THE ISO-DUAL TESSERACT

Nathan O. Schmidt
Department of Mathematics
Boise State University
1910 University Drive
Boise, ID 83725, USA
nathanschmidt@u.boisestate.edu

November 16, 2013

Abstract

In this work, we deploy Santilli’s iso-dual iso-topic lifting and Inopin’s holographic ring (IHR) topology as a platform to introduce and assemble a tesseract from two inter-locking, iso-morphic, iso-dual cubes in Euclidean triplex space. For this, we prove that such an “iso-dual tesseract” can be constructed by following a procedure of simple, flexible, topologically-preserving instructions. Moreover, these novel results are significant because the tesseract’s state and structure are directly inferred from the one initial cube (rather than two distinct cubes), which identifies a new iso-geometrical inter-connection between Santilli’s exterior and interior dynamical systems.

Keywords: Santilli iso-number; Inopin holographic ring; Iso-geometry; Tesseract.
1 Introduction

 Everybody knows what the square is: a square is a 2D object in 2D space with 4 equal edges, 4 equal angles, and 4 vertices. Most people know what the cube is: a cube is a 3D object in 3D space—the 3D analog of the square—with 12 equal edges, 6 square faces, and 8 vertices, where 3 edges meet at each vertex. But few people know what the tesseract is: a tesseract is a 4D object in 4D space—the 4D analog of the cube—with 32 edges, 24 faces, and 16 vertices, where 4 edges meet at each vertex. Basically, the tesseract is to the cube just as the cube is to the square.

 To date, there are numerous geometrical procedures of tesseract construction that operate with conventional mathematics. However, in this paper, we disclose the first iso-geometrical procedure of tesseract construction that operates with Santilli’s new iso-mathematics [1, 2, 3, 4, 5, 6].

 To introduce and illustrate this notion, lets consider one approach to build a tesseract. First, we know that the cube has 8 vertices and the tesseract has 16 vertices, therefore a tesseract has two times as many vertices as a cube. For this method, this value of two is of interest to us—but why?
 Well, suppose that two distinct cubes are positioned in a 3D space, where the sum of the vertices of these two cubes is 16. These resulting 16 vertices indicate that a tesseract can be assembled from the two cubes by introducing 8 additional edges to inter-connect the 8 vertex pairs in a pairwise fashion. Now lets take this one step further: what if one could build a tesseract from a one cube instead of two? In conventional mathematics, this question may seem irrelevant because the 8 vertices of a single cube is insufficient to synthesize a tesseract of 16 vertices. However, in the realm of Santilli’s iso-mathematics [1, 2, 3, 4, 5, 6], this question becomes legitimate when we consider the concept of iso-duality.

 In this paper, we attack the said inquiry and prove that it is possible to build a tesseract from one initial cube by iso-topically lifting [1, 2, 3, 4, 5, 6] its 8 vertices to simultaneously infer an exterior cube and an interior cube to generate the required 16 vertices, where the double-projected cubes are iso-dual and are both iso-morphic, inter-locking, and synchronized to the initial cube. Consequently, the 16 generated vertices are inter-connected in a pairwise fashion to iso-mathematically synthesize the iso-tesseract. Thus, for this investigation, Section 2 presents the main procedure and results,
while Section 3 recapitulates the significance of our discovery and suggests future modes of exploration along this research trajectory.

2 Procedure
In this main section, we launch our exploration by instantiating the dual 4D space-time IHR topology [6, 7, 8, 9] so we can subsequently generalize it to encompass the exterior and interior IHR iso-duality [10] and thereby assemble the iso-dual tesseract from one cube through a step-by-step process.

2.1 Preparation: initializing the dual 4D space-time IHR topology
Here, we prepare for the iso-dual tesseract construction of Section 2.2 by first recalling the dual 4D space-time IHR topology [6, 7, 8, 9, 10] via the following procedure:

1. First, given eq. (18) of [6] we identify \( Y \equiv \mathbb{T} \) as the set of all triplex numbers, the Euclidean triplex space, and the dual 3D Cartesian-spherical coordinate-vector state space. Here, a triplex number \( \vec{y} \in Y \) is a dual 3D Cartesian-spherical coordinate-vector state that is expressed via eq. (17) of [6] as

\[
y = \vec{y} = \vec{y}_R + \vec{y}_I + \vec{y}_Z = (\vec{y}) = (|\vec{y}|, \langle \vec{y} \rangle, [\vec{y}])_S = (\vec{y}_R, \vec{y}_I, \vec{y}_Z)_C , \quad \forall \vec{y} \in Y ,
\]

