Graceful Labeling for Trees

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Abstract

We define so called *n*-delta lattice containing (*n*-1) lattice points in first (topmost) row, (*n*-2) lattice points in second row, and so on. Each time the count of lattice points decreases by unity as we move down by one row till we reach the last (bottommost) row containing single lattice point. We label these lattice points in two different ways and obtain two different labeled lattices. In the first kind of labeling we associate vertex pairs in a particular way as labels for points of the lattice and so call it edge-labeled *n*-delta lattice. In the second kind of labeling we associate integers as labels with lattice points in each row to indicate the position of that lattice point in the row and so call it position-labeled *n*-delta lattice. This defining of position-labeled *n*-delta lattice enables us to associate a lexicographic ordering with lattice paths. We define distinct as well as different lattice paths and further see that for proving graceful tree conjecture one needs to show that the count of distinct lattice paths corresponding to trees in the edge-labeled *n*-delta lattice is same as the count of nonisomorphic trees with *n* vertices. We verify this for some (small) values of *n*. We further see that existence of graceful labeling for an unlabeled tree with *n* vertices follows from the existence of a lattice path representing this same tree in the edge-labeled *n*-delta lattice. It is possible to generate all (*n*, *n*-1)-trees from all (*n*-1, *n*-2)-trees by attaching an edge that emerges from each of the inequivalent vertices of (*n*-1, *n*-2)-trees and entering in the new vertex taken outside. We show that extending all lattice paths by adding a lattice point in the paths sitting in the sub-lattice of *n*-delta lattice in all possible ways is same as the above mentioned generation of trees from lower trees.

1. Introduction: A tree on *n* vertices is said to be graceful or said to have a graceful labeling if when its vertices are labeled with integers {1, 2, …, n} and lines (edges) are labeled by the difference of their respective end vertex labels then all the edge labels taken together constitute the set {1, 2, …, n−1}.

In the year 1964 Ringel [1] proposed the following

Conjecture 1.1(Ringel): If T is a fixed tree with m lines, then $K_{(2m+1)}$, the complete graph on (*2m+1*) vertices, can be decomposed into (*2m+1*) copies of T.

Attempts to prove Ringel's conjecture have focused on a stronger conjecture about trees [2], called the Graceful Tree Conjecture:

Conjecture 1.2 (Graceful Tree Conjecture): Every (unlabeled) tree is graceful, i.e. has a graceful labeling.

2. Graceful Tree Conjecture: In this section we define certain lattices and show that all possible graceful trees in a complete graph can be seen as certain lattice paths in a triangular shaped lattice of points each of whose lattice point is labeled by a unique vertex pair (forming the edge).

Definition 2.1: A **delta lattice (*n*-delta lattice)** is a triangular shaped lattice of points, having shape of inverted triangle (like symbol delta), containing (n-1) lattice points in first (topmost) row, and each time this number of lattice points in the respective rows decreases by unity as one moves down to next row till one reaches the last row containing single lattice point.

Definition 2.2: The **edge-labeled *n*-delta lattice** is the same lattice above whose lattice points are labeled as follows: the lattice points in the top row have associated labels (i, i+1), where i goes from 1 to n-1, the lattice points in the second row below it have associated labels (i, i+2), where letter i goes from 1 to n-2, …., the lattice points in the k-th row, reached by successively creating rows downwards, have associated labels (i, i+k), where i goes from 1 to n-k, … the last row has a single lattice point with vertex pair (1, n) as the associated label.

We give below as an illustration the representation of this lattice with associated labels for n = 2,3,4,5,6 (we don't draw here the associated lattice points and it is to be understood that they are there) as follows:

1) For n = 2, the 2-delta lattice consists of single lattice point labeled by the associated vertex pair (1,2) :

(1,2)

2) For n = 3, the 3-delta lattice consists of three lattice points, two in first row and one in second row:

(1,2) (2,3)

(1,3)

3) For n = 4, the 4-delta lattice is:

(1,2) (2,3) (3,4)

(1,3) (2,4)

(1,4)

4) For n = 5, the 5-delta lattice is:

(1,2) (2,3) (3,4) (4,5)

(1,3) (2,4) (3,5)

(1,4) (2,5)

(1,5)

5) For n = 6, the 6-delta lattice is:

(1,2) (2,3) (3,4) (4,5) (5,6)

(1,3) (2,4) (3,5) (4,6)

(1,4) (2,5) (3,6)

(1,5) (2,6)

(1,6)

Definition 2.3: An imaginary vertical line starting from lattice point associated with pair (1,n) and going upwards passing through the lattice points (2,n-1), (3,n-2), ….., extending and incorporating the lattice points on the rows, and rising up to first row is called **line of symmetry**.

