MEASUREING COMPLEXITY BY USING REDUCTION TO SOLVE P VS NP AND NC & PH HIERARCHY

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1. Abstract

This article describes about that NC/Polynomial hierarchy difference and P is not NP by using problem reduction. If L is not P, we can prove P is not NP by using difference between logarithm space reduction and polynomial time reduction. Like this, we can also prove NC hierarchy by using difference between AL0 and NC1. This means L is not P. Therefore P is not NP. And we can also prove Polynomial hierarchy by using P is not NP.

2. P is not NP if L is not P

Theorem 1. $L \subsetneq P \rightarrow P \subsetneq NP$

Proof. To prove it by using contraposition $P = NP \rightarrow L = P$. $P = NP$ then

$\forall A \in NP \exists B \in P\,(A = B)$

As we all know $NP \circ FP \in NP$. From assumption $P = NP$, all $NP \circ FP$ correspond to P. Therefore

$P = NP \rightarrow \forall C \in NP \forall D \in FP \exists E \in P\,(C \circ D = E)$

Mentioned [1] Theorem 10.43, all P are closed under logarithm space reduction FL. Therefore

$\exists F \in P \forall H \in P \exists G \in FL\,(F \circ G = H)$

That is,

$P = NP$

$\rightarrow \exists F \in P \forall C \in NP \forall D \in FP \exists G \in FL\,(C \circ D = F \circ G)$

$\rightarrow \forall D \in FP \exists G \in FL\,(D = G)$

This means $L = P$. Therefore, this theorem was shown. □

3. NC hierarchy

And we use circuit problem as follows;

Definition 2. We will use the term "AC^i" as uniform circuits family set that compute AC^i problem, "NC^i" as uniform circuits family set that compute NC^i problem, "RC^i" as reversible circuits family that compute NC^i problem. "$f \circ g$" as connected circuit that g outputs connect to f inputs. In this case, we also use circuits family or circuits family set. For example, $A \circ BB$ of circuits family A and circuits family set BB means a circuit that $a \circ b \mid a \in A, b \in B \in BB$. Circuits family uniformity is that these circuits can compute AC^0.

Theorem 3. *AC^i has Universal Circuits Family that can emulate all AC^i circuits family.*

Proof. To prove this theorem by making universal circuit family $A^i \in AC^i$ that emulate circuit family $\{C_j\} \in AC^i$ by using "depth circuit tableau". Universal circuit $U_j \in A^i$ have partial circuit $u_{k,d}$ that emulate all C_j gates $g_{k\in n}$ (include input value) and connected wires $w_{p,q}$ from g_p output to g_q input in every depth d. ($w_{p,p}$ always exist)

$u_{v\in n,d}$ have inputs from all $u_{u\in n,d-1}$ and g_u information that mean

a) validity of $u_{u,d-1}$

b) $u_{u,d-1}$ output (true if g_u output true)

c) existence of $w_{u,v}$ (true if $w_{u,v}$ is exists)

d) negation of $w_{u,v}$ (true if $w_{u,v}$ include not gate)

e) gate type of g_v (Or gate or And gate)

and outputs to $u_{w\in n,d+1}$ that mean

A) validity of $u_{v,d}$

B) $u_{v,d}$ output

These $u_{v,d}$ compute output like this;

If $u_{u,d-1}$ a) or c) input false then $u_{v,d}$ ignore $u_{u,d-1}$.

If $u_{u,d-1}$ a) and c) input true then $u_{v,d}$ A) output true and $u_{v,d}$ B) output g_k value that compute from e), b), d). b), d) include another $u_{w\in n,d-1}$ b), d).

If all a) input false then $u_{k,d}$ A) output false.

If all c) input false then $u_{k,d}$ A) output false.

And depth 0 circuit compute additional condition;

If $u_{k,0}$ is C_j input then $u_{k,0}$ A) output true and $u_{i,d}$ B) output C_j input value, else $u_{k,0}$ A) output false.

This U_j that consists of u emulate C_j. We can make every u in AC^0, so that A^i in AC^i.

Therefore, this theorem was shown. □

Definition 4. We will use the term "A^i" as universal circuits family that compute AC^i problem, "N^i" as universal circuits family that compute NC^i problem.