where \((\vec{y}_R, \vec{y}_I, \vec{y}_Z)_C\) is a 3D Cartesian coordinate-vector state in the 3D Cartesian coordinate-vector state space \( Y_C \) so \((\vec{y}_R, \vec{y}_I, \vec{y}_Z)_C \in Y_C \), while simultaneously \((|\vec{y}|, \langle \vec{y} \rangle, [\vec{y}])_S\) is a 3D spherical coordinate-vector state in the 3D spherical coordinate-vector state space \( Y_S \) so \((|\vec{y}|, \langle \vec{y} \rangle, [\vec{y}])_S \in Y_S \), such that \( Y_C \) and \( Y_S \) are dual, iso-morphic, synchronized, and interlocking in \( Y \) [6]. Hence, eq. (1) satisfies with the constraints imposed by eqs. (19–28) of [6]—see Figures 4 and 5 of [6].

2. Second, given eq. (33) of [6] we recall that

\[
T^2_r = \{ \vec{y} \in Y : |\vec{y}| = r \},
\]

where \( T^2_r \subset Y \) is the 2-sphere IHR of amplitude-radius \( r > 0 \) that is centered on the origin \( O \in Y \) and is iso-metrically embedded in \( Y \); \( T^2_r \)
is the multiplicative group of all non-zero triplex numbers with the amplitude-radius $r$, which is simultaneously dual to the two triplex sub-spaces $[6, 7, 8]$: the “micro sub-space 3-brane” $Y_\subset Y$ and the “macro sub-space 3-brane” $Y_\subset Y$—see Figure 7 of [6]. Here, we note that the 1-sphere IHR $T_r^1 \subset T_r^2$ of amplitude-radius $r > 0$ (and curvature $\frac{1}{r}$) from eq. (16) of [6] is the great circle of $T_r^2$.

At this point, we’ve initialized Inopin’s dual 4D space-time IHR topology $[6, 7, 8, 9, 10]$ and are therefore prepared to assemble the iso-dual tesseract of Section 2.2.

2.2 Engagement: constructing the iso-dual tesseract

Here, equipped with the dual 4D space-time IHR topology of Section 2.1, we introduce, define, and assemble the proposed iso-dual tesseract via the following procedure:

1. First, we recall that in conventional mathematics the number 1 is the multiplicative identity which satisfies the original number field axioms [11]. Thus, in general, the number 1 plays a crucial and diverse role throughout the various branches of mathematics such as, for example, normalization in statistics. Therefore, we begin by setting the amplitude-radius $r = 1$ for $T_r^1$ and $T_r^2$.

2. Second, we construct the initial cube from 8 triplex vertices that are confined to $T_r^2$. For this cube, we define the underlying set of 8 triplex vertices as

$$\{\vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r, \vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r\} \equiv V_{T_r^2} \subset T_r^2 \subset Y$$

such that

$$\{\vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r\} \equiv V_{T_r^2} \subset V_{T_r^2} \subset T_r^2 \subset Y$$

are the “top vertices” for the “top square surface” and

$$\{\vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r\} \equiv V_{T_r^2} \subset V_{T_r^2} \subset T_r^2 \subset Y$$
are the “bottom vertices” for the “bottom square surface”, which comply with the cubic vertex triplex amplitude-radius constraints

\[
1 \equiv r \equiv |\vec{a}_r| \equiv |\vec{b}_r| \equiv |\vec{c}_r| \equiv |\vec{d}_r| \equiv |ar{\vec{a}}_r| \equiv |ar{\vec{b}}_r| \equiv |ar{\vec{c}}_r| \equiv |ar{\vec{d}}_r|,
\]

(6)

the cubic vertex triplex phase constraints

\[
\langle \vec{a}_r \rangle \equiv \langle \bar{\vec{b}}_r \rangle - \frac{\pi}{2} \equiv \langle \bar{\vec{c}}_r \rangle \pm \pi \equiv \langle \bar{\vec{d}}_r \rangle - \frac{3\pi}{2}
\]

\[
\langle \bar{\vec{a}}_r \rangle \equiv \langle \bar{\vec{b}}_r \rangle - \frac{\pi}{2} \equiv \langle \bar{\vec{c}}_r \rangle \pm \pi \equiv \langle \bar{\vec{d}}_r \rangle - \frac{3\pi}{2}
\]