In the above illustrations of delta lattices:

1) For $n = 2$ the line of symmetry passes through lattice point associated with vertex pair (1, 2), i.e. through the only lattice point.
2) For $n = 3$ the line of symmetry passes through lattice point associated with vertex pair (1, 3), since there is no other lattice point on this vertical line.
3) For $n = 4$ the line of symmetry passes through lattice point associated with vertex pairs (1, 4), (2, 3) since there is no lattice point on this vertical line from second row.
4) For $n = 5$ the line of symmetry passes through lattice point associated with vertex pairs (1, 5), (2, 4) since there is no lattice point on this vertical line from first and third row.
5) For $n = 6$ the line of symmetry passes through lattice point associated with vertex pairs (1, 6), (2, 5), (3, 4) since there is no lattice point on this vertical line from second and fourth row.

Definition 2.3: If we choose one entry (lattice point in terms of vertex pair) from each row of the edge-labeled n-delta lattice (and consider the graph produced by edges in this choice taken together) then this assembly of vertex pairs taken together in a set is called a **lattice path.** Actually, the lattice path sitting in the lattice can be shown by starting from bottommost row and selecting and joining a lattice point from each row above in succession till one reaches the topmost row where the lattice path terminates at some lattice point in the first (topmost) row.

Remark 2.1: It is easy to check that in the edge-labeled n-delta lattice corresponding to graph of n vertices, each lattice path among all possible $(n-1)!$ lattice paths represents a graceful graph, i.e. a labeled graph having graceful labeling. Lattice path can be visually shown by starting with bottommost row containing lattice point labeled by vertex pair $(1, n)$ we join to selected point among points associated with some vertex pair $\{(1, n-1), (2, n)\}$ in the row above it. We then join to selected point among points associated with some vertex pair $\{(1, n-2), (2, n-1), (3, n)\}$ in the row above it, and so on. We continue this procedure of joining to the selected point each time among the points associated with some vertex pair in the row just above till we reach the top row and join to some lattice point associated with some vertex pair for the topmost (first) row.

Definition 2.4: If a lattice path formed by choosing vertex pairs such that each row of the lattice contributes exactly one vertex pair and all vertex pairs taken together contain all the vertices labeled as {1, 2, 3, ….., n} then each of such lattice paths represent a **graceful tree** and these are the only lattice paths representing graceful trees**.** All other lattice paths formed by choosing one vertex pair from each row of the triangular lattice but which vertex pairs taken together do not contain all the vertices are not trees though they are **graceful graphs**.

We now proceed to give examples of the lattice paths in the above mentioned lattices for $n = 2, 3, 4, 5, 6$

1) Case $n = 2$: In this case, there is only one lattice point with associated vertex label (1, 2). So, the lattice path is of zero length. Note that there exist only one tree, made up of single edge, and this is graceful labeling
2) Case $n = 3$: In this case, there two lattice paths formed by vertex pairs {(1, 3), (1, 2)} and {(1, 3), (2, 3)}. Note that these lattice paths are mirror images of each other in the line of symmetry mentioned above. So, they are graceful graphs. Moreover, since each together contain all vertices {1, 2, 3}, so, they are graceful trees.
3) Case $n = 4$: In this case, there are in all six (3!) lattice paths formed by vertex pairs {(1, 4), (1, 3), (1, 2)}, {(1, 4), (1, 3), (2, 3)}, {(1, 4), (1, 3), (3, 4)}, {(1, 4), (2, 4), (1, 2)}, {(1, 4), (2, 4), (2, 3)}, {(1, 4), (2, 4), (3, 4)}. Out of these the first, second, fourth, and fifth lattice paths are graceful trees, while paths third and sixth are only graceful graphs but not graceful trees.
4) Case $n = 5$: The three lattice paths which are nonisomorphic graceful trees are

   {(1,5), (1,4), (1,3), (1,2)}
   {(1,5), (1,4), (1,3), (2,3)}
   {(1,5), (1,4), (2,4), (2,3)}

5) Case $n = 6$. In this case we get following six distinct lattice paths which correspond to nonisomorphic trees and chosen entries in successive rows are shown to be joined by an arrow to bring clarity about path structure that will be seen in the 6-delta lattice when lattice points will be joined in the 6 copies of 6-delta lattices:

