y}(1-\cos(ty))}{y}dy -$$

$$\int_{\epsilon}^{p_{r1}} \frac{e^{(1-\sigma)y}}{y}dy + \int_{\epsilon}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y}dy$$

where, ϵ is an arbitrary small positive number. With the variable substantiations $z_1 = y/\log p_{r1}$ and $z_2 = y/\log p_{r2}$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1}) z_1} (1 - \cos(t(\log p_{r1}) z_1))}{z_1} dz_1 -$$

$$\int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2}) z_2} (1 - \cos(t(\log p_{r2}) z_2))}{z_2} dz_2 -$$

$$\int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1}) z_1}}{z_1} dz_1 + \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2}) z_2}}{z_2} dz_2$$

By the virtue of the following identity ([9], page 230)

$$\int_0^1 \frac{e^{at}(1 - \cos(bt))}{t} dt = \frac{1}{2} \log(1 + b^2/a^2) + \mathrm{Li}(a) + \Re[E_1(-a + ib)],$$

where $a > 0$, we then obtain the following

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \Re[E_1((s-1)\log p_{r1})] + \mathrm{Li}((1-\sigma)\log p_{r1}) -$$

$$\Re[E_1((s-1)\log p_{r2})] - \mathrm{Li}((1-\sigma)\log p_{r2}) -$$

$$\int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1}) z_1}}{z_1} dz_1 + \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2}) z_2}}{z_2} dz_2$$

With the variable substantiations $w_1 = (1-\sigma)(\log p_{r1}) z_1$ and $w_1 = (1-\sigma)(\log p_{r1}) z_1$ and by adding and subtracting the terms $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}$, we then have

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \Re[E_1((s-1)\log p_{r1})] + \mathrm{Li}((1-\sigma)\log p_{r1}) -$$

$$\Re[E_1((s-1)\log p_{r2})] - \mathrm{Li}((1-\sigma)\log p_{r2}) +$$

$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{e^{w_2} - 1}{w_2} dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{e^{w_1} - 1}{w_1} dw_1 +$$

$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}.$$

Using the following identity [9, page 230]

$$\int_0^a \frac{e^t - 1}{t} dt = \mathrm{Ei}(a) - \log(a) - \gamma$$

where $a > 0$, we then obtain for $\sigma < 1$,

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \Re[E_1((s-1)\log p_{r1})] - \Re[E_1((s-1)\log p_{r2})]$$

Similarly, using the identity [9, page 230]

$$\int_{p_0}^{1} \frac{e^{at}\sin(bt)}{t}dt = \pi - \arctan(b/a) + \Im[E_1(-a+ib)],$$

where $a > 0$, we can show that for $\sigma < 1$, we have

$$-\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x}\sin(t\log x)}{\log x}dx = \Im[E_1((s-1)\log p_{r1})] - \Im[E_1((s-1)\log p_{r2})].$$

Therefore, for $\Re(s) > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon$$

where, $\varepsilon = O\left(\frac{|s|+1}{(\sigma-0.5)^2}{p_{r1}}^{1/2-\sigma}\log p_{r1}\right)$.

Appendix 4

To show that

$$\left|\sum_{n=1}^{N} \frac{\mu(n,p_r)}{n}\right| \leq 2$$

we first note that

$\sum_{d/n}\mu(d,p_r) = 1$, if $n = 1$,
$\sum_{d/n}\mu(d,p_r) = 1$, if all the prime factors of n are less than p_r,
$\sum_{d/n}\mu(d,p_r) = 0$, if any of the prime factors of n is greater than p_r.

Adding all the terms $\sum_{d/n}\mu(d,p_r)$ for $1 \leq n \leq N$, we then obtain

$$0 < \sum_{n=1}^{N} \mu(n,p_r)\left\lfloor\frac{N}{n}\right\rfloor \leq N,$$

where $\lfloor x \rfloor$ refers to the integer value of x. Define r_n as

$$r_n = \frac{N}{n} - \left\lfloor\frac{N}{n}\right\rfloor,$$

where $0 \leq r_n < 1$. Hence, we have

$$\sum_{n=1}^{N} \mu(n,p_r)r_n < \sum_{n=1}^{N} \mu(n,p_r)\left\lfloor\frac{N}{n}\right\rfloor + \sum_{n=1}^{N} \mu(n,p_r)r_n \leq \sum_{n=1}^{N} \mu(n,p_r)r_n$$