(7)

such that

\[
\langle \vec{a}_r \rangle \equiv \langle \bar{\vec{a}}_r \rangle \pm \pi
\]

\[
\langle \vec{b}_r \rangle \equiv \langle \bar{\vec{b}}_r \rangle \pm \pi
\]

\[
\langle \vec{c}_r \rangle \equiv \langle \bar{\vec{c}}_r \rangle \pm \pi
\]

\[
\langle \vec{d}_r \rangle \equiv \langle \bar{\vec{d}}_r \rangle \pm \pi,
\]

(8)

and the cubic vertex triplex inclination constraints

\[
[\vec{a}_r] \equiv [\vec{b}_r] \equiv [\vec{c}_r] \equiv [\vec{d}_r]
\]

\[
[\bar{\vec{a}}_r] \equiv [\bar{\vec{b}}_r] \equiv [\bar{\vec{c}}_r] \equiv [\bar{\vec{d}}_r]
\]

(9)

such that

\[
[\vec{a}_r] \equiv [\bar{\vec{a}}_r] \pm \pi
\]

\[
[\vec{b}_r] \equiv [\bar{\vec{b}}_r] \pm \pi
\]

\[
[\vec{c}_r] \equiv [\bar{\vec{c}}_r] \pm \pi
\]

\[
[\vec{d}_r] \equiv [\bar{\vec{d}}_r] \pm \pi,
\]

(10)

to establish the cubic vertex triplex antisymmetric constraints

\[
\vec{a}_r \equiv -\bar{\vec{a}}_r
\]

\[
\vec{b}_r \equiv -\bar{\vec{b}}_r
\]

\[
\vec{c}_r \equiv -\bar{\vec{c}}_r
\]

\[
\vec{d}_r \equiv -\bar{\vec{d}}_r.
\]

(11)

Therefore, the cube built from the 8 triplex vertices comprising $V_{T_2^2}$ of eq. (3)—which satisfy eqs. (6–11) and are confined to $T_2^2$—is depicted in Figures 1 and 2.
Fig. 1: The 8 triplex vertices of the initial cube comprise the set 
\{\vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r\} \equiv V_T^2, which are confined to \( T_r^2 \) (not shown).
Fig. 2: The 8 triplex vertices of the initial cube comprise the set \( \{ \vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r, \bar{\vec{a}}_r, \bar{\vec{b}}_r, \bar{\vec{c}}_r, \bar{\vec{d}}_r \} \equiv V_{T^2_r} \), which are confined to \( T^2_r \) (shown).
3. Third, in isomathematics \([1, 2, 3, 4, 5]\), Santilli proved that the standard multiplicative unit which satisfies the conventional number field axioms \([11]\) is not limited to the number 1, and can thus be replaced with a positive-definite iso-multiplicative iso-unit \(\hat{r}_+ > 0\) for isonumbers. Thus, in accordance to Santilli’s methodology \([1, 2, 3, 4, 5]\), we select some \(\hat{r}_+\) with the corresponding iso-unit inverse \(\hat{r}_- = \frac{1}{\hat{r}_+}\), such that

\[
\hat{r}_+ > r > \hat{r}_- > 0. \tag{12}
\]

4. Fourth, we engage \(\hat{r}_+\) to iso-topically lift \([1, 2, 3, 4, 5]\) \(T^2_r\) to the exterior iso-2-sphere IHR \(T^2_{\hat{r}_+}\) via the transition

\[
f(T^2_r, \hat{r}_+) : T^2_r \rightarrow T^2_{\hat{r}_+} \tag{13}
\]

and its corresponding inverse

\[
f^{-1}(T^2_{\hat{r}_+}, \hat{r}_+) : T^2_{\hat{r}_+} \rightarrow T^2_r, \tag{14}
\]

such that the iso-unit \(\hat{r}_+\) is the exterior iso-radius of \(T^2_{\hat{r}_+}\), which is “outside” of \(T^2_r\) because eq. (2) becomes

\[
T^2_{\hat{r}_+} \equiv \{\hat{y}_{\hat{r}_+} \in Y : |\hat{y}_{\hat{r}_+}| = r \times \hat{r}_+\} \tag{15}
\]

for

\[
\hat{y}_{\hat{r}_+} \equiv \hat{y} \times \hat{r}_+, \; \forall \hat{y} \in T^2_r \rightarrow \forall \hat{y}_{\hat{r}_+} \in T^2_{\hat{r}_+}, \tag{16}
\]

where \(T^2_r\) and \(T^2_{\hat{r}_+}\) are locally iso-morphic \([6, 10]\). Therefore, given that \(V_{T^2_r} \subset T^2_r\), the iso-topic lifting of eqs. (13–16) indicates

