Convergence of Quadratic Sequences

Roberto Luis Recalde Gutiérrez

Taking the Definition of the Derivative:

$$f'_{(x)} = \lim_{h\to 0}\frac{f_{(x+h)} - f_{(x)}}{h}$$

We can say:

$$f_{(x+h)} \cong f_{(x)} + hf'_{(x)}$$

And the smaller h, the more precise the approximation.

Let:

$$f_{(x)} = \sqrt{x}$$

Let's suppose that we have a perfect square and, therefore, we know its square root.

$$b = \sqrt{a} \qquad b \in \mathbb{N}; a \in \mathbb{N}$$

But also we have an integer that is not a perfect square and we want to calculate the approximate value of its square root.

$$d = \sqrt{c} \qquad d \in \mathbb{R}; c \in \mathbb{N}$$

Let's assume: $c > a$

Then:

$$c = a + h \qquad h \in \mathbb{N}$$

This means: $d > b$

Then:

$$d = b + m \qquad m \in \mathbb{R}$$

Now we can do the following replacements:

$$\sqrt{c} = \sqrt{a} + m$$

$$c = \left(\sqrt{a} + m\right)^2 = a + 2m\sqrt{a} + m^2$$

But we said that:

$$c = a + h$$

Then:
$h = c - a$

$h = a + 2m\sqrt{a} + m^2 - a$

$h = 2m\sqrt{a} + m^2$

On the other hand, the derivative of $f_{(x)}$ is:

$$f'_{(x)} = \frac{d}{dx}\sqrt{x}$$

$$f'_{(x)} = \frac{1}{2\sqrt{x}}$$

$$f'_{(x)} = \frac{1}{2f_{(x)}}$$

Hence, we can say that:

$$f_{(c)} = f_{(a+h)} \cong f_{(a)} + hf'_{(a)}$$

$$f_{(c)} \cong f_{(a)} + \frac{2m\sqrt{a} + m^2}{2f_{(a)}}$$

But:
$m = d - b$

$m = \sqrt{c} - \sqrt{a}$

$m = f_{(c)} - f_{(a)}$

Then:

$$f_{(c)} \cong f_{(a)} + \frac{2\left(f_{(c)} - f_{(a)}\right)f_{(a)} + \left(f_{(c)} - f_{(a)}\right)^2}{2f_{(a)}}$$

$$f_{(c)} \cong f_{(a)} + f_{(c)} - f_{(a)} + \frac{\left(f_{(c)} - f_{(a)}\right)^2}{2f_{(a)}}$$

$$f_{(c)} \cong f_{(c)} + \frac{\left(f_{(c)} - f_{(a)}\right)^2}{2f_{(a)}} \qquad \textbf{\textit{Eq. 1}}$$

And here we could find the "error" that we made when we did the approximation with the formula of the Derivative.

$$e = \frac{\left(f_{(c)} - f_{(a)}\right)^2}{2f_{(a)}} \qquad \textbf{\textit{Eq. 2}}$$

Now, with this error function in our hand, let's try to find out a better approximation of $f(c)$.
To do this, we are going to assume that $c = 2a$.
So, let's rewrite the error function as a function of c.

$$f_{(a)} = f_{\left(\frac{c}{2}\right)}$$

But:

$$f_{(x)} = \sqrt{x}$$

So:

$$f_{(a)} = \frac{f_{(c)}}{\sqrt{2}}$$

Then:

$$e_{(c)} = \frac{\left(f_{(c)} - \frac{f_{(c)}}{\sqrt{2}}\right)^2}{\frac{2f_{(c)}}{\sqrt{2}}}$$

$$e_{(c)} = \frac{\left(\frac{\sqrt{2}f_{(c)} - f_{(c)}}{\sqrt{2}}\right)^2}{\sqrt{2}f_{(c)}}$$

$$e_{(c)} = \frac{\frac{\left(f_{(c)}(\sqrt{2}-1)\right)^2}{2}}{\sqrt{2}f_{(c)}}$$

$$e_{(c)} = \frac{f_{(c)}^2\left(\sqrt{2}-1\right)^2}{2\sqrt{2}f_{(c)}}$$

$$e_{(c)} = f_{(c)}\left(\frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}}\right)$$

Now we have our error function as a function of c, the first approach of $f_{(c)}$ will be:

$$f_{(c)_0} = f_{(c)} + e_{(c)}$$

$$f_{(c)_0} = f_{(c)} + f_{(c)}\left(\frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}}\right)$$

$$f_{(c)_0} = f_{(c)}\left(1 + \frac{\left(\sqrt{2}-1\right)^2}{2\sqrt{2}}\right)$$

$$f_{(c)_0} = f_{(c)}\frac{3}{2\sqrt{2}}$$ ***Eq. 3***

So, using the definition of the Derivative, we found in ***Eq. 3*** the first approach of $f_{(c)}$.

