Convergence of Quadratic Sequences

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Taking the Definition of the Derivative:

$$f'_{(x)} = \lim_{h \to 0} \frac{f_{(x+h)} - f_{(x)}}{h}$$

We can say:

$$f_{(x+h)} \cong f_{(x)} + hf'_{(x)}$$

And the smaller h, the more precise the approximation.

Let:

$$f_{(x)} = \sqrt{x}$$

Let's suppose that we have a perfect square and, therefore, we know its square root.

$$b = \sqrt{a} \qquad b \in \mathbb{N}; a \in \mathbb{N}$$

But also we have an integer that is not a perfect square and we want to calculate the approximate value of its square root.

$$d = \sqrt{c} \qquad d \in \mathbb{R}; c \in \mathbb{N}$$

Let's assume: $c > a$

Then:

$$c = a + h \qquad h \in \mathbb{N}$$

This means: $d > b$

Then:

$$d = b + m \qquad m \in \mathbb{R}$$

Now we can do the following replacements:

$$\sqrt{c} = \sqrt{a} + m$$

$$c = \left(\sqrt{a} + m\right)^2 = a + 2m\sqrt{a} + m^2$$

But we said that:

$$c = a + h$$

Then:

$$h = c - a$$

$$h = a + 2m\sqrt{a} + m^2 - a$$

$$h = 2m\sqrt{a} + m^2$$

On the other hand, the derivative of $f_{(x)}$ is:

$$f'_{(x)} = \frac{d}{dx}\sqrt{x}$$

$$f'_{(x)} = \frac{1}{2\sqrt{x}}$$

$$f'_{(x)} = \frac{1}{2f_{(x)}}$$

Hence, we can say that:

$$f_{(c)} = f_{(a+h)} \cong f_{(a)} + hf'_{(a)}$$

$$f_{(c)} \cong f_{(a)} + \frac{2m\sqrt{a} + m^2}{2f_{(a)}}$$

But:

$$m = d - b$$

$$m = \sqrt{c} - \sqrt{a}$$

$$m = f_{(c)} - f_{(a)}$$

Then:

$$f_{(c)} \cong f_{(a)} + \frac{2\left(f_{(c)} - f_{(a)}\right)f_{(a)} + \left(f_{(c)} - f_{(a)}\right)^2}{2f_{(a)}}$$

$$f_{(c)} \cong f_{(a)} + f_{(c)} - f_{(a)} + \frac{\left(f_{(c)} - f_{(a)}\right)^2}{2f_{(a)}}$$

$$f_{(c)} \cong f_{(c)} + \frac{\left(f_{(c)} - f_{(a)}\right)^2}{2f_{(a)}} \qquad \textbf{\textit{Eq. 1}}$$

And here we could find the "error" that we made when we did the approximation with the formula of the Derivative.

$$e = \frac{\left(f_{(c)} - f_{(a)}\right)^2}{2f_{(a)}}$$

Now, with this error function in our hand, let's try to find out a better approximation of $f(c)$.
To do this, we are going to assume that $c = 2a$.
So, let's rewrite the error function as a function of c.

$$f_{(a)} = f_{\left(\frac{c}{2}\right)}$$

But:

$$f_{(x)} = \sqrt{x}$$

So:

$$f_{(a)} = \frac{f_{(c)}}{\sqrt{2}}$$

Then:

$$e_{(c)} = \frac{\left(f_{(c)} - \frac{f_{(c)}}{\sqrt{2}}\right)^2}{\frac{2f_{(c)}}{\sqrt{2}}}$$

$$e_{(c)} = \frac{\left(\frac{\sqrt{2}f_{(c)} - f_{(c)}}{\sqrt{2}}\right)^2}{\sqrt{2}f_{(c)}}$$

$$e_{(c)} = \frac{\frac{\left(f_{(c)}(\sqrt{2}-1)\right)^2}{2}}{\sqrt{2}f_{(c)}}$$

$$e_{(c)} = \frac{f_{(c)}^2\left(\sqrt{2}-1\right)^2}{2\sqrt{2}f_{(c)}}$$

$$e_{(c)} = f_{(c)}\left(\frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}}\right) \qquad \textbf{\textit{Eq. 2}}$$

Now we have our error function as a function of c, we are going to calculate a new value for the error, introducing the first approach of $f_{(c)}$ that we found in ***Eq. 1***.

Recall ***Eq. 1***, rewritten with $e_{(c)}$:

$$f_{(c)} \cong f_{(c)} + e_{(c)}$$

And we are going to use a subindex for this first approach.

$$f_{(c)_0} \cong f_{(c)_0} + e_{(c)_0}$$

So, from ***Eq.2***:

$$e_{(c)_1} = \left(f_{(c)_0} + e_{(c)_0}\right)\left(\frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}}\right)$$

$$e_{(c)_1} = f_{(c)_0}\left(1 + \frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}}\right)\left(\frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}}\right)$$

As:

$$\left(\frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}}\right) < 1$$

Then:

$$\left(1 + \frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}}\right)\left(\frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}}\right) < 2\left(\frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}}\right)$$

So:

$$e_{(c)_1} < 2\, e_{(c)_0}$$

But we know that our first approach of $f_{(c)}$ was:

$$f_{(c)} \cong f_{(a)} + hf'_{(a)}$$

And this is equal to:

$$f_{(c)} \cong f_{(c)} + e_{(c)}$$

So, we are going to stand that the first approach of $f_{(c)}$ will be:

$$f_{(c)_0} = f_{(a)} + hf'_{(a)}$$

Next, the second approach will be:

$$f_{(c)_1} = f_{(a)} + hf'_{(a)} - e_{(c)_0}$$

And, as we are substracting $e_{(c)_0}$ from the first approach, we are substracting a value wich is less than 2 times the value of the error added in the first approach. So, this second approach is better than the first.
But, while the first approach was greater than the real value of $f_{(c)}$, the second approach will be less than the real value of $f_{(c)}$.