   (1,6)→(1,5)→(1,4)→ (1,3)→(1,2) …..(1)

   (1,6)→(1,5)→(1,4)→ (1,3)→ (2,3) ….(2)

   (1,6)→(1,5)→(1,4)→ (2,4)→ (2,3) .....(3)

$(1,6) \rightarrow (1,5) \rightarrow (1,4) \rightarrow (2,4) \rightarrow (3,4)$ .....(4)

$(1,6) \rightarrow (1,5) \rightarrow (2,5) \rightarrow (2,4) \rightarrow (3,4)$ .....(5)

$(1,6) \rightarrow (1,5) \rightarrow (2,5) \rightarrow (1,3) \rightarrow (3,4)$ .....(6)

Further, it is easy to see that if we take some lattice path and consider the path **formed as mirror image in the line of symmetry** of the chosen path then both these paths represent graceful graphs which are isomorphic. The graceful nature of mirror image is clear (since the mirror image itself is again a lattice path). More clearly, the mirror image of a lattice point with associated vertex pair (i, j) we get the lattice point with associated vertex pair $(n\text{-}i+1,\ n\text{-}j+1)$ and so by the below given simple theorem 2.1 the result follows.

Theorem 2.1 Every graceful (n, n−1) tree remains graceful under the transformation (mapping) of vertex labels:
$j \rightarrow (n - j + 1)$.
Proof: Let i, k be the vertex labels of two adjacent vertices of the tree. Then the edge label for this edge will be $| i - k |$. Now under the mentioned transformation the edge labels
$| i - k | \rightarrow | (n - i + 1) - (n - k + 1) | = | i - k |$, hence etc.

□

Remark 2.2: It is easy to visualize that edge-labeled *n*-delta lattice is essentially a representation for complete graph on *n* vertices where these vertices are labeled by numbers {1, 2, 3, …, n}.
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Remark 2.3: A **lattice path** is a path obtained by selecting some one lattice point on each row of edge-labeled *n*-delta lattice and joining these lattice points in sequence starting with the lattice point on the last row and moving up in succession incorporating the chosen lattice point on each row till the path finally terminates at the selected lattice point on the first row.

Remark 2.4: In the above definition by starting with the selected lattice point on the first row and moving down in succession incorporating the chosen lattice point on each row till the path finally terminates at the selected lattice point on the last row we will construct the same lattice path.

Remark 2.5: It is easy to visualize that a lattice path in n-delta lattice, when corresponds to a tree, is essentially equivalent to showing existence of a gracefully labeled isomorphic copy (for an unlabeled tree of some isomorphic type) in the complete graph on n vertices where these vertices are labeled by numbers {1, 2, 3, …, n}.

It is easy to see that when we take a lattice path and use it to construct a graph by taking the vertex pairs that appear in that path as edges and the numbers that appear in the totality in these vertex pairs as vertex labels we get essentially a graceful graph.

If this graceful graph is an (n, n-1) connected graph or (n,n-1) acyclic graph then the lattice path represents a graceful tree. Otherwise, the associated graph, though graceful, obtained from that lattice path is not a tree.

Consider following two **straight** lattice paths which are symmetrically placed (mirror images of each other) around line of symmetry, namely,

$$(1,n)\rightarrow(1,n\text{-}1)\rightarrow(1,n\text{-}2)\rightarrow\ldots\ldots\ldots\rightarrow(1,2)$$

and

$$(1,n)\rightarrow(2,n)\rightarrow(3,n)\rightarrow\ldots\ldots\ldots\rightarrow(n\text{-}1,n)$$

It is easy to check that these lattice paths lying at left and right boundary of n-delta lattice correspond as a graph to gracefully labeled (n, n-1) **star-trees**.

Consider following two **zigzag** lattice paths which are symmetrically placed (mirror images of each other) around line of symmetry, namely,

$$(1,n)\rightarrow(1,n\text{-}1)\rightarrow(2,n\text{-}1)\rightarrow(2,n\text{-}2)\rightarrow(3,n\text{-}2)\rightarrow\ldots\ldots$$

and

$$(1,n)\rightarrow(2,n)\rightarrow(2,n\text{-}1)\rightarrow(3,n\text{-}1)\rightarrow(3,n\text{-}2)\rightarrow(4,n\text{-}2)\rightarrow\ldots$$

It is easy to check that these lattice paths going away from and coming towards line of symmetry by unit distance at each alternate move and passing in a zigzag way close to line of symmetry of n-delta lattice correspond as a graph to gracefully labeled (n,n-1) **path-trees**.