From now we are going to find better approximations of $f_{(c)}$, replacing the previous approximation into the error function of the new approximation.
In other words, we are going to build a Sequence wich converges to $f_{(c)}$.

To do this, we will take the first approach and substract a new error value each time.
The subsequent values of $f_{(c)}$ will be:

$$f_{(c)_1} = f_{(c)}\frac{3}{2\sqrt{2}} - e_{(c)_0}$$

$$f_{(c)_2} = f_{(c)}\frac{3}{2\sqrt{2}} - e_{(c)_1}$$

$$f_{(c)_3} = f_{(c)}\frac{3}{2\sqrt{2}} - e_{(c)_2}$$

And so on…
Where $e_{(c)_n}$ is a function of $f_{(c)_n}$.

So, we can say:

$$f_{(c)_n} = f_{(c)}k_n$$

Where k_n is a factor that produces each approximation of $f_{(c)}$.

Then, using ***Eq. 2*** for the error function:

$$f_{(c)_n} = f_{(c)}\frac{3}{2\sqrt{2}} - \frac{\left(f_{(c)_{n-1}} - f_{(a)}\right)^2}{2f_{(a)}}$$

$$f_{(c)_n} = f_{(c)}\frac{3}{2\sqrt{2}} - \frac{\left(f_{(c)_{n-1}} - \frac{f_{(c)}}{\sqrt{2}}\right)^2}{\frac{2f_{(c)}}{\sqrt{2}}}$$

$$f_{(c)_n} = f_{(c)}\frac{3}{2\sqrt{2}} - \frac{\left(\frac{\sqrt{2}f_{(c)_{n-1}} - f_{(c)}}{\sqrt{2}}\right)^2}{\sqrt{2}f_{(c)}}$$

$$f_{(c)_n} = f_{(c)}\frac{3}{2\sqrt{2}} - \frac{2{f_{(c)}}_{n-1}^{2} - 2\sqrt{2}f_{(c)_{n-1}}f_{(c)} + f_{(c)}^2}{2\sqrt{2}f_{(c)}}$$

$$f_{(c)_n} = f_{(c)}\frac{3}{2\sqrt{2}} - \frac{2k_{n-1}^2 f_{(c)}^2}{2\sqrt{2}f_{(c)}} + k_{n-1}f_{(c)} - \frac{f_{(c)}}{2\sqrt{2}}$$

$$f_{(c)_n} = f_{(c)}\left(\frac{3}{2\sqrt{2}} - \frac{k_{n-1}^2}{\sqrt{2}} + k_{n-1} - \frac{1}{2\sqrt{2}}\right)$$

$$f_{(c)_n} = f_{(c)}\left(-\frac{k_{n-1}^2}{\sqrt{2}} + k_{n-1} + \frac{1}{\sqrt{2}}\right)$$

So, we can say:

$$k_n = -\frac{k_{n-1}^2}{\sqrt{2}} + k_{n-1} + \frac{1}{\sqrt{2}}$$

But, this will ensure that ${f_{(c)}}_n$ is a better approach than ${f_{(c)}}_{n-1}$?
What's the relationship between k_n and k_{n-1} ?

To answer these questions we have to analize two scenarios:

1) $k_{n-1} > 1$
2) $k_{n-1} < 1$

So first, let $k_{n-1} > 1$.
Which means:

$$k_{n-1} = 1 + r$$

Then:

$$f_{(c)_n} = f_{(c)}\left(-\frac{(1+r)^2}{\sqrt{2}} + 1 + r + \frac{1}{\sqrt{2}}\right)$$

$$f_{(c)_n} = f_{(c)}\left(-\frac{1+2r+r^2}{\sqrt{2}} + 1 + r + \frac{1}{\sqrt{2}}\right)$$

$$f_{(c)_n} = f_{(c)}\left(-\frac{1}{\sqrt{2}} - \sqrt{2}\,r - \frac{r^2}{\sqrt{2}} + 1 + r + \frac{1}{\sqrt{2}}\right)$$

$$f_{(c)_n} = f_{(c)}\left(-\frac{r^2}{\sqrt{2}} + r\left(1-\sqrt{2}\right) + 1\right)$$