Next, the third approach will be:

$$f_{(c)_2} = f_{(a)} + hf'_{(a)} - e_{(c)_1}$$

As $e_{(c)_1}$ is based on $f_{(c)_1}$ and there is a smaller distance between $f_{(c)_1}$ and $f_{(c)}$ than $f_{(c)_0}$ and $f_{(c)}$, then the error introduced (added) in $f_{(c)_2}$ will be smaller than the error added in the first approach. And, moreover, $e_{(c)_1}$ is less than 2 times $e_{(c)_0}$.
So, $f_{(c)_2}$ is a better approach than $f_{(c)_1}$.

And we can keep going with this technique.
The next approach will be the first approach minus the error based on the approach before.

So, we can build a Sequence using this technique.

$$f_{(c)_n} = f_{(c)_0} - \frac{\left(f_{(c)_{n-1}} - f_{(a)}\right)^2}{2f_{(a)}} \quad \textit{\textbf{Eq. 3}}$$

Where $c = a + h;\ c = 2a = 2h.$

Remember we said that a is a perfect square, and b is an integer wich is the square root of a.
So c is 2 times a perfect square.
Then:

$$f_{(c)} = b\sqrt{2}$$

And:

$$f_{(c)_0} = f_{(a)} + hf'_{(a)} = f_{(a)} + \frac{h}{2f_{(a)}} = b + \frac{b^2}{2b} = \frac{3}{2}b$$

So, from ***Eq. 3***:

$$f_{(c)_n} = \frac{3}{2}b - \frac{\left(f_{(c)_{n-1}} - b\right)^2}{2b} = \frac{3}{2}b - \frac{f_{(c)}{}_{n-1}^{2} - 2bf_{(c)_{n-1}} + b^2}{2b}$$

$$f_{(c)_n} = \frac{3}{2}b - \frac{f_{(c)}{}_{n-1}^{2}}{2b} + f_{(c)_{n-1}} - \frac{1}{2}b$$

$$f_{(c)_n} = b - \frac{f_{(c)}{}_{n-1}^{2}}{2b} + f_{(c)_{n-1}}$$

In other words, we can say that the Quadratic Sequence:

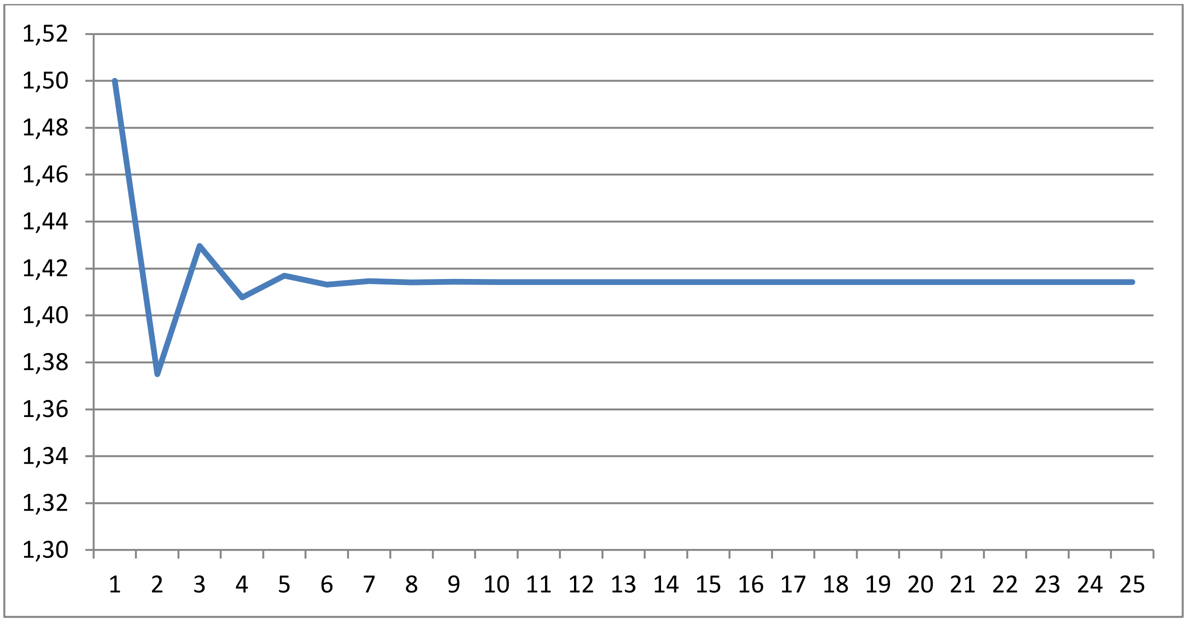
$$a_n = b - \frac{a_{n-1}^2}{2b} + a_{n-1}$$

Converges to:
$b\sqrt{2}$

With:
$a_0 = \frac{3}{2}b;\ b \in \mathbb{N}$

The following graphics show the shapes of these convergences.

Convergence to $\sqrt{2}$:



Convergence to $2\sqrt{2}$: