MEASURING COMPLEXITY BY USING REDUCTION TO SOLVE P VS NP AND NC & PH

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1. Abstract

This article describes about that NC and PH is proper (especially P is not NP) by using problem reduction. If L is not P, we can prove P is not NP by using difference between logarithm space reduction and polynomial time reduction. Like this, we can also prove that NC is proper by using difference between AL0 and NC1. This means L is not P. Therefore P is not NP. And we can also prove that PH is proper by using P is not NP.

2. P is not NP if L is not P

Definition 1. We will use the term "L", "P", "NP", "FL", "FP" as each complexity classes. These complexity classes also use Turing Machine (TM) set that compute target complexity classes problems. "$f \circ g$" as composite TM that accepting configurations of g are starting configurations of f. In this case, we also use complexity classes to show target TM. For example, $a \circ bb$ when a is TM and bb is complexity class mean that $a \circ b \mid b \in bb$.

Theorem 2. $L \subsetneq P \rightarrow P \subsetneq NP$

Proof. To prove it by using contraposition $P = NP \rightarrow L = P$. As we all know $NP \circ FP \in NP$. From assumption $P = NP$, all $NP \circ FP$ correspond to P. Therefore

$P = NP \rightarrow \forall C \in NP \forall D \in FP \exists E \in P \, (C \circ D = E)$

Mentioned [1] Theorem 10.43, $CIRCUIT-VALUE$ are closed under logarithm space reduction FL. That is,

$\forall H \in P \exists G \in FL \, (CIRCUIT-VALUE \circ G = H)$

Therefore

$P = NP$

$\rightarrow \forall C \in NP \forall D \in FP \exists G \in FL \, (C \circ D = CIRCUIT-VALUE \circ G)$

$\rightarrow \forall D \in FP \exists G \in FL \, (CIRCUIT-VALUE \circ D = CIRCUIT-VALUE \circ G)$

$\rightarrow \forall D \in FP \exists G \in FL \, (D = G)$

This means $L = P$. Therefore, this theorem was shown. □

3. NC is proper

And we use circuit problem as follows;

Definition 3. We will use the term "AC^i", "NC^i" as each complexity decision problems classes. "FAC^i" as function problems class of "AC^i". These complexity classes also use uniform circuits family set that compute target complexity classes problems. "$f \circ g$" as composite circuit that output of g are input of f. In this case, we also use complexity classes to show target circuit. For example, $A \circ BB$ when A is circuits family and BB is circuits family set mean that $a \circ b \mid a \in A, b \in B \in BB$. Circuits family uniformity is that these circuits can compute FAC^0.

Theorem 4. $NL \leq_{AC^0} NC^2$

Proof. Mentioned [1] Theorem 10.40, all NC^2 are closed by FL reduction. This reduction is validity of (c_1, c_2) transition function. Transition function change $O(1)$ memory and keep another memory. Therefore this validity can compute AC^0 and we can replace FL to FAC^0. □

Theorem 5. *AC^i has Universal Circuits Family that can emulate all AC^i circuits family.*

Proof. To prove this theorem by making universal circuit family $A^i \in AC^i$ that emulate circuit family $\{C_j\} \in AC^i$ by using "depth circuit tableau". Universal circuit $U_j \in A^i$ have partial circuit $u_{k,d}$ that emulate all C_j gates $g_{k \in n}$ (include input value) and connected wires $w_{p,q}$ from g_p output to g_q input in every depth d. ($w_{p,p}$ always exist)

$u_{v \in n,d}$ have inputs from all $u_{u \in n,d-1}$ and g_u information that mean

a) validity of $u_{u,d-1}$

b) $u_{u,d-1}$ output (true if g_u output true)

c) existence of $w_{u,v}$ (true if $w_{u,v}$ is exists)

d) negation of $w_{u,v}$ (true if $w_{u,v}$ include not gate)

e) gate type of g_v (Or gate or And gate)

and outputs to $u_{w \in n,d+1}$ that mean

A) validity of $u_{v,d}$

B) $u_{v,d}$ output

These $u_{v,d}$ compute output like this;

If $u_{u,d-1}$ a) or c) input false then $u_{v,d}$ ignore $u_{u,d-1}$.

If $u_{u,d-1}$ a) and c) input true then $u_{v,d}$ A) output true and $u_{v,d}$ B) output g_k value that compute from e), b), d). b), d) include another $u_{w \in n,d-1}$ b), d).

If all a) input false then $u_{k,d}$ A) output false.

If all c) input false then $u_{k,d}$ A) output false.

And depth 0 circuit compute additional condition;

If $u_{k,0}$ is C_j input then $u_{k,0}$ A) output true and $u_{i,d}$ B) output C_j input value, else $u_{k,0}$ A) output false.

This U_j that consists of u emulate C_j. We can make every u in FAC^0, so that A^i in AC^i.

Therefore, this theorem was shown. □

Definition 6. We will use the term "A^i" as universal circuits family that compute AC^i problem, "N^i" as universal circuits family that compute NC^i problem.