Since

$$-N \leq \sum_{n=1}^{N} \mu(n,p_r)r_n \leq N,$$

thus, for every p_r we have

$$-N < \sum_{n=1}^{N} \mu(n, p_r) \frac{N}{n} \leq 2N,$$

or

$$-1 < \sum_{n=1}^{N} \frac{\mu(n, p_r)}{n} \leq 2.$$

Appendix 5

For $\sigma < 1$ and on RH, to show that

$$\int_1^{\log a} \frac{d\text{Li}({p_r}^y)}{p_r^{\sigma y}} M(\sigma, {p_r}^y; 1, p_r^{a-y}) \approx \int_1^{\log a} \frac{e^{\beta y}}{y} \exp(-E_1(-\beta y)) dy,$$

we first recall that

$$M(\sigma, p_r; 1, p_r^a) = F(a, (\sigma - 1) \log p_r) + R(\sigma, p_r; 1, p_r^a),$$

and

$$M(\sigma, {p_r}^y; 1, p_r^{a-y}) = F(a/y - 1, (\sigma - 1) p_r^y)) + R(\sigma, {p_r}^y; 1, p_r^{a-y}),$$

where $F(a, (\sigma - 1) \log p_r)$ is the regular component of $M(\sigma, p_r; 1, p_r^a)$ and it is given by

$$F(a, (\sigma - 1) \log p_r) = 1 + \int_0^a \rho'(x) e^{x(1-\sigma) \log p_r} dx,$$

or,

$$F(a, (\sigma - 1) \log p_r) = 1 + \int_0^a \frac{\rho(x - 1)}{x} e^{x(1-\sigma) \log p_r} dx.$$

To achieve our objective, we first need to show that as a approaches infinity, the difference $F((\sigma - 1) \log p_r^y)) - F(a/y - 1, (\sigma - 1) \log p_r^y))$ is negligible compared with $F((\sigma - 1) p_r^y))$ for values of y in the range $1 \leq y \leq \log a$. If we choose $b = (1 - \sigma) \log p_r$ in the range $0 < b < 1$, then it can be easily shown that as a approaches infinity, the difference $F((\sigma - 1) \log p_r^y)) - F(a/y - 1, (\sigma - 1) \log p_r^y))$ approaches zero for values of y in the range $1 \leq y \leq \log a$. For $y = \log a$, this difference is given by

$$F((\sigma - 1) \log p_r^{\log a})) - F(a/\log a - 1, (\sigma - 1) \log p_r^{\log a})) = \int_{a/\log a - 1}^{\infty} \frac{\rho(x - 1)}{x} e^{x(1-\sigma) \log {p_r}^{\log a}} dx.$$

For sufficiently large x, $\rho(x) < x^{-x}$. Hence

$$\int_{a/\log a - 1}^{\infty} \frac{\rho(x - 1)}{x} e^{x(1-\sigma) \log {p_r}^{\log a}} dx < \int_{a/\log a - 1}^{\infty} \frac{e^{-(x-1) \log(x-1) + x(1-\sigma) \log {p_r}^{\log a}}}{x} dx.$$

For $b = (1 - \sigma) \log p_r < 1$ and for sufficiently large a, we then have

$$F((\sigma-1)\log p_r^{\log a})) - F(a/\log a - 1, (\sigma-1)\log p_r^{\log a})) < \int_{a/\log a-1}^{\infty} \frac{e^{-x(1-b)}}{x} dx.$$

Thus, as a approaches infinity, the above difference approaches zero. Since for $\sigma < 1$, $F((\sigma - 1)\log p_r^{\log a}))$ approaches infinity as a approaches infinity, thus $F(a/y - 1, (\sigma - 1)p_r^y))$ can be given by $F((\sigma - 1)p_r^y))$ for values of y in the range $1 \leq y \leq \log a$. In other words; for $1 \leq y \leq \log a$, we have

$$M(\sigma, {p_r}^y; 1, p_r^{a-y}) = F((\sigma-1)\log p_r^y)) + R(\sigma, {p_r}^y; 1, p_r^{a-y})$$

Furthermore, $M(\sigma, p_r; 1, p_r^a)$ can be written as

$$M(\sigma, p_r; 1, p_r^a) = \int_1^a \frac{p_r}{p_r^{\sigma y}} M'(1, p_r; 1, p_r^y).$$

Hence,

$$R(\sigma, p_r; 1, p_r^a) = \int_1^a p_r^{(1-\sigma)y} R'(1, p_r; 1, p_r^y).$$

Hence, on RH, $R(1, p_r; 1, p_r^a)$ should decay faster than $1/p_r^{a/2}$ and $R(\sigma, p_r; 1, p_r^a)$ is negligible compared to $F((\sigma - 1)\log p_r^y))$.

Therefore, on RH and for $y < \log a$, we have as a approaches infinity

$$\lim_{p_r \to \infty} \frac{F((\sigma-1)\log p_r^y))}{M(\sigma, {p_r}^y; 1, p_r^{a-y})} = 1$$

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