\[
\begin{align*}
\tilde{a}_{\hat{r}_+} &\equiv \tilde{a}_r \times \hat{r}_+ \\
\tilde{b}_{\hat{r}_+} &\equiv \tilde{b}_r \times \hat{r}_+ \\
\tilde{c}_{\hat{r}_+} &\equiv \tilde{c}_r \times \hat{r}_+ \\
\tilde{d}_{\hat{r}_+} &\equiv \tilde{d}_r \times \hat{r}_+ \\
\tilde{\bar{a}}_{\hat{r}_+} &\equiv \tilde{\bar{a}}_r \times \hat{r}_+ \\
\tilde{\bar{b}}_{\hat{r}_+} &\equiv \tilde{\bar{b}}_r \times \hat{r}_+ \\
\tilde{\bar{c}}_{\hat{r}_+} &\equiv \tilde{\bar{c}}_r \times \hat{r}_+ \\
\tilde{\bar{d}}_{\hat{r}_+} &\equiv \tilde{\bar{d}}_r \times \hat{r}_+.
\end{align*} \tag{17}
\]
enabling us to rewrite eq. (6) to establish the \textit{exterior cubic iso-vertex}
\textit{iso-triplex amplitude-radius constraints}

\begin{align}
\hat{r}_+ & \equiv |\vec{a}_{\hat{r}_+}| \equiv |\vec{b}_{\hat{r}_+}| \equiv |\vec{c}_{\hat{r}_+}| \equiv |\vec{d}_{\hat{r}_+}| \\
& \equiv |\vec{\bar{a}}_{\hat{r}_+}| \equiv |\vec{\bar{b}}_{\hat{r}_+}| \equiv |\vec{\bar{c}}_{\hat{r}_+}| \equiv |\vec{\bar{d}}_{\hat{r}_+}|
\end{align}

(18)

with the \textit{exterior iso-vertex} directional-preservations

\begin{align}
\langle \vec{a}_{\hat{r}_+} \rangle & \equiv \langle \vec{a}_{\hat{r}} \rangle \quad |\vec{a}_{\hat{r}_+}| \equiv |\vec{a}_{\hat{r}}| \\
\langle \vec{b}_{\hat{r}_+} \rangle & \equiv \langle \vec{b}_{\hat{r}} \rangle \quad |\vec{b}_{\hat{r}_+}| \equiv |\vec{b}_{\hat{r}}| \\
\langle \vec{c}_{\hat{r}_+} \rangle & \equiv \langle \vec{c}_{\hat{r}} \rangle \quad |\vec{c}_{\hat{r}_+}| \equiv |\vec{c}_{\hat{r}}| \\
\langle \vec{d}_{\hat{r}_+} \rangle & \equiv \langle \vec{d}_{\hat{r}} \rangle \quad |\vec{d}_{\hat{r}_+}| \equiv |\vec{d}_{\hat{r}}| \\
\langle \vec{\bar{a}}_{\hat{r}_+} \rangle & \equiv \langle \vec{\bar{a}}_{\hat{r}} \rangle \quad |\vec{\bar{a}}_{\hat{r}_+}| \equiv |\vec{\bar{a}}_{\hat{r}}| \\
\langle \vec{\bar{b}}_{\hat{r}_+} \rangle & \equiv \langle \vec{\bar{b}}_{\hat{r}} \rangle \quad |\vec{\bar{b}}_{\hat{r}_+}| \equiv |\vec{\bar{b}}_{\hat{r}}| \\
\langle \vec{\bar{c}}_{\hat{r}_+} \rangle & \equiv \langle \vec{\bar{c}}_{\hat{r}} \rangle \quad |\vec{\bar{c}}_{\hat{r}_+}| \equiv |\vec{\bar{c}}_{\hat{r}}| \\
\langle \vec{\bar{d}}_{\hat{r}_+} \rangle & \equiv \langle \vec{\bar{d}}_{\hat{r}} \rangle \quad |\vec{\bar{d}}_{\hat{r}_+}| \equiv |\vec{\bar{d}}_{\hat{r}}|
\end{align}

(19)

that continue to satisfy the generalized constraints of eqs. (7–11) to establish

\begin{equation}
\{\vec{a}_{\hat{r}_+, \hat{r}}, \vec{b}_{\hat{r}_+, \hat{r}}, \vec{c}_{\hat{r}_+, \hat{r}}, \vec{d}_{\hat{r}_+, \hat{r}}, \vec{\bar{a}}_{\hat{r}_+, \hat{r}}, \vec{\bar{b}}_{\hat{r}_+, \hat{r}}, \vec{\bar{c}}_{\hat{r}_+, \hat{r}}, \vec{\bar{d}}_{\hat{r}_+, \hat{r}}\} \equiv V_{T^2_{\hat{r}_+}} \subset T^2_{\hat{r}_+} \subset Y_+
\end{equation}

(20)

for the implied exterior vertex iso-topic lifting \(V_{T^2_{\hat{r}_+}} \rightarrow V_{T^2_{\hat{r}_+}}\), where \(V_{T^2_{\hat{r}_+}}\) is the exterior set of 8 iso-triplex iso-vertices that are confined to \(T^2_{\hat{r}_+}\) and form the \textit{exterior cube} of the tesseract for the \textit{exterior dynamical system} of the macro sub-space 3-brane \(Y_+\).