Theorem 7. *FAC^0 can reduce all AC^i to A^i. That is, A^i is closed under FAC^0 reduction.*

Proof. Mentioned above 35, we can make all AC^i by using AC^0 and we can connect these AC^i to A^i. That is, we can emulate all AC^i circuit by using $A^i \circ AC^0$. From the view of A^i, AC^0 is input reduction from AC^i to A^i. Therefore, this theorem was shown. □

As we all know, all NC^1 decision problems can embed NC^1 function problems. To simplify, we define "Padding" that embed decision problems in function problems.

Definition 8. We will use the term "Padding function" and "$Pad_{N^i}\left(NC^1\right)$" as function that change decision problem NC^1 to function problems $Pad_{N^i}\left(NC^1\right)$ that outputs fit to N^i inputs. This $Pad_{N^i}\left(NC^1\right)$ output must include NC^1 output in head. (Other output make additional AC^0 circuit that input is some NC^1 gate output.)

Theorem 9. $NC^i \subsetneq NC^{i+1}$

Proof. We can prove this theorem like mentioned above 2.

To prove it using reduction to absurdity. We assume that $NC^i = AC^i = NC^{i+1}$. From assumption $NC^i = AC^i$, there is N^i that equal A^i.

$NC^i = AC^i \rightarrow \forall A^i \in AC^i \exists N^i \in NC^i \left(A^i = N^i\right)$

From view of circuit structure, it is trivial that $N^i \circ Pad_{N^i}\left(NC^1\right) \in NC^{i+1}$. From assumption $NC^i = AC^i = NC^{i+1}$, all $N^i \circ Pad_{N^i}\left(NC^1\right)$ correspond to NC^i. Therefore

$NC^i = AC^i = NC^{i+1} \rightarrow \forall C \in NC^1 \exists D \in NC^i \left(N^i \circ Pad_{N^i}(C) = D\right)$

Mentioned above 7, all AC^i are closed by FAC^0 reduction to universal circuit A^i. That is,

$\forall H \in AC^i \exists G \in FAC^0 \left(A^i \circ G = H\right)$

Therefore

$NC^i = AC^i = NC^{i+1}$

$\rightarrow \forall C \in NC^1 \exists D \in NC^i \forall H \in AC^i \exists G \in FAC^0 \left(A^i \circ G = H\right) \wedge \left(N^i \circ Pad_{N^i}(C) = D\right)$

$\rightarrow \forall C \in NC^1 \exists G \in FAC^0 \left(A^i \circ G = N^i \circ Pad_{N^i}(C)\right)$

$\rightarrow \forall C \in NC^1 \exists G \in FAC^0 \left(G = Pad_{N^i}(C)\right)$

But this means $AC^0 = NC^1$ because head of $Pad_{N^i}(C)$ output is C output. It is contradict $AC^0 \subsetneq NC^1$.

Therefore, this theorem was shown than reduction to absurdity. □

4. P is not NP

Theorem 10. $P \neq NP$

Proof. Mentioned above 2, $L \subsetneq P \rightarrow P \subsetneq NP$. And mentioned above 9, $L \subset NC^i \subsetneq NC^{i+1} \subset P$. Therefore $P \subsetneq NP$. □

5. PH is proper

Theorem 11. $\Pi_k \subsetneq \Sigma_{k+1}$

Proof. We can prove this theorem like mentioned above 9.

To prove it using reduction to absurdity. We assume that $\Pi_k = \Sigma_{k+1}$. As we all know $\Pi_k \circ \Sigma_1 \in \Sigma_{k+1}$. From assumption, all $\Pi_k \circ \Sigma_1$ correspond to Π_k. Therefore

$\Pi_k = \Sigma_{k+1} \rightarrow \forall C \in \Pi_k \forall D \in \Sigma_1 \exists E \in \Pi_k (C \circ D = E)$

Mentioned [2] Theorem 6.26, $QSAT'_k$ are $\Pi_k - Complete$ under polynomial time reduction. That is,

$\forall H \in \Pi_k \exists G \in FP \left(QSAT'_k \circ G = H\right)$

Therefore

$\Pi_k = \Sigma_{k+1}$

$\rightarrow \forall C \in \Pi_k \forall D \in \Sigma_1 \exists G \in P \left(C \circ D = QSAT'_k \circ G\right)$

$\rightarrow \forall D \in \Sigma_1 \exists G \in FP\left(QSAT_k' \circ D = QSAT_k' \circ G\right)$
$\rightarrow \forall D \in \Sigma_1 \exists G \in FP\left(D = G\right)$
But this means $P = NP$ and contradict $P \neq NP$. Therefore $\Pi_k \subsetneq \Sigma_{k+1}$.
Therefore, this theorem was shown than reduction to absurdity. □

