Can the efficiency of an arbitrary reversible cycle be equal to the efficiency of the enclosing Carnot cycle? Part – A

P. Radhakrishnamurty
282, Duo Marvel Layout, Ananthapuragate, Yelahanka,
Bangalore -560064, India.E-Mail: padyala1941@yahoo.com

Abstract

One of the major issues that remained controversial in classical thermodynamics is resolved. The issue is: Is it possible for the efficiency of an arbitrary reversible heat engine cycle to be equal to the efficiency of the enclosing Carnot cycle? Taking the simplest case of a reversible cycle that involves heat interactions at three different temperatures, we demonstrate that the answer is in the affirmative. We also show that if it is possible for the efficiency of an arbitrary reversible cycle to be lower than the efficiency of the enclosing Carnot cycle, then it is also possible for the efficiency of an arbitrary reversible cycle to be greater than the efficiency of the enclosing Carnot cycle. If the later is impossible, the former, too, is impossible. The later, however, is impossible according to Carnot's corollary. Therefore, inequality of efficiencies of the two cycles is impossible. The only option left is equality of their efficiencies.

Keywords: Thermodynamics, Reversible cycle efficiency, Carnot cycle efficiency

1. Introduction

Ever since the efficiency of Carnot heat engine cycle (Carnot cycle) is defined in such a way that its value is less than one, troubles started[1]. Efficiency of arbitrary reversible cycles is an issue discussed in physics and chemistry education journals from time to time [1-5]. Arbitrary reversible cycles can be considered as combinations of Carnot cycles. Such combinations of Carnot cycles are used to demonstrate many important results in thermodynamics. A few among such results are, for example: (1) the demonstrations of Clausius' theorem that entropy of an arbitrary reversible cyclic process is zero [6], (2) the development of the concept of Kelvin temperature scale [7], and (3) the demonstrations that Carnot heat engine has the maximum possible efficiency, of all heat engines interacting with the same pair of heat reservoirs (HRs) at temperatures T_H and T_L [8]. When applied to combinations of Carnot cycles, the definition of efficiency leads to many problems. One such problem is connected with the question: Is it possible for the efficiency of an arbitrary reversible cycle to be equal to the value of efficiency of the enclosing Carnot cycle[2]? We address this question here in part A. Another problem with efficiency of reversible cycles is connected with the question: Do different reversible cycles operating between maximum and minimum temperatures T_H and T_L respectively, have different values of efficiency? We address this question in part B using Fermi's method [8].

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[1] In fact, such a definition forced itself as a necessary consequence of rejecting the caloric theory of heat, according to which heat was a conserved quantity.

[2] An enclosing Carnot cycle is one which brings about a loss of heat in HR at T_H and a gain of heat in HR at T_L (T_H and T_L being the highest and the lowest temperatures of heat interaction between the system and the surroundings in the given arbitrary cycle).

2. n-T Cycle

A heat engine transforms heat into mechanical work. It works around a cycle. In general, an arbitrary reversible heat engine cycle involves heat interaction between the system and the surroundings at two or n (>2) temperatures. When n = 2 we call it Carnot cycle and when n > 2, we call it n-T cycle. We depict the cyclic processes pictorially using temperature-entropy (T-S) diagrams instead of the usual pressure-volume (P-V) diagrams, since the former are easier to draw. A typical 5-T cycle is shown in figure 1.

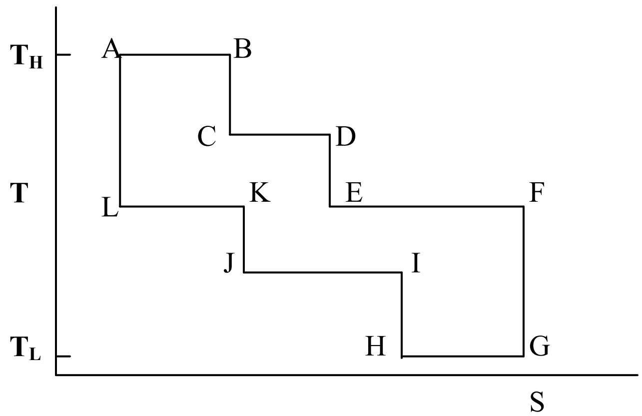

Fig. 1. T-S Diagram of a n-T reversible cycle.