We now proceed with an algorithm to generate all possible gracefully labeled trees in terms of the totality of all lattice paths in n-delta lattice.

For the sake of clarity let us state some more definitions:

Definition 2.5: A tree is called a **star-tree** or simply a **star** if it is a tree with one vertex of degree k, k bigger than one, and all other vertices are adjacent to it and have degree exactly equal to one.

Definition 2.6: A tree is called a **path-tree** or simply a **path** if it is tree with all vertices have degree two except two (end) vertices (where the path terminates) and they have degree one.

Definition 2.7: The **position-labeled *n*-delta lattice** is the same lattice above whose lattice points are labeled as follows: the lattice points in the top row have associated labels i, where i goes from 1 to n-1, the lattice points in the second row below it have associated labels i, where letter i goes from 1 to n-2, …., the lattice points in the k-th row, reached by successively creating rows downwards, have associated labels i, where i goes from 1 to n-k, … the last row has a single lattice point with vertex pair 1 as the associated label.

We give below as an illustration the representation of this position-labeled lattice with associated labels for n = 2,3,4,5,6 (we don't draw here the associated lattice points and it is to be understood that they are there) as follows:

1)For n = 2, the position-labeled 2-delta lattice is:

1

2) For n = 3, the position-labeled 3-delta lattice is:

1 2

1

3) For n = 4, the position-labeled 4-delta lattice is:

1 2 3

1 2

1

4) For n = 5, the position-labeled 5-delta lattice is:

1 2 3 4

1 2 3

1 2

1

5) For n = 6, the 6-delta lattice is:

1 2 3 4 5

1 2 3 4

1 2 3

1 2

1

Definition 2.8: A lattice path in position-labeled *n*-delta lattice is produced similarly as was produced in edge-labeled *n*-delta lattice by starting at the only lattice point in the last row that exists at bottom of the and choosing one lattice point per row in the rows above in succession in the lattice and joining these lattice points in succession.

Remark 2.6: Note that a lattice path in position-labeled *n*-delta lattice provides a natural lexicographic ordering for the corresponding lattice path containing same lattice points in edge-labeled *n*-delta lattice. It is easy to see that the lattice path 1→1→1→1→….. →1 is the lattice path with smallest lexicographic order in the position-labeled *n*-delta lattice, representing star-tree having vertex labeled 1 as central vertex and vertices labeled 2, 3, …, *n*

as pendant vertices. Similarly, the lattice path 1→2→3→4→….. → *n* is the lattice path with largest lexicographic order in the position-labeled *n*-delta lattice which is mirror image of lattice path just considered with smallest lexicographic order in the line of symmetry, again representing star-tree having vertex labeled 1 as central vertex and vertices labeled 2, 3, …, *n* as pendant vertices. Also, it is easy to see that lattice paths
1→1→2→2→3→3→….. →*i*→*i*→(*i*+1)→(*i*+1)→….. →[*n*/2] and
1→2→2→3→3→4….. →*i*→(*i*+1)→(*i*+1)→(*i*+2)→….. →[*n*/2] are lattice paths in position-labeled lattice representing path-trees in edge-labeled lattice which are mirror images of each other in the line of symmetry.

Definition 2.9: Two lattice paths in edge-labeled lattice are called **different** if as lattice paths in position-labeled lattice they are different, i.e. they are different lexicographic sequences (but may be representing graceful graphs which are isomorphic thus may not be different as unlabeled graphs).

Definition 2.10: Two lattice paths in edge-labeled lattice are called **distinct** if they represent two graceful graphs which are nonisomorphic.

3. Graceful Labeling for Trees: Graceful tree conjecture can be settled if one can show that the count of **distinct lattice paths corresponding to graceful trees** in the edge-labeled *n*-delta lattice is same as the count of **unlabelled (*n*, *n*-1)-trees**.

Conjecture 3.1: For every unlabeled (*n*, *n*-1)-tree there exists a lattice path in the edge-labeled *n*-delta lattice such that the lexicographic order of the sequence corresponding to same lattice path viewed in the position-labeled *n*-delta lattice lies in between the lexicographic orders of lattice paths for star-tree and path-tree.