Theorem 5. *AC^0 can reduce all AC^i to A^i. That is, A^i is closed under AC^0 reduction.*

Proof. Mentioned above 23, we can make all AC^i by using AC^0 and we can connect these AC^i to A^i. That is, we can emulate all AC^i circuit by using $A^i \circ AC^0$. From the view of A^i, AC^0 is input reduction from AC^i to A^i. Therefore, this theorem was shown. □

Theorem 6. $NC^i \subsetneq NC^{i+1}$

Proof. We can prove this theorem like mentioned above 1.

To prove it using reduction to absurdity. We assume that $NC^i = AC^i = NC^{i+1}$. From assumption, there is;

$\forall A \in NC^{i+1} \exists B \in NC^i \, (A = B)$

$\forall C \in AC^i \exists D \in NC^i \, (C = D)$

As we all know $NC^i \circ NC^1 \in NC^{i+1}$. From assumption $NC^i = AC^i = NC^{i+1}$, all $NC^i \circ NC^1$ correspond to NC^i. Therefore

$NC^i = AC^i = NC^{i+1} \rightarrow \forall C \in NC^i \forall D \in NC^1 \exists E \in NC^i \, (C \circ D = E)$

Mentioned above 5, all AC^i are closed by AC^0 reduction to universal circuit A^i. Therefore

$\forall H \in AC^i \exists G \in AC^0 \left(A^i \circ G = H \right)$

That is,
$NC^i = AC^i = NC^{i+1}$
$\rightarrow \forall C \in NC^i \forall D \in NC^1 \exists G \in AC^0 \left(C \circ D = A^i \circ G\right)$
$\rightarrow \forall D \in NC^1 \exists G \in AC^0 \left(D = G\right)$
But this means $AC^0 = NC^1$ and contradict $AC^0 \subsetneq NC^1$.
Therefore, this theorem was shown than reduction to absurdity. □

4. P is not NP

Theorem 7. $P \neq NP$

Proof. Mentioned above 1, $L \subsetneq P \rightarrow P \subsetneq NP$. And mentioned above 6, $L \subset NC^i \subsetneq NC^{i+1} \subset P$. Therefore $P \subsetneq NP$. □

5. Polynomial hierarchy

Theorem 8. $\Pi_k \subsetneq \Sigma_{k+1}$, $\Sigma_k \subsetneq \Pi_{k+1}$

Proof. We can prove this theorem like mentioned above 6.

To prove it using reduction to absurdity. We assume that $\Pi_k = \Sigma_{k+1}$. From assumption, there is;

$\forall A \in \Sigma_{k+1} \exists B \in \Pi_k \left(A = B\right)$

As we all know $\Pi_k \circ \Sigma_1 \in \Sigma_{k+1}$. From assumption $\Pi_k = \Sigma_{k+1}$, all $\Pi_k \circ \Sigma_1$ correspond to Π_k. Therefore

$\Pi_k = \Sigma_{k+1} \rightarrow \forall C \in \Pi_k \forall D \in \Sigma_1 \exists E \in \Pi_k \left(C \circ D = E\right)$

Mentioned [2] Theorem 6.21 and 6.22, all Σ_k and Π_k are closed under polynomial time reduction Δ_1. Therefore

$\exists F \in \Pi_k \forall H \in \Pi_k \exists G \in \Delta_1 \left(F \circ G = H\right)$

That is,
$\Pi_k = \Sigma_{k+1}$
$\rightarrow \exists F \in \Pi_k \forall C \in \Pi_k \forall D \in \Sigma_1 \exists G \in \Delta_1 \left(C \circ D = F \circ G\right)$
$\rightarrow \forall D \in \Sigma_1 \exists G \in \Delta_1 \left(D = G\right)$
But this means $\Delta_1 = \Sigma_1$ and contradict $P \subsetneq NP$. Therefore $\Pi_k \subsetneq \Sigma_{k+1}$.
We can prove $\Sigma_k \subsetneq \Pi_{k+1}$ like this.
Therefore, this theorem was shown than reduction to absurdity. □

References

[1] Michael Sipser, (translation) OHTA Kazuo, TANAKA Keisuke, ABE Masayuki, UEDA Hiroki, FUJIOKA Atsushi, WATANABE Osamu, Introduction to the Theory of COMPUTATION Second Edition, 2008
[2] OGIHARA Mitsunori, Hierarchies in Complexity Theory, 2006