A 2-T cycle (n = 2) is the well known Carnot cycle. A 3-T cycle is the simplest of non-Carnot cycles. It is important to note that, though heat interaction occurs at n (>2) temperatures during a cycle, it is possible in principle, that HRs at (n-1), ((n-1) ≥ 2) different temperatures only suffer change; the other HRs suffer no change as their gains and losses of heat being equal. For example, heat interaction may occur during a cycle at three different temperatures but it is possible in principle for HRs at two temperatures only suffer change. Similarly heat interaction may occur at four different temperatures but it is possible in principle for HRs at two or three temperatures only suffer change.

3. Efficiency of an arbitrary reversible heat engine cycle

According to the standard practice, the efficiency η_R, of an arbitrary n-T reversible cycle is defined [1] by the equations (1):

$$\eta_R = \frac{W}{Q_{in}} = \frac{Q_{in} - Q_{out}}{Q_{in}} = \frac{\int dQ_{in} - \int dQ_{out}}{\int dQ_{in}} = 1 - \frac{\int dQ_{out}}{\int dQ_{in}} \qquad (1)$$

Where, Q_{in}, W and Q_{out} are respectively, the heat input, work output, and heat rejected, per cycle.

Literature study [1-4] shows that the efficiency of such an n-T reversible cycle is lower than the efficiency of the enclosing Carnot cycle. We demonstrate in this article that:

a) It is not true that the efficiency of an n-T reversible cycle is lower than the efficiency of the enclosing Carnot cycle, but the two efficiencies are equal.

b) If it is possible for the efficiency of an n-T cycle with heat interaction at highest temperature T_H and lowest temperature T_L, to be lower than the efficiency of a Carnot cycle interacting with HRs at T_H and T_L (the enclosing Carnot cycle), then it is also possible for the efficiency of an n-T cycle involving heat interaction at highest temperature T_H and lowest temperature T_L, to be greater than the efficiency of the enclosing Carnot cycle. If the latter is impossible, then the former, too, is impossible; leaving the only option that an n-T cycle and the enclosing Carnot cycle must have the same efficiency.

For these demonstrations, we consider the simplest non-Carnot cycles – the 3-T (n=3) cycles. Each such cycle can be considered as a combination of two Carnot cycles with a common isotherm. For the sake of simplicity, we assume that each of the two Carnot cycles lies between a pair of adiabats that have the same value of ΔS, that is $(S_i - S_j) = (S_j - S_k) = \Delta S$. (This restrictive assumption is removed and the result generalized in part B). Though these cycles involve heat interactions at three different temperatures, only two HRs suffer change at the end of the cycle, in view of our simplifying assumption.

4. Demonstration for (a)

Fig. 2 depicts a Carnot heat engine cycle ABCDA. In this cycle the system interacts with HRs at absolute temperatures T_H and T_L ($<T_H$). The system receives Q_H units of heat at T_H and rejects Q_L units of heat at T_L. W (= $Q_H - Q_L$) units of work is delivered. The efficiency of this Carnot cycle η_C, is given by

$$\eta_C = \frac{W}{Q_H} = \frac{Q_H - Q_L}{Q_H} = \left[1 - \frac{Q_L}{Q_H}\right] < 1 \tag{2}$$