5. Fifth, given eqs. (13–16), the relation \(\hat{r}_- = \frac{1}{\hat{r}_+}\) is the foundation of the exterior and interior IHR iso-duality of [10], where the iso-unit inverse \(\hat{r}_-\) is the \textit{interior iso-radius} of the \textit{interior iso-2-sphere} IHR \(T^2_{\hat{r}_-}\) that is “inside” of \(T^2_{\hat{r}}\), such that \(T^2_{\hat{r}_-}\) is simultaneously iso-topically lifted to \(T^2_{\hat{r}_-}\) via the transition

\[ f(T^2_{\hat{r}}, \hat{r}_-) : \ T^2_{\hat{r}} \rightarrow T^2_{\hat{r}_-} \]

(21)

and its corresponding inverse

\[ f^{-1}(T^2_{\hat{r}_-}, \hat{r}_-) : \ T^2_{\hat{r}_-} \rightarrow T^2_{\hat{r}}, \]

(22)
because eq. (2) becomes

\[ T^2_{\hat{r}_-} = \{ \hat{y} \in Y : |\hat{y}| = r \times \hat{r}_- \} \]  

(23)

for

\[ \hat{y} \equiv \hat{y} \times \hat{r}_-, \forall \hat{y} \in T^2_r \rightarrow \forall \hat{y} \in T^2_{\hat{r}_-}, \]  

(24)

where \( T^2_r \) and \( T^2_{\hat{r}_-} \) are locally iso-morphic [6, 10]. Thus, the \( T^2_{\hat{r}_-} \) of eqs. (13–16) is iso-dual to the \( T^2_r \) of eqs. (21–24) with respect to \( T^2_r \) in accordance to the exterior and interior IHR iso-duality of [10]. Therefore, given that \( V_{\hat{T}^2} \subset T^2_r \), the iso-topic lifting of eqs. (21–24) indicates

\[
\begin{align*}
\bar{\mathbf{a}}_{\hat{r}_-} & \equiv \bar{\mathbf{a}}_r \times \hat{r}_- \\
\bar{\mathbf{b}}_{\hat{r}_-} & \equiv \bar{\mathbf{b}}_r \times \hat{r}_- \\
\bar{\mathbf{c}}_{\hat{r}_-} & \equiv \bar{\mathbf{c}}_r \times \hat{r}_- \\
\bar{\mathbf{d}}_{\hat{r}_-} & \equiv \bar{\mathbf{d}}_r \times \hat{r}_-
\end{align*}
\]  

(25)

enabling us to rewrite eq. (6) to establish the interior cubic iso-vertex iso-triplex amplitude-radius constraints

\[
\hat{r}_- \equiv |\bar{\mathbf{a}}_{\hat{r}_-}| \equiv |\bar{\mathbf{b}}_{\hat{r}_-}| \equiv |\bar{\mathbf{c}}_{\hat{r}_-}| \equiv |\bar{\mathbf{d}}_{\hat{r}_-}|
\]  

(26)

with the interior iso-vertex directional-preservations

\[
\begin{align*}
\langle \bar{\mathbf{a}}_{\hat{r}_-} \rangle & \equiv \langle \bar{\mathbf{a}}_r \rangle \equiv \langle \bar{\mathbf{a}}_{\hat{r}_+} \rangle \quad | \quad |\bar{\mathbf{a}}_{\hat{r}_-}| \equiv |\bar{\mathbf{a}}_r| \equiv |\bar{\mathbf{a}}_{\hat{r}_+}| \\
\langle \bar{\mathbf{b}}_{\hat{r}_-} \rangle & \equiv \langle \bar{\mathbf{b}}_r \rangle \equiv \langle \bar{\mathbf{b}}_{\hat{r}_+} \rangle \\
\langle \bar{\mathbf{c}}_{\hat{r}_-} \rangle & \equiv \langle \bar{\mathbf{c}}_r \rangle \equiv \langle \bar{\mathbf{c}}_{\hat{r}_+} \rangle \\
\langle \bar{\mathbf{d}}_{\hat{r}_-} \rangle & \equiv \langle \bar{\mathbf{d}}_r \rangle \equiv \langle \bar{\mathbf{d}}_{\hat{r}_+} \rangle
\end{align*}
\]  

