THE RIEMANN HYPOTHESIS

Bertrand Wong
Eurotech, S'pore
Email: bwong8@singnet.com.sg

ABSTRACT

The Riemann hypothesis is an important outstanding problem in number theory as its validity will affirm the manner of the distribution of the prime numbers. It posits that all the non-trivial zeros of the zeta function ζ lie on the critical strip between Re(s) = 0 and Re(s) = 1 at the critical line Re(s) = 1/2. The important question is whether there would be zeros appearing at other locations on this critical strip, e.g., at Re(s) = 1/4, 1/3, 3/4, or, 4/5, etc., which would disprove the Riemann hypothesis. This paper provides a proof of the Riemann hypothesis.

MSC: 11-XX (Number Theory)

Keywords: prime counting function; zeta function; estimate of quantity of primes; primes; non-trivial zeros; contradiction

Theorem:- The Riemann hypothesis is true.

Proof:-

According to the precepts of fractal geometry, phenomena which appear random when viewed en masse display some orderliness and pattern which could be regarded as a fractal characteristic. For instance, the prime numbers are very random and haphazard entities, yet, when viewed en masse they display a regularity in the way they thin out, whereby it is affirmed that the number of primes not exceeding a given natural number n is approximately $n/\log n$, in the sense that the ratio of the number of such primes to $n/\log n$ eventually approaches 1 as n becomes larger and larger, log n being the natural logarithm (to the base e) of n (vide the prime number theorem proved in 1896 by Hadamard and de la Vallee-Poussin). In other words, the prime number theorem, which is the direct outcome of the Riemann hypothesis, states that the limit of the quotient of the 2 functions $\pi(n)$ and $n/\log n$ as n approaches infinity is 1, which is expressed by the formula:

$$\lim_{n\to\infty} \pi(n)/(n/\log n) = 1$$

the larger the number n is, the better is the approximation of the quantity of primes, as is implied by the above formula where $\pi(n)$ is the prime counting function (π here is not the π which is the constant 3.142 used to compute perimeters and areas of circles, but is only a convenient symbol adopted to denote the prime counting function)

All this is in spite of the fact that the primes are scarcer and scarcer as n is larger and larger.

The prime number theorem could in fact be regarded as a weaker version of the Riemann hypothesis which posits that all the non-trivial zeros of the zeta function ζ on the critical strip bounded by $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ would be at the critical line $\mathrm{Re}(s) = 1/2$. For a better understanding of the close connection between the prime number theorem and the Riemann hypothesis, it should be noted that Hadamard and de la Vallee Poussin had in 1896 independently proven that none of the non-trivial zeros lie on the very edge of the critical strip, on the lines $\mathrm{Re}(s) = 0$ or $\mathrm{Re}(s) = 1$ - this was enough for deducing the prime number theorem. The locations of these non-trivial zeros on the critical strip could be described by a complex number $1/2 + bi$ where the real part is 1/2 and i represents the square root of -1. It had already been proven that there is an infinitude of non-trivial zeros at the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$. The moot question is whether there would be any zeros off the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, e.g., at $\mathrm{Re}(s) = 1/4, 1/3, 3/4$, or, 4/5, etc., the presence of any of which would disprove the Riemann hypothesis. So far, no such "off-the-critical-line" zeros has been found.

The validity of the Riemann hypothesis would evidently imply the validity of the prime number theorem (which as described above is the offspring and weaker version of the Riemann hypothesis) though the validity of the prime number theorem does not imply the former. Nevertheless, both of them have one thing in common in that they are both concerned with the estimate of the quantity of primes less than a given number, with the Riemann hypothesis positing a more exact estimate of the quantity of primes less than a given number. But, on the other hand, what would be the result if the Riemann hypothesis were false? We will come back to this later.