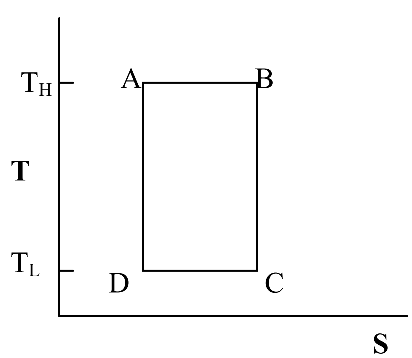

Fig. 2. T-S diagram of a Carnot Cycle

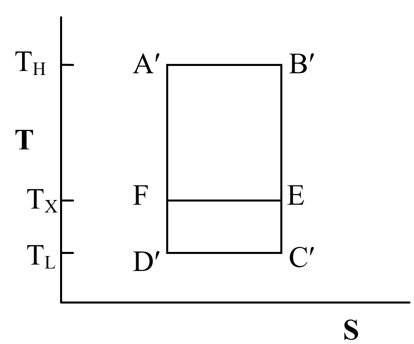

Fig. 3. Combination of two Carnot Cycles

It is possible to consider the Carnot cycle ABCDA as a combination of two Carnot cycles, A′B′EFA′ and FEC′D′F, as shown in Fig. 3. In the cycle A′B′EFA′ the system interacts with HRs at T_H and T_X, and in FEC'D′F the system interacts with HRs at T_X and T_L, ($T_H > T_X > T_L$). When these two cycles are described in clockwise direction once, the changes in the surroundings are the same as those produced by

Carnot cycle ABCDA (Fig. 1). Since the work output and heat input are measured from the changes that occurred in the surroundings only; cycles which produce identical changes in the surroundings must have the same efficiency. Hence, efficiency of cycles shown in Fig. 2 and Fig. 3 must be equal.

Again, it is also possible to show the operation of the composite cycle in Fig. 3 in the form of the composite cycle ABEE′C′D′F′FA shown in Fig. 4 and ABEF′′C′′D′′EFA shown in Fig. 5. Figures 4, 5 are obtained by sliding the component Carnot cycles ABEFA and FECDF (Fig. 3) relative to each other along their common isotherm, to different extents. When these two cycles are described in clockwise direction

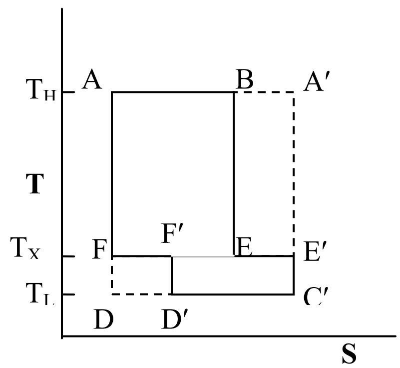

Fig. 4. Cycle CDFE is slid along FE to C′D′F′E′

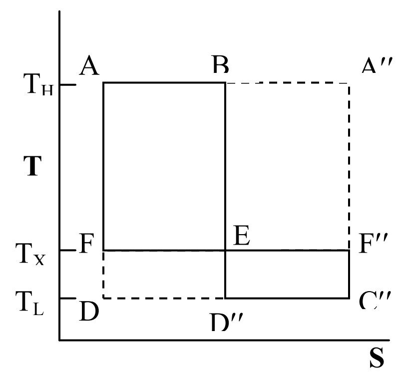

Fig. 5.Cycle CDFE is slid along FE to C′′D′′EF′′

once, the changes in the surroundings are the same as those produced by Carnot cycle ABCDA in Fig. 2. Such being the case, it is impossible for those cycles to have different values of efficiency. Therefore, it follows that the efficiency of the cycles shown in Figs. 4 and 5 are equal to the efficiency of the cycle shown in Fig. 2, which equals the efficiency of the Carnot cycle in Fig.1. The enclosing Carnot cycles (shown on dotted line in Figs. 4 and 5) also have the same efficiency as the Carnot cycle in Fig. 1 since they also interact with HRs at T_H and T_L only.