Theorem 12. $\Delta_k \subsetneq \Sigma_k, \Sigma_k \neq \Pi_k$

Proof. Mentioned [2] Theorem 6.12,
$\Sigma_k = \Pi_k \rightarrow \Sigma_k = \Pi_k = PH$
$\Delta_k = \Sigma_k \rightarrow \Delta_k = \Sigma_k = \Pi_k = PH$
This contraposition is,
$(\Sigma_k \subsetneq PH) \vee (\Pi_k \subsetneq PH) \rightarrow \Sigma_k \neq \Pi_k$
$(\Delta_k \subsetneq PH) \vee (\Sigma_k \subsetneq PH) \vee (\Pi_k \subsetneq PH) \rightarrow \Delta_k \neq \Sigma_k$
From mentioned above 11,
$\Sigma_k \subsetneq \Pi_{k+1} \subset PH$
Therefore, $\Delta_k \neq \Sigma_k, \Sigma_k \neq \Pi_k$.
Mentioned [2] Theorem 6.10,
$\Sigma_k \subset \Sigma_{k+1}$, $\Pi_k \subset \Pi_{k+1}$,$\forall k \geq 1\left(\Delta_k \subset \left(\Sigma_k \cap \Pi_k\right) \subset \left(\Sigma_k \cup \Pi_k\right) \subset \Delta_{k+1}\right)$
Therefore, $\Delta_k \subsetneq \Sigma_k, \Sigma_k \neq \Pi_k$. □

Theorem 13. $\Pi_k \not\subset \Sigma_k, \Sigma_k \not\subset \Pi_k$

Proof. To prove it using reduction to absurdity. We assume that $\Pi_k \subset \Sigma_k$. This means that all $\overline{\Sigma_k} = \Pi_k$ is also Σ_k.
$\Pi_k \subset \Sigma_k \rightarrow \forall A \in \Sigma_k \left(\overline{A} \in \Pi_k \subset \Sigma_k\right)$
Mentioned [2] Theorem 6.21, all Σ_k are closed under polynomial time conjunctive reduction. We can emulate these reduction by using Π_1. That is,
$\exists B \in \Sigma_k \forall C \in \Sigma_k \exists D \in \Pi_1 \left(B \circ D = C\right)$
Therefore,
$\Pi_k \subset \Sigma_k$
$\rightarrow \exists B \in \Sigma_k \forall C \in \Sigma_k \exists D \in \Pi_1 \forall A \in \Sigma_k \left(B \circ D = C\right) \wedge \left(\overline{A} \in \Pi_k \subset \Sigma_k\right)$
$\rightarrow \exists B \in \Sigma_k \forall C \in \Sigma_k \exists D \in \Pi_1 \left(B \circ D = C\right) \wedge \left(\overline{B} \in \Sigma_k\right)$
$\rightarrow \exists B \in \Sigma_k \forall C \in \Sigma_k \exists D \in \Pi_1 \left(B \circ D = C\right) \wedge \left(B \in \Pi_k\right)$
Therefore $\Sigma_k \subset \Pi_k$ because $B \circ D \in \Pi_k$. But this means $\Sigma_k = \Pi_k$ and contradict mentioned above 12 $\Sigma_k \neq \Pi_k$. Therefore $\Pi_k \not\subset \Sigma_k$.
We can prove $\Sigma_k \not\subset \Pi_k$like this.
Therefore, this theorem was shown than reduction to absurdity. □

Theorem 14. $\Delta_k \subsetneq \Pi_k$

Proof. To prove it using reduction to absurdity. We assume that $\Delta_k = \Pi_k$.
Mentioned [2] Theorem 6.10,
$\Sigma_k \subset \Sigma_{k+1}$, $\Pi_k \subset \Pi_{k+1}$,$\forall k \geq 1\left(\Delta_k \subset \left(\Sigma_k \cap \Pi_k\right) \subset \left(\Sigma_k \cup \Pi_k\right) \subset \Delta_{k+1}\right)$
Therefore
$\Delta_k = \Pi_k$
$\rightarrow \Delta_k = \Pi_k \subset \left(\Sigma_k \cap \Pi_k\right) \subset \Sigma_k \subset \left(\Sigma_k \cup \Pi_k\right) \subset \Delta_{k+1}$
$\rightarrow \Pi_k \subset \Sigma_k$
But this result contradict mentioned above 13.
Therefore, this theorem was shown than reduction to absurdity. □

Theorem 15. $\Sigma_k \subsetneq \Delta_{k+1}, \Pi_k \subsetneq \Delta_{k+1}$

Proof. To prove it using reduction to absurdity. We assume that $\Sigma_k = \Delta_{k+1}$.

Mentioned [2] Theorem 6.10,

$\forall k \geq 1\,(\Delta_k \subset (\Sigma_k \cap \Pi_k) \subset (\Sigma_k \cup \Pi_k) \subset \Delta_{k+1})$

Therefore

$\Sigma_k = \Delta_{k+1}$

$\rightarrow \Delta_k \subset (\Sigma_k \cap \Pi_k) \subset \Pi_k \subset (\Sigma_k \cup \Pi_k) \subset \Sigma_k = \Delta_{k+1}$

$\rightarrow \Pi_k \subset \Sigma_k$

But this result contradict mentioned above 13. Therefore $\Sigma_k \subsetneq \Delta_{k+1}$.

We can prove $\Pi_k \subsetneq \Delta_{k+1}$ like this.

Therefore, this theorem was shown than reduction to absurdity. □

References

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