Meanwhile, more about the non-trivial zeros of the zeta function $\zeta(s)$ defined by a power series shown below:

$$\zeta(s) = \sum_{n=1} 1/n^s = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \ldots$$

At the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ all the non-trivial zeros would be found on an oscillatory sine wave which oscillates in spirals, there being an infinitude of these spirals (representing the so-called complex plane). All the properties of the prime counting function $\pi(n)$ are in some way coded in the properties of the zeta function ζ, evidently resulting in the primes and the non-trivial zeros being some sort of mirror images of one another - the regularity in the way the primes progressively thin out and the progressively better approximation of the quantity of primes towards infinity by the prime counting function $\pi(n)$ mirror or reflect the regularity in the way the non-trivial zeros of the zeta function ζ line up at the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, the non-trivial zeros becoming progressively closer together there, with no zeros appearing anywhere else on the critical strip, and, all this has been found to be true for the 1^{st}. 10^{13} non-trivial zeros.

Riemann had posited that the margin of error in the estimate of the quantity of primes less than a given number with the prime counting function $\pi(n)$ could be eliminated by utilizing the following J function which is a step function involving the non-trivial zeros expressed in terms of the zeta function ζ, which has been shown to be effective (2 steps are involved here - first, the prime counting function $\pi(n)$ is expressed in terms of the $J(n)$ function, then the $J(n)$ function is expressed in terms of the zeta function ζ, with the $J(n)$ function forming the link between the counting of the prime counting function $\pi(n)$ and the measuring (involving analysis and calculus) of the zeta function ζ, which would result in the properties of the prime counting function π(*n*) somehow encoded in the properties of the zeta function ζ):

$$J(n) = Li(n) - \sum_{p} Li(n^p) - \log 2 + \int_{x}^{\infty} dt/(t(t^2 - 1) \log t)$$

where the 1st. term *Li*(*n*) is generally referred to as the "principal term" and the 2nd. term $\sum_{p} Li(n^p)$ had been called the "periodic terms" by Riemann, *Li* being the logarithmic integral

The above formula may look intimidating but is actually not. The 3rd. term log 2 is a number which is 0.69314718055994… while the 4th. term $1/(t(t^2 - 1) \log t)$ which is an integral representing the area under the curve of a certain function from the argument all the way out to infinity can only have a maximum value of 0.1400101011432869…. Since these 2 terms taken together (and minding the signs) are limited to the range from -0.6931… to -0.5531…, and since the prime counting function π(*n*) deals with really large quantities up to millions and trillions they are much inconsequential and can be safely ignored. The 1st. term or principal term *Li*(*n*), where *n* is a real number, should also be not much of a problem as its value can be obtained from a book of mathematical tables or computed by some math software package such as *Mathematica* or *Maple*. However, special attention should be given to the 2nd. term $\sum_{p} Li(n^p)$

which concerns the sum of the non-trivial zeros of the zeta function ζ (*p* in this 2nd. term is a "rho", which is the 17th. letter of the Greek alphabet, and it means "root" - a root is a non-trivial zero of the Riemann zeta function ζ - a root here is a solution or value of an unknown of an equation which could be factorized). Riemann had evidently called the 2nd. term "periodic terms" as the components there vary irregularly.

The prime number theorem asserts that $\pi(n) \sim Li(n)$ (technically $Li(n) = \int_{2}^{n} dx/\log(x)$) which also

implies the weaker result that $\pi(n) \sim n/\log n$. However, with *Li*(*n*) the prime count estimate would have a margin of error. The Riemann hypothesis asserts that the difference between the true number of primes *p*(*n*) and the estimated number of primes *q*(*n*) would be not much larger than $\sqrt{n}$. With the above *J*(*n*) function we could eliminate this margin of error and obtain an exact estimate of the quantity of primes less than a given number:

$$J(n) = \text{exact quantity of primes less than a given number}$$

Since the 3rd. and 4th. terms of the *J*(*n*) function are inconsequential and can be safely ignored, as is described above, deducting the 2nd. term from the 1st. term should be sufficient:

$$J(n) = Li(n) - \sum_{p} Li(n^p) = \text{exact quantity of primes less than a given number}$$

The above in a nutshell shows the intimate relationship between the primes and the non-trivial zeros of the zeta function ζ, the primes and the non-trivial zeros being some sort of mirror images of one another as is described above, with the distribution of the non-trivial zeros being regarded as the music of the primes by mathematicians.

We return to the question of the consequence of the falsity of the Riemann hypothesis. Let us here assume that the Riemann hypothesis is false, i.e., there are also zeros found off the critical line Re(s) = 1/2 on the critical strip between Re(s) = 0 and Re(s) = 1, e.g., at Re(s) = 1/4, 1/3, 3/4, or, 4/5, etc., and see the consequence. What would be the significant implication of this assumption? The falsity of the Riemann hypothesis would imply that the distribution of the zeros of the zeta function ζ on the critical strip between Re(s) = 0 and Re(s) = 1 has lost the regularity of pattern which is characteristic of the non-trivial zeros at the critical line Re(s) = 1/2 and which is described above, and is now disorderly and irregular. This would in turn imply that the distribution of the primes is also similarly disorderly and irregular since the primes and the non-trivial zeros of the zeta function ζ are intimately linked and are some sort of mirror images of one another - any changes in one of them would be reflected in the other on account of their intimate link - note that the zeta function ζ has the property of prime sieving (compare: sieve of Eratosthenes) encoded within it, the properties of the prime counting function $\pi(n)$ being somehow encoded in the properties of the zeta function ζ, so that if the zeros generated were disorderly and irregular it would mean that the distribution of the primes were also similarly disorderly and irregular - the characteristic of the primes on the input side of the function determines the characteristic of the zeros on the output side of the function (i.e., the distribution of the primes determines the distribution of the zeros, so that from a study of the distribution of the zeros the distribution of the primes could be deduced and vice versa), which is expected for a function. The overall result would be that the more orderly the distribution of the zeros is the more orderly would be the corresponding distribution of the primes, the more disorderly the distribution of the zeros is the more disorderly would be the corresponding distribution of the primes, and, vice versa. But, according to the prime number theorem, or, prime counting function $\pi(n)$, which is closely connected with the Riemann hypothesis itself being an offspring and weaker version of it as is described above, there is instead actually a regularity in the way the primes thin out, with the prime counting function $\pi(n)$ even providing a progressively better estimate of the quantity of primes towards infinity - this progressively better estimate would not be possible if the primes behave really badly and are really highly disorderly and irregular - there is no such really great disorder or irregularity among the primes, a state of affair which is evidently affirmed by the fact that the corresponding non-trivial zeros at the critical line Re(s) = 1/2 on the critical strip between Re(s) = 0 and Re(s) = 1 display regularity in the way they line up at the critical line Re(s) = 1/2, the non-trivial zeros becoming progressively closer together there with no zeros appearing anywhere else on the critical strip (all of which has been found to be true for the 1st. 10^{13} non-trivial zeros - an important point to note is that though the non-trivial zeros at the critical line Re(s) = 1/2 become more and more closely packed together the farther along we move up this critical line while the primes occur farther and farther along the number line, the density of the one is approximately the reciprocal of the density of the other wherein the complementariness, regularity, symmetry is evident), this regularity of the distribution of the non-trivial zeros mirroring the regularity of the distribution of the primes as is explained above. Our assumption of the falsity of the Riemann hypothesis has thus resulted in a contradiction of the actual distribution of the primes and the actual distribution of the corresponding non-trivial zeros at the critical line Re(s) = 1/2 on the critical strip between Re(s) = 0 and Re(s) = 1. If our assumption that the Riemann hypothesis is false is correct, the prime number theorem would be false as there would be great disorder and irregularity among the primes with no regularity in the way the primes thin out and without the prime counting function $\pi(n)$ providing a progressively better estimate of the quantity of primes towards infinity (this progressively better estimate of the quantity of primes actually implies some regularity in the distribution of the primes). However, as is explained just above the prime number theorem is not false; it had in fact been proven through both non-elementary methods (by Hadamard and de la Vallee Poussin) and elementary methods (by Erdos and Selberg later) and is indubitably true. Therefore, our assumption of the falsehood of the Riemann hypothesis is at fault. The Riemann hypothesis cannot be false and has to be true.

REFERENCES

[1] G. H. Hardy and E. M. Wright, 1979, An Introduction To Theory Of Numbers, Oxford, England: Clarendon Press

[2] H. M. Edwards, 2001, Riemann's Zeta Function, Dover Publications, Inc.

[3] B. B. Mandelbrot, 1977, The Fractal Geometry Of Nature, W. H. Freeman

[4] B. Riemann, 1859, On The Number Of Prime Numbers Less Than A Given Quantity