According to the definition of efficiency (eq. (1)), η_R of cycles shown in Figs. 4, 5 is expressed such that the heat input $\int dQ_{in}$ in these two cycles is different and is also different from that of the Carnot cycle in Fig. 1. The heat input in cycle-4 is considered to be ($Q_{AB} + Q_{EE'}$), while that in cycle-5 is considered to be ($Q_{AB} + Q_{EF''}$). Thus the heat inputs in cycles 2, 4, 5 are different, with the least in cycle-2 and the highest in cycle-5, for the same work output. Consequently, efficiencies of the cycles are different and are in the order: efficiency of cycle-2 > efficiency of cycle-4 > efficiency of cycle-5. This, however, is not true; for, the changes in the surroundings produced by all these three cycles are same.

This completes the demonstration (a) that it is not true that the efficiency of an n-T, (n>2) reversible cycle is lower than the efficiency of the enclosing Carnot cycle, but that the two efficiencies are equal.

5. Demonstration for (b)

Let us now consider the cycles shown in Figs. 6, 7. In these two cycles, the system follows the

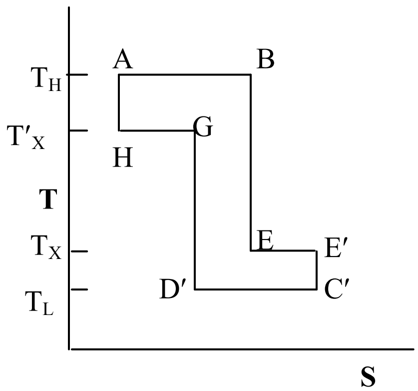

Fig. 6. Heat is rejected at T_L and $T'_X > T_X$

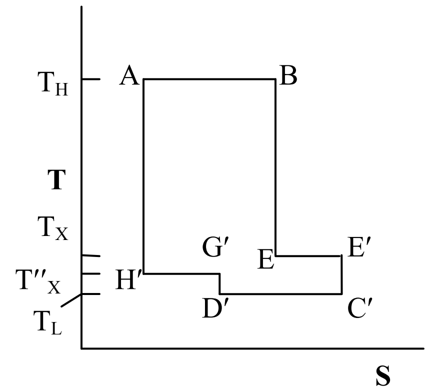

Fig. 7. Heat is rejected at T_L and $T''_X < T_X$

same path - from state A to state D′ as it did in the cycle shown in Fig. 4. Then, in the cycle in Fig. 6, the system changes adiabatically from state D′ to state G at temperature T'_X ($> T_X$). The system then follows the reversible isothermal path GH, rejecting heat to HR at T'_X followed by a reversible adiabatic path along HA, and completes the cycle. In the cycle in Fig. 7, the system changes adiabatically from state D′ to G′ at temperature T''_X ($< T_X$). The system then follows the reversible isothermal path G′H′, rejecting heat to HR at T''_X followed by a reversible adiabatic path along H′A, and completes the cycle.

Comparing cycles depicted in Figs. 4 and 6, and following the conventional analysis, we find that the heats absorbed in the two cycles are the same but the work delivered in the cycle in Fig. 6 is less than that in the cycle in Fig. 4 leading to a lower efficiency for cycle in Fig. 6. Again, comparing cycles depicted in Figs. 4 and 7, we find that the heats absorbed in the two cycles are the same but the work delivered in the cycle in Fig. 7 is more than that in the cycle in Fig. 4 leading to a higher efficiency for cycle in Fig. 7. Since we already showed above that the efficiency of cycle in Fig. 4 is equal to the efficiency of the enclosing Carnot cycle, it follows that if the efficiency of a reversible cycle is less than the efficiency of the enclosing Carnot cycle, then it is also possible for the efficiency of a reversible cycle to be more than the efficiency of the enclosing Carnot cycle. If one is impossible the other is also impossible. This proves that the efficiency of an arbitrary reversible cycle is equal to the efficiency of the enclosing Carnot cycle. This completes our demonstration.

The above results hold potential to open new vistas for exploration of the nature of heat.

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