We now verify this conjecture for some small values of *n*. For *n* = 1, 2, 3, 4 the conjecture is clear. So, we check the case *n* = 5. The edge-labeled 5-delta lattice is

(1,2) (2,3) (3,4) (4,5)

(1,3) (2,4) (3,5)

(1,4) (2,5)

(1,5)

For this lattice the corresponding position-labeled 5-delta lattice is

1 2 3 4

1 2 3

1 2

1

The lattice path (1, 5)→(1, 4)→(1, 3)→(1, 2) in the edge-labeled lattice is **star-tree** and from its associated lattice path 1→1→1→1 in the position-labeled lattice and so has associated lexicographic sequence (1, 1, 1, 1). Further, the lattice path (1, 5)→(1, 4)→(2, 4)→(2, 3) in the edge-labeled lattice is **path-tree** and from its associated lattice path 1→1→2→2 in the position-labeled lattice and so has associated lexicographic sequence (1, 1, 2, 2). As we know, there exists only one tree in addition to star-tree and path-tree mentioned above with lattice path (1, 5)→(1, 4)→(1, 3)→(2, 3) in the edge-labeled lattice and has its associated lattice path 1→1→1→2 in the position-labeled lattice and so has lexicographic sequence (1, 1, 1, 2). It is clear to see that

$$(1, 1, 1, 1) < (1, 1, 1, 2) < (1, 1, 2, 2)$$

We now proceed with one more check: the case $n = 6$. In this case we get the lattice paths in edge-labeled n-delta lattice and their associated lexicographic sequences from corresponding lattice paths in position-labeled n-delta lattice as follows:

Lattice Path: (1, 6)→(1, 5)→(1, 4)→(1, 3)→(1, 2)
Lexicographic Sequence: (1, 1, 1, 1, 1)
Lattice Path: (1, 6)→(1, 5)→(1, 4)→(1, 3)→(2, 3)
Lexicographic Sequence: (1, 1, 1, 1, 2)
Lattice Path: (1, 6)→(1, 5)→(1, 4)→(2, 4)→(2, 3)
Lexicographic Sequence: (1, 1, 1, 2, 2)
Lattice Path: (1, 6)→(1, 5)→(1, 4)→(2, 4)→(3, 4)
Lexicographic Sequence: (1, 1, 1, 2, 3)
Lattice Path: (1, 6)→(1, 5)→(2, 5)→(1, 3)→(3, 4)
Lexicographic Sequence: (1, 1, 2, 1, 3)
Lattice Path: (1, 6)→(1, 5)→(2, 5)→(2, 4)→(3, 4)

Lexicographic Sequence: (1, 1, 2, 2, 3)

It is clear from above that the lexicographic order of the sequence associated with any tree other than star-tree and path-tree lies in between the sequence corresponding to star-tree and path-tree.

We now proceed to discuss two procedures. Showing equivalence of these two procedures will imply Graceful Tree Conjecture.

1) The procedure for generating all possible unlabeled (n, n—1) trees from all possible unlabeled (n—1, n—2) trees by extension at every inequivalent vertex.
2) The procedure of extending all possible lattice paths representing (n—1, n—2) trees sitting inside sub-lattice of edge-labeled n-delta lattice that results after deleting first row of edge-labeled n-delta lattice.

Conjecture 3.2: The procedures 1) and 2) are equivalent.

Remark 3.1: Note that in the procedure mentioned in 1) we need to extend once at each inequvalent vertex (defined below), i.e. we need to emerge one edge from each inequvalent vertex and join it to a new vertex taken outside. We proceed with some definitions which will make precise the first procedure mentioned above.

Definition 3.1: Let *G* be an unlabelled (p, q) graph and let G^e be a supergraph of *G* obtained by taking a (new) vertex outside of the vertex set *V(G)* and joining it to some (unspecified) vertex of *G* by an (new) edge not in the edge set of G, E*(G)*, is called the **extension** of *G* to G^e.

Definition 3.2: The subset V_j of vertices $\{u_1{}^j, u_2{}^j, \cdots, u_r{}^j\}$ in a tree T is called a **set of equivalent vertices** or simply an **equivalent set** if all the trees T + $vu_s{}^j$, 1 ≤ s ≤ r, obtained from T by adding an edge $vu_s{}^j$, obtained by joining vertex $u_s{}^j$ in set V_j to a new vertex v not in *V(G)*, are isomorphic.

Definition 3.3: The subset of vertices (vertices) of *V(G)* is called a **set of equivalent vertices** or simply an **equivalent set** if the extension of graph *G* at any vertex among these vertices, achieved by joining any one vertex among these vertices to a (new) vertex taken outside not in *V(G)*, leads to graphs which are all isomorphic.