So, the term that represents k_n is a parabola.
Let's find its global extrema.

$$\frac{d}{dr}\left(-\frac{r^2}{\sqrt{2}} + r\left(1-\sqrt{2}\right) + 1\right) = -\sqrt{2}\,r + 1 - \sqrt{2}$$

$$-\sqrt{2}\,r + 1 - \sqrt{2} = 0$$

$$r = \frac{1-\sqrt{2}}{\sqrt{2}}$$

Is it a Maximum or a Minimum?

$$\frac{d}{dr}\left(-\sqrt{2}\,r + 1 - \sqrt{2}\right) = -\sqrt{2}$$

The parabola is concave down, so it's a Maximum.
And this Maximum is negative. So, the graphic will be:

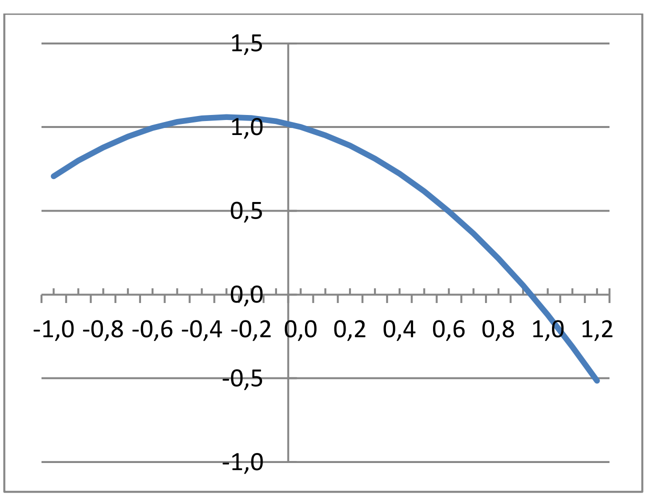


Figure 1

As $r > 0$ (Because we said that $k_{n-1} > 1$), we are going to analize only the interval $(0, \infty)$ for r.

In this interval, the slope of the tangent lines of the curve is always negative. And as r increases, the absolute value of the slope increases too (because it's a concave down parabola).

But let's go further.
For which value of r, the slope of its tangent line is -1?

$$-\sqrt{2}\,r + 1 - \sqrt{2} = -1$$

$$r = \frac{2-\sqrt{2}}{\sqrt{2}}$$

So, in the interval $\left(0, \frac{2-\sqrt{2}}{\sqrt{2}}\right)$ the variations in k_n are less than the variations in r.
And also we know that k_n is always less than 1 when r is greater than 0 (from the formula of k_n represented in the graphic).

Then, we can say:

$$k_n = 1 - s$$

And if:

$$0 < r < \frac{2-\sqrt{2}}{\sqrt{2}} \rightarrow \boldsymbol{s < r} \;\; \textit{because when } r = \frac{2-\sqrt{2}}{\sqrt{2}}, s = \frac{\sqrt{2}-1}{\sqrt{2}} \textit{ wich is less than } r.$$

So, if s is always less than r (in the interval we mentioned), even though k_n is less than 1, ${f_{(c)}}_n$ will be a better approach than ${f_{(c)}}_{n-1}$.

Now, let $k_{n-1} < 1$.
Which means:

$$k_{n-1} = 1 - r$$

Then:

$$f_{(c)_n} = f_{(c)}\left(-\frac{(1-r)^2}{\sqrt{2}} + 1 - r + \frac{1}{\sqrt{2}}\right)$$

$$f_{(c)_n} = f_{(c)}\left(-\frac{1-2r+r^2}{\sqrt{2}} + 1 - r + \frac{1}{\sqrt{2}}\right)$$

$$f_{(c)_n} = f_{(c)}\left(-\frac{1}{\sqrt{2}} + \sqrt{2}\,r - \frac{r^2}{\sqrt{2}} + 1 - r + \frac{1}{\sqrt{2}}\right)$$

$$f_{(c)_n} = f_{(c)}\left(-\frac{r^2}{\sqrt{2}} - r(1-\sqrt{2}) + 1\right)$$

So, the term that represents k_n is a parabola.
Let's find its global extrema.

$$\frac{d}{dr}\left(-\frac{r^2}{\sqrt{2}} - r(1-\sqrt{2}) + 1\right) = -\sqrt{2}\,r - 1 + \sqrt{2}$$

$$-\sqrt{2}\,r - 1 + \sqrt{2} = 0$$

$$r = \frac{\sqrt{2}-1}{\sqrt{2}}$$

Is it a Maximum or a Minimum?