(27)
that incorporate eq. (19) and continue to satisfy the generalized constraints of eqs. (7–11) to establish

\[
\{ \vec{a}_r, \vec{b}_r, \vec{c}_r, \vec{d}_r, \vec{\bar{a}}_r, \vec{\bar{b}}_r, \vec{\bar{c}}_r, \vec{\bar{d}}_r \} \equiv V_{T^2_{\hat{r}_-}} \subset T^2_{\hat{r}_-} \subset Y_-	ag{28}
\]

for the implied interior vertex iso-topic lifting \( V_{T^2_+} \rightarrow V_{T^2_{\hat{r}_-}} \), where \( V_{T^2_{\hat{r}_-}} \) is the interior set of 8 iso-triplex iso-vertices that are confined to \( T^2_{\hat{r}_-} \) and form the interior cube of the tesseract for the interior dynamical system of the micro sub-space 3-brane \( Y_- \).

6. Sixth, given the 8 exterior iso-triplex iso-vertices of \( T^2_{\hat{r}_+} \) in eq. (20) and the 8 interior iso-triplex iso-vertices of \( T^2_{\hat{r}_-} \) in eq. (28), we identify the 16 iso-triplex iso-vertices of the iso-dual tesseract as

\[
V_{T^2_{\hat{r}_\pm}} \equiv V_{T^2_{\hat{r}_+}} \cup V_{T^2_{\hat{r}_-}},
\]

where 8 additional edges are inserted to inter-link the iso-vertex pairs in a pairwise fashion to inter-connect Santilli’s exterior and interior dynamical systems for \( Y_+ \) and \( Y_- \), respectively. See Figure 3 for a depiction of the iso-dual tesseract.

7. Seventh, it is straightforward to assign triplex order parameters \([6, 7, 9]\) to the iso-vertices of eq. (29) to topologically deform the tesseract. For example, suppose that one layer of triplex order parameters \([6, 7, 9]\) is assigned to the 8 vertices of \( V_{T^2_+} \) as

\[
\{ \vec{\psi}(\vec{a}_r), \vec{\psi}(\vec{b}_r), \vec{\psi}(\vec{c}_r), \vec{\psi}(\vec{d}_r), \vec{\psi}(\vec{\bar{a}}_r), \vec{\psi}(\vec{\bar{b}}_r), \vec{\psi}(\vec{\bar{c}}_r), \vec{\psi}(\vec{\bar{d}}_r) \} \equiv \vec{\psi}_{T^2_+}
\]

to encode topological deformations that comply with the antisymmetric constraints

\[
\begin{align*}
\vec{\psi}(\vec{a}_r) &\equiv -\vec{\psi}(\vec{\bar{a}}_r) \\
\vec{\psi}(\vec{b}_r) &\equiv -\vec{\psi}(\vec{\bar{b}}_r) \\
\vec{\psi}(\vec{c}_r) &\equiv -\vec{\psi}(\vec{\bar{c}}_r) \\
\vec{\psi}(\vec{d}_r) &\equiv -\vec{\psi}(\vec{\bar{d}}_r)
\end{align*}
\]

(31)

that are depicted in Figure 4.
Fig. 3: The 8 triplex vertices of $V^2_T \subset T^2_r$ are iso-topically lifted via the double-projection iso-dual transition $V^2_T \subset T^2_r \leftrightarrow V^2_T \rightarrow V^2_T \subset T^2_r$ to generate the 16 iso-triplex iso-vertices of $V^2_T \pm$ for the iso-dual tesseract. Here, the exterior cube’s 8 exterior iso-vertices in $V^2_T$ are confined to the exterior IHR $T^2_{r+} \subset Y_+$ (not shown) in the exterior dynamical system while the interior cube’s 8 interior iso-vertices in $V^2_T$ are confined to the interior IHR $T^2_{r-} \subset Y_-$ (not shown) in the interior dynamical system, which are iso-dual to each other and are both iso-morphic, inter-locking, and synchronized to the initial cube [10].
Fig. 4: The 8 triplex vertices of $V_{T^2} \subset T^2_r$ are assigned one layer of triplex order parameters $[6, 7, 9]$ to encode topological deformations. These order parameter states can be iso-topically lifted $[1, 2, 3, 4, 5, 6]$ to iso-triplex iso-vertex order parameter states in a double-projective fashion for the iso-dual tesseract.
8. Finally, we can simply select some positive-definite iso-unit with a corresponding inverse (i.e. we can reuse $\hat{r}_+$ and $\hat{r}_-$ or select alternative quantities) and repeat the iso-dual iso-topic lifting of Steps 1–6 for the vertice’s triplex order parameters of eqs. (30–31) to define iso-triplex order parameters for the iso-dual tesseract. Thus, if we opt to redeploy $\hat{r}_+$ and $\hat{r}_-$ we define the iso-dual iso-topic liftings

$$\tilde{\psi}(\hat{a}_r) \equiv \tilde{\psi}(\bar{a}_r) \times \hat{r}_+ \quad | \quad \tilde{\psi}(\bar{a}_r) \equiv \tilde{\psi}(-\bar{a}_r) \times \hat{r}_-$$

$$\tilde{\psi}(\hat{b}_r) \equiv \tilde{\psi}(\bar{b}_r) \times \hat{r}_+ \quad | \quad \tilde{\psi}(\bar{b}_r) \equiv \tilde{\psi}(-\bar{b}_r) \times \hat{r}_-$$

$$\tilde{\psi}(\hat{c}_r) \equiv \tilde{\psi}(\bar{c}_r) \times \hat{r}_+ \quad | \quad \tilde{\psi}(\bar{c}_r) \equiv \tilde{\psi}(-\bar{c}_r) \times \hat{r}_-$$

$$\tilde{\psi}(\hat{d}_r) \equiv \tilde{\psi}(\bar{d}_r) \times \hat{r}_+ \quad | \quad \tilde{\psi}(\bar{d}_r) \equiv \tilde{\psi}(-\bar{d}_r) \times \hat{r}_-$$

$$\tilde{\psi}(\hat{a}_r) \equiv \tilde{\psi}(\bar{a}_r) \times \hat{r}_+ \quad | \quad \tilde{\psi}(\bar{a}_r) \equiv \tilde{\psi}(-\bar{a}_r) \times \hat{r}_-$$

$$\tilde{\psi}(\hat{b}_r) \equiv \tilde{\psi}(\bar{b}_r) \times \hat{r}_+ \quad | \quad \tilde{\psi}(\bar{b}_r) \equiv \tilde{\psi}(-\bar{b}_r) \times \hat{r}_-$$

$$\tilde{\psi}(\hat{c}_r) \equiv \tilde{\psi}(\bar{c}_r) \times \hat{r}_+ \quad | \quad \tilde{\psi}(\bar{c}_r) \equiv \tilde{\psi}(-\bar{c}_r) \times \hat{r}_-$$

$$\tilde{\psi}(\hat{d}_r) \equiv \tilde{\psi}(\bar{d}_r) \times \hat{r}_+ \quad | \quad \tilde{\psi}(\bar{d}_r) \equiv \tilde{\psi}(-\bar{d}_r) \times \hat{r}_-$$

for the double-projection iso-morphic transitions

$$\tilde{\psi}(\hat{a}_r) \leftarrow \tilde{\psi}(\bar{a}_r) \rightarrow \tilde{\psi}(\bar{a}_r)$$

$$\tilde{\psi}(\hat{b}_r) \leftarrow \tilde{\psi}(\bar{b}_r) \rightarrow \tilde{\psi}(\bar{b}_r)$$

$$\tilde{\psi}(\hat{c}_r) \leftarrow \tilde{\psi}(\bar{c}_r) \rightarrow \tilde{\psi}(\bar{c}_r)$$

$$\tilde{\psi}(\hat{d}_r) \leftarrow \tilde{\psi}(\bar{d}_r) \rightarrow \tilde{\psi}(\bar{d}_r)$$

$$\tilde{\psi}(\hat{a}_r) \leftarrow \tilde{\psi}(\bar{a}_r) \rightarrow \tilde{\psi}(\bar{a}_r)$$

$$\tilde{\psi}(\hat{b}_r) \leftarrow \tilde{\psi}(\bar{b}_r) \rightarrow \tilde{\psi}(\bar{b}_r)$$

$$\tilde{\psi}(\hat{c}_r) \leftarrow \tilde{\psi}(\bar{c}_r) \rightarrow \tilde{\psi}(\bar{c}_r)$$

$$\tilde{\psi}(\hat{d}_r) \leftarrow \tilde{\psi}(\bar{d}_r) \rightarrow \tilde{\psi}(\bar{d}_r)$$
and the corresponding inverses