Definition 3.4: The collection of subsets $\{V_1, V_2, \cdots, V_m\}$ of *V(G)*, the vertex set of graph *G* is called a **partitioning of *V(G)* into equivalent sets** if all the subsets V_i , i = 1, 2,…, m are equivalent sets,

$$V_i \bigcap V_j = \phi, \forall i \neq j, and\ V(G) = \bigcup_{i=1}^{m} V_i$$, where ϕ is a null set.

Definition 3.5: The collection of all possible unlabelled (nonisomorphic) (n, n─1) trees is called **(n, n─1)-stock**.

Definition 3.6: The set of unlabelled (n+1, n)-trees obtained by extension at (any) one vertex belonging to every set of equivalent vertices in the partitioning of *V(T)* into equivalent sets for a tree *T* is called **Complete Extension of *T*,** and is denoted by ***CE(T).***

Theorem 3.1: The collection of nonisomorphic trees contained in $\bigcup_j^{/} CE(T_j)$, T_j belongs to (*n*, *n*─1)-stock, and $^{/}$ indicates that the union is over nonisomorphic trees, forms a (*n*+1, *n*)-stock.

Proof: For every tree *T* belonging to (n+1, n)-stock there exists trees *T** of some isomorphism type in (n, n─1)-stock obtained by deleting pendant vertex of *T*. So, *T* can be considered as arrived at by extension of some tree like *T** belonging to (n, n─1)-stock, and trees isomorphic to T arriving from more than one *T** is taken only once in the union.

□

Definition 3.7: Two vertices in an unlabeled graph are called **inequivalent** if they belong to **two different** equivalent sets.

Thus, extending at each inequivalent vertex of trees in the (n-1, n-2)-stock we will be able to generate the entire (n, n-1)-stock.

To understand procedure in 2) we begin with some simple observations:

1) The (n+1)-delta lattice can be obtained from n-delta lattice by just appending new diagonal made up of n new lattice points labeled by vertex pairs {(1, n+1), (2, n+1), (3, n+1), …., (n, n+1)} parallel to diagonal {(1, n), (2, n), (3, n), …., (n-1, n)} of n-delta lattice.
2) We consider sub-lattice of (n+1)-delta lattice that results after deleting first row of the (n+1)-delta lattice and consider all lattice paths sitting in this sub-lattice and representing (n, n-1)-trees with edge labeling made up of labels {2, 3, 4, … , n} and contain vertex labels {1, 2, …., i-1, i+1, …., n+1}, where missing label $i \in \{1,2,3,\cdots,n+1\}$.
3) Now, it is clear to see that we can extend each of lattice path considered in 2) in two ways: first by appending an edge (i-1, i) to get first variety of graceful tree in (n+1)-delta lattice and then by appending an edge (i, i+1) to get second variety of graceful tree in (n+1)-delta lattice.

We claim that we capture all possible unlabeled (n+1, n)-trees that forms the entire (n+1, n)-stock, in their graceful avatar!

Illustration: Consider sub-lattice of 5–delta lattice after deleting its first row and lattice paths in it representing (4, 3)-trees below:

(1,3) (2,4) (3,5)

(1,4) (2,5)

(1,5)

(i) The lattice paths in this sub-lattice of 5–delta lattice which are (4, 3)-trees are {(1, 5), (1, 4), (1, 3)}, {(1, 5), (1, 4), (2, 4)}, {(1, 5), (1, 4), (3, 5)}, {(1, 5), (2, 5), (1, 3)},{(1, 5), (2, 5), (2, 4)}, {(1, 5), (2, 5), (3, 5)}.
(ii) By extending these lattice paths obtained in (ii) in 5–delta lattice we have following totality of lattice paths: {(1, 5), (1, 4), (1, 3),

(1, 2)}, {(1, 5), (1, 4), (1, 3), (2, 3)}, {(1, 5), (1, 4), (2, 4), (2, 3)}, {(1, 5), (1, 4), (2, 4), (3, 4)}, {(1, 5), (2, 5), (2, 4), (2, 3)}, {(1, 5), (2, 5), (2, 4), (3, 4)}, {(1, 5), (2, 5), (3, 5), (3, 4)}, {(1, 5), (2, 5), (3, 5), (4, 5)}

(iii) It can be easily checked that in the procedure explained in this illustration to get all possible distinct trees in 5–delta lattice by attaching an edge to the lattice paths in sub-lattice of 5–delta lattice we are essentially attaching edge to at least some one vertex belonging to every set of equivalent vertices in the partitioning of vertex set *V(T)* into equivalent sets for every tree *T* under consideration.

References

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