$$\frac{d}{dr}\left(-\sqrt{2}\,r - 1 + \sqrt{2}\right) = -\sqrt{2}$$

The parabola is concave down, so it's a Maximum.
And this Maximum is positive. So, the graphic will be:

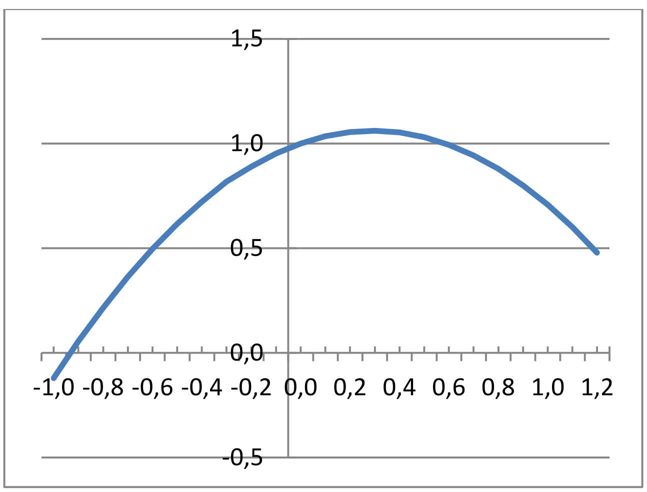


Figure 2

As $r > 0$ (Because we said that $k_{n-1} < 1$), we are going to analize only the interval $(0, \infty)$ for r.

In this interval, the slope of the tangent lines of the curve is positive when r is less than the Maximum, and negative when it is greater. Plus, when $r = 0$, the slope is less than 1.

So, in the interval $\left(0, \frac{\sqrt{2}-1}{\sqrt{2}}\right)$ the variations in k_n are less than the variations in r. And k_n is greater than 1.

Then, we can say:

$$k_n = 1 + s$$

And if:

$$0 < r < \frac{\sqrt{2}-1}{\sqrt{2}} \rightarrow \boldsymbol{s} < \boldsymbol{r} \ \ because\ when\ r = \frac{\sqrt{2}-1}{\sqrt{2}}, s = \frac{2\sqrt{2}-3}{2\sqrt{2}}\ wich\ is\ less\ than\ r.$$

So, if s is always less than r (in the interval we mentioned), even though k_n is greater than 1, ${f_{(c)}}_n$ will be a better approach than ${f_{(c)}}_{n-1}$.

Given the intervals we mentioned, we can say that the Sequence we proposed converges to $f_{(c)}$.

Let's do some more math!

The first approach of $f_{(c)}$ is:

$$f_{(c)_0} = f_{(c)}\frac{3}{2\sqrt{2}}$$

So:

$$k_0 = \frac{3}{2\sqrt{2}}$$

In this case, k_0 > 1.
Then:

$$k_0 = 1 + r_0 \ \rightarrow r_0 = \frac{3-2\sqrt{2}}{2\sqrt{2}}\ \ which\ is\ less\ than\ \frac{2-\sqrt{2}}{\sqrt{2}}$$

So, r_0 is in the interval that makes ${f_{(c)}}_1$ a better approach.

The second approach would be:

$$f_{(c)_1} = f_{(c)}\left(-\frac{k_0^2}{\sqrt{2}} + k_0 + \frac{1}{\sqrt{2}}\right)$$

$$f_{(c)_1} = f_{(c)}\left(-\frac{9}{8\sqrt{2}} + \frac{3}{2\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$f_{(c)_1} = f_{(c)}\frac{11}{8\sqrt{2}}$$

In this case, k_1 < 1.
Then:

$$k_1 = 1 - r_1 \ \rightarrow r_1 = \frac{8\sqrt{2} - 11}{8\sqrt{2}} \ \ which\ is\ less\ than\ \frac{\sqrt{2} - 1}{\sqrt{2}}$$

So, r_1 is in the interval that makes ${f_{(c)}}_2$ a better approach.

As ${f_{(c)}}_2$ will be greater than $f_{(c)}$ and a better approach than ${f_{(c)}}_0$, r_2 wil also be in the interval that makes ${f_{(c)}}_3$ a better approach.
And, as ${f_{(c)}}_3$ will be less than $f_{(c)}$ and a better approach than ${f_{(c)}}_1$, r_3 will also be in the interval that makes ${f_{(c)}}_4$ a better approach.