\[
\begin{align*}
\vec{\psi}(\vec{a}_{\hat{r}}^-) & \rightarrow \vec{\psi}(\vec{a}_r) \leftarrow \vec{\psi}(\vec{a}_{\hat{r}}^+) \\
\vec{\psi}(\vec{b}_{\hat{r}}^-) & \rightarrow \vec{\psi}(\vec{b}_r) \leftarrow \vec{\psi}(\vec{b}_{\hat{r}}^+) \\
\vec{\psi}(\vec{c}_{\hat{r}}^-) & \rightarrow \vec{\psi}(\vec{c}_r) \leftarrow \vec{\psi}(\vec{c}_{\hat{r}}^+) \\
\vec{\psi}(\vec{d}_{\hat{r}}^-) & \rightarrow \vec{\psi}(\vec{d}_r) \leftarrow \vec{\psi}(\vec{d}_{\hat{r}}^+) \\
\vec{\psi}(\bar{\vec{a}}_{\hat{r}}^-) & \rightarrow \vec{\psi}(\bar{\vec{a}}_r) \leftarrow \vec{\psi}(\bar{\vec{a}}_{\hat{r}}^+) \\
\vec{\psi}(\bar{\vec{b}}_{\hat{r}}^-) & \rightarrow \vec{\psi}(\bar{\vec{b}}_r) \leftarrow \vec{\psi}(\bar{\vec{b}}_{\hat{r}}^+) \\
\vec{\psi}(\bar{\vec{c}}_{\hat{r}}^-) & \rightarrow \vec{\psi}(\bar{\vec{c}}_r) \leftarrow \vec{\psi}(\bar{\vec{c}}_{\hat{r}}^+) \\
\vec{\psi}(\bar{\vec{d}}_{\hat{r}}^-) & \rightarrow \vec{\psi}(\bar{\vec{d}}_r) \leftarrow \vec{\psi}(\bar{\vec{d}}_{\hat{r}}^+).
\end{align*}
\]

(34)

At this point, we’ve completed the construction of the iso-dual tesseract by generalizing the dual 4D space-time IHR topology of Section 2.1 with the exterior and interior iso-duality [10].

3 Conclusion

In this research investigation, we deployed Santilli’s iso-mathematics [1, 2, 3, 4, 5, 6] and Inopin’s dual 4D space-time IHR topology [6, 7, 8, 9] as a platform to assemble the iso-dual tesseract from two inter-locking, iso-morphic, iso-dual cubes in Euclidean triplex space that fundamentally comply with exterior and interior IHR iso-duality [10]. To prove that such a tesseract can be built from one cube (rather than two distinct cubes), we presented the step-by-step procedure of Section 2 with simple, flexible, topologically-preserving instructions, where the single, initial cube was iso-topically lifted to simultaneously infer the exterior cube and the interior cube via double-projection. Subsequently, the exterior cube and the interior cube were inter-linked together in a point-by-point fashion by inter-linking the 8 iso-vertex pairs with 8 additional edges to superstruct the iso-dual tesseract. In total, the outcomes of this exploration are significant because an original iso-geometrical inter-connection between Santilli’s exterior and interior dynamical systems has been established, which advances the application of iso-mathematics [1, 2, 3, 4, 5, 6] in a new direction.

We suggest that the next logical step of this research process should be to assign triplex order parameters [6, 7, 8, 9] to further encode topologi-
cal deformations and thereby define a complete “iso-dual tesseract wave-
function”. From there, we may continue to launch from this platform to
explore this frontier along various trajectories and assess the application
of geno-mathematics and hyper-mathematics [1, 2, 3, 4, 5]. Thus, this
developing iso-geometrical framework warrants further development, scrutiny,
collaboration, and hard work in order to advance it for future application
in the discipline of science.

References
[1] R. M. Santilli. Isonumbers and genonumbers of dimensions 1, 2, 4, 8,
their isoduals and pseudoduals, and ”hidden numbers” of dimension 3,
Proceedings-American Institute of Physics, volume 1479, page 1013,
2012.
to triplex numbers, fractals, and Inopin’s holographic ring: prelimi-
nary assessment and new lemmas. Accepted in the Hadronic Journal
[viXra:1308.0051], 36, 2013.
Riemannian dual space-time topology equipped with order parameter
dual space-time. Accepted in the Hadronic Journal [viXra:1308.0052],
36, 2013.
space-time: hypothesis and preliminary construction. Accepted in the
[10] N. O. Schmidt. Dynamic iso-sphere holographic rings with ex-
terior and interior iso-duality. Accepted in the Hadronic Journal [viXra:1311.0031], 2013.