So, the Sequence we proposed really converges to $f_{(c)}$.

$$ {f_{(c)}}_n = f_{(c)} \frac{3}{2\sqrt{2}} - \frac{\left({f_{(c)}}_{n-1} - f_{(a)}\right)^2}{2f_{(a)}} $$

Remember we said that a is a perfect square, and b is an integer wich is the square root of a.
Also we said that $c = 2a$.
So c is 2 times a perfect square.
Then:

$$ f_{(c)} = b\sqrt{2} $$

$$ {f_{(c)}}_n = \frac{3}{2}b - \frac{\left({f_{(c)}}_{n-1} - f_{(a)}\right)^2}{2f_{(a)}} $$

$$ {f_{(c)}}_n = \frac{3}{2}b - \frac{\left({f_{(c)}}_{n-1} - b\right)^2}{2b} = \frac{3}{2}b - \frac{{f_{(c)}}_{n-1}^2 - 2b{f_{(c)}}_{n-1} + b^2}{2b} $$

$$ {f_{(c)}}_n = \frac{3}{2}b - \frac{{f_{(c)}}_{n-1}^2}{2b} + {f_{(c)}}_{n-1} - \frac{1}{2}b $$

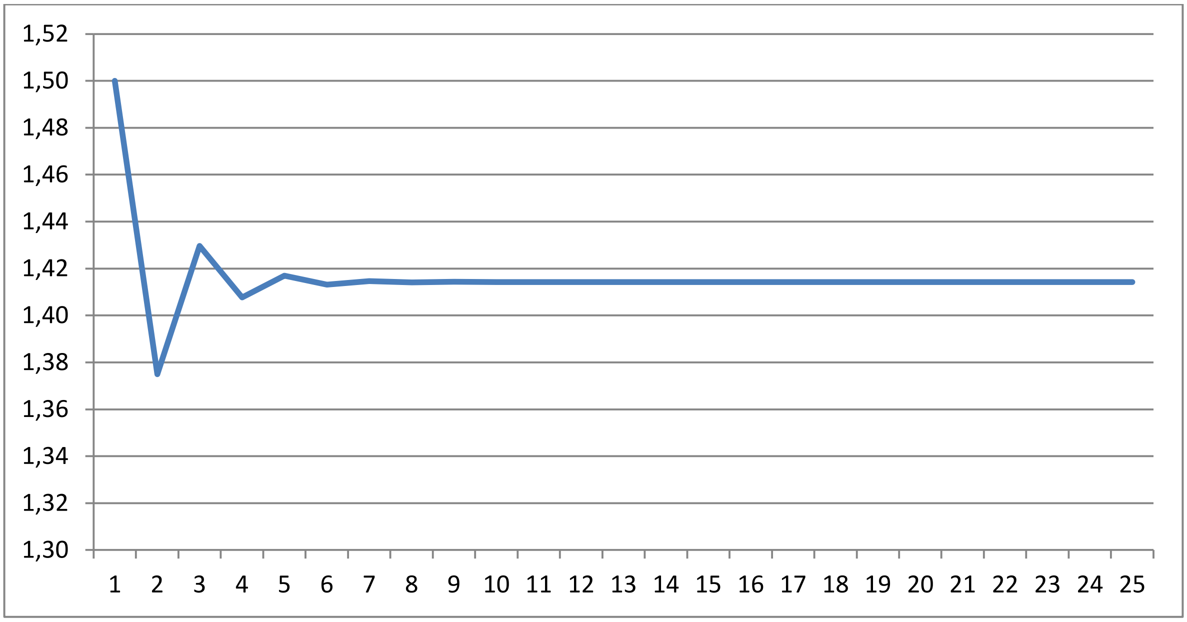
$$ {f_{(c)}}_n = b - \frac{{f_{(c)}}_{n-1}^2}{2b} + {f_{(c)}}_{n-1} $$

In other words, we can say that the following Quadratic Sequence converges to $b\sqrt{2}$:

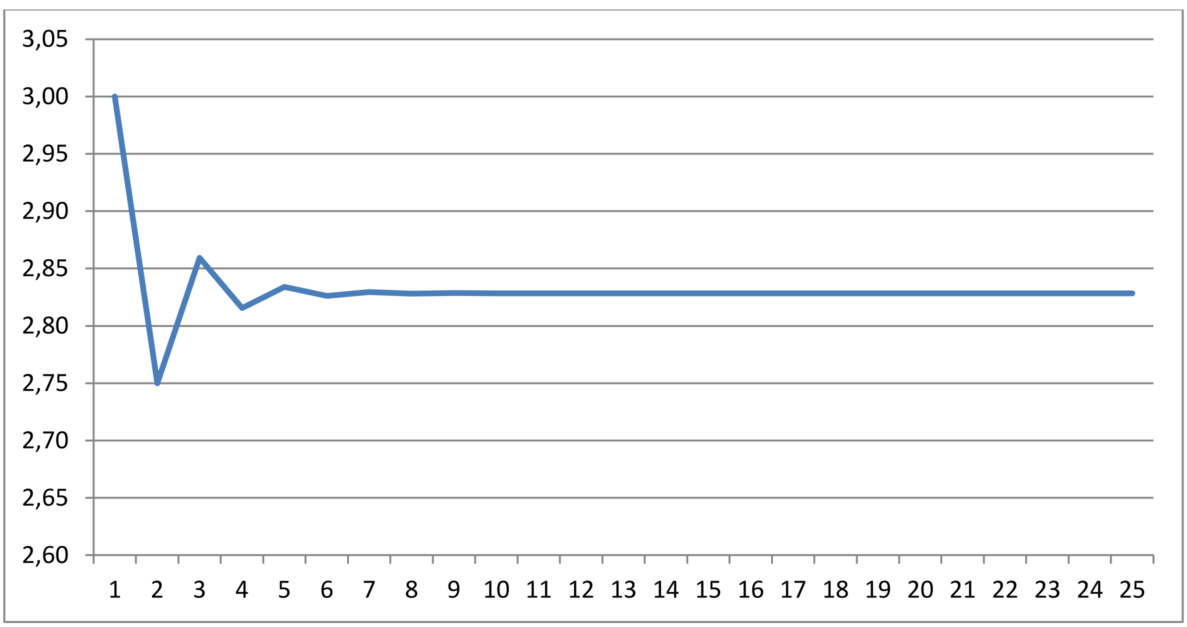
$$ a_n = b - \frac{a_{n-1}^2}{2b} + a_{n-1} \qquad\qquad a_0 = \frac{3}{2}b;\ b \in \mathbb{N} $$

The following graphics show the shapes of these convergences.

Convergence to $\sqrt{2}$:



Convergence to $2\sqrt{2}$:



Let's go further to find a nicer Quadratic Sequence.

Let:

$$a_n = \frac{1}{d_n} + b \qquad\qquad d_0 = \frac{2}{b}$$

Then:

$$\frac{1}{d_n} + b = b - \frac{\left(\frac{1}{d_{n-1}} + b\right)^2}{2b} + \frac{1}{d_{n-1}} + b$$

$$\frac{1}{d_n} + b = b - \frac{\frac{1}{d_{n-1}^2} + \frac{2b}{d_{n-1}} + b^2}{2b} + \frac{1}{d_{n-1}} + b$$

$$\frac{1}{d_n} + b = b - \frac{1}{2bd_{n-1}^2} - \frac{1}{d_{n-1}} - \frac{b}{2} + \frac{1}{d_{n-1}} + b$$

$$\frac{1}{d_n} = \frac{b}{2} - \frac{1}{2bd_{n-1}^2}$$

$$\frac{1}{d_n} = \frac{b^2 d_{n-1}^2 - 1}{2bd_{n-1}^2}$$

$$d_n = \frac{2bd_{n-1}^2}{b^2 d_{n-1}^2 - 1}$$

Let:

$$d_n = \frac{1}{g_n} \qquad\qquad g_0 = \frac{1}{2}b$$

Then:

$$\frac{1}{g_n} = \frac{\frac{2b}{g_{n-1}^2}}{\frac{b^2}{g_{n-1}^2} - 1}$$

$$\frac{1}{g_n} = \frac{\frac{2b}{g_{n-1}^2}}{\frac{b^2 - g_{n-1}^2}{g_{n-1}^2}}$$

$$\frac{1}{g_n} = \frac{2b}{b^2 - g_{n-1}^2}$$

$$g_n = \frac{b^2 - g_{n-1}^2}{2b}$$

As $g_n = \frac{1}{d_n}$ and $d_n = \frac{1}{a_n - b}$:

We can say that the following Quadratic Sequence converges to $b(\sqrt{2} - 1)$:

$$g_n = \frac{1}{2}b - \frac{g_{n-1}^2}{2b} \qquad g_0 = \frac{1}{2}b;\ b \in \mathbb{N}$$