

# Erdős Conjecture I.

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ABSTRACT. An old conjecture of Paul Erdős [6] states that there exist only 7 integers  $A = 4, 7, 15, 21, 45, 75$  and 105 such that the difference  $A - 2^B$  is a prime for all  $B$  for which it is at least two. It is known that the conjecture is true for all  $A < 2^{77}$ , as Uchiyama and Yorinaga have verified in 1977 ([21]), and in this short paper I show how it is related to other famous unsolved problems in prime number theory. In order to do this, I formulate the main hypothetical result of this paper - a useful upper bound conjecture (Conjecture 3.), describing one aspect of the distribution of primes in various special forms, paying a brief attention to Fermat, Mersenne, Fibonacci, Lucas and Smarandache sequences, and I debate some side effects of the most surprising results it implies. At the end I also give connections of the questions discussed to other important areas of prime number theory, such as topics from the theory of distribution of primes in denser sequences, and along the way I mention some further conjectures of Erdős that have relevant applications there.

## 1. Sorbents.

Let me introduce the following notation:

DEFINITION 1. *Let  $f$  and  $g$  be two functions such that for every real number  $k > 1$  there exists an integer constant  $x_0$  such that for all  $x > x_0$  we have  $f^k(x) > g(x)$ , then we will write  $g \lll f$ . If  $f \lll g$  and  $g \lll f$  at the same time, we'll say that  $f$  and  $g$  belong to the same sorbent,  $S$  say, and we'll write  $f \equiv g$ .*

Sometimes we might also write  $f = s(g)$ ,  $s$  denoting the sorbent allocating map, or simply  $f = g$  in cases when there is no possibility

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of confusion in notation. To sketch the use of sorbents I give a very elementary, but strikingly far reaching application.

Let  $f$  and  $g$  be two polynomials such that  $f \equiv g$ , then

$$(1) \quad s(\pi(f)) = \varsigma(f, g) \cdot s(\pi(g)),$$

where  $s$  is again the sorbent function, while  $\varsigma(f, g)$  is either 0 or 1 according to whether  $f$  and  $g$  have the same irreducibility properties or not. One can write this equivalently as

CONJECTURE 2. *If  $A$  and  $B$  denote any one variable functions (polynomials in particular), then we have*

$$(2) \quad \rho(C = A.B) \equiv \varsigma(f, g) \cdot \rho(A) \cdot \rho(B),$$

where  $\rho(H)$  is the density function corresponding to function  $H$ , and  $\varsigma(f, g)$  is here again either 0 or 1 like before, depending only on whether  $C$  is trivial or not.

This also covers the case of the problem of occurrences of primes of a given special form (A) as values of a given function (B), and happens to be in a close connection to the arithmetical functions theory, linking distribution of primes to things as diverse as odd perfect numbers.

At the same time one also immediately sees its direct relation to the results of the famous Bateman-Horn [1] quantitative form of Schinzel's [17] Hypothesis H.

## 2. The Upper Bound Conjecture.

In order to develop possible implications of the Conjecture 2 into something more precise and useful, it is necessary to recall the basic property of the simple prime density function. The Prime Number Theorem ([13]) states that

$$\pi(x) \sim \text{Li}(x) = \lim_{\epsilon \rightarrow 0} \left( \int_0^{1-\epsilon} + \int_{1+\epsilon}^x \right) \frac{dt}{\log t} \sim \text{Ls}(x) = \sum_{n=2}^x (\log x)^{-1},$$

the first estimation being due to Gauß, the second one due to Dirichlet. By definition, the local prime density, or equivalently the probability of primality of an integer in a small nbd of  $x$ ,  $\rho_1(x)$  can be recovered from this equation by (see [11]) differentiating the corresponding prime distribution function. In general hold

$$(3) \quad \rho_k(x) \equiv D'_k(x) \equiv (\log x)^{-k},$$

and considering further a generalization to problems concerning the distribution of primes in "sparse" sequences and special forms, noticing

that  $D(x) < x$  for all  $D$ , implying  $0 \leq \rho < 1$ , gives us the simple inequation

$$(4) \quad \rho(W = \{n_1, n_1, \dots, n_k\}) \leq \prod_{i=1}^k \rho(n_i),$$

where the equality in (4) occurs iff  $W$  is a stochastically independent set of integers. This is in turn equivalent to saying that if we let  $C$  be the set of all primality restrictions that are put upon a sequence  $S(n)$ , or conditions the sequence  $S(n)$  must obey, then there exists a function  $f$  such that

$$(5) \quad D(S(n)) = \pi(x, S(n)) \sim f(C) \sum_{i=1}^x \rho(S(i)) \sim f(C) \int_1^x \rho(S(t)) dt,$$

where  $f$  is *bounded* and depends only on the size and the structure of the condition set  $C$ . Furthermore  $f$  has a regular, approximable behaviour in all fixed, non-trivial cases, and its global properties can be deduced from arithmetical structure of integers in the particular sequence  $S(n)$ .

In terms of *sorbents* we in fact explicitly conjecture that  $f \ll 1$ , and that for a dense set of  $f$  we also have<sup>1</sup>  $f \equiv 1$ . The first, weak assumption gives us that for all sequences  $S(n)$  defined in a closed arithmetical manner the corresponding distribution function  $D(S)$  satisfies the inequality

$$(6) \quad \pi(x, S) \ll \int_1^{S^{-1}(x)} (\log S^{-1}(t))^{-1} dt,$$

$S^{-1}$  denoting the inverse map of  $S$ , and it is plausible to conjecture that the “order” inequality in (6) could be replaced by the standard one for all sufficiently large  $x$ , and therefore hypothesize that we always have correctness of the following **UB Conjecture**, in the above notation written as

**CONJECTURE 3.** *For all  $S(n)$ , and all  $x > X_0$  hold*

$$(7) \quad \pi(x, S) < \int_1^{S^{-1}(x)} (\log S^{-1}(t))^{-1} dt.$$

This sort of a thing is however justified only by establishing a deeper connection of the result (c.f. [15]) to a different “maximal prime density” conjecture of Erdős [7].

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<sup>1</sup>the situation is a bit more delicate than one could presume, as Graham demonstrated in [9]. His result can also be used to justify the density conjecture mentioned.

### 3. Mersenne and Fermat Primes.

As far as applications of the  $UBC$  are concerned, sparse sequences, such as functions of powers of integers, that are obviously arithmetically closed, are now evidently a very suitable point to start at, for if their local density  $\rho^*(x)$  satisfies, say, the inequality

$$\rho^*(x) < e(x^{-1-\epsilon}) = e^{x^{-1-\epsilon}}$$

then

$$\int_2^x \log \rho^*(t) dt < \int_2^\infty \log \rho^*(t) dt < \int_2^\infty \frac{dt}{t^{1+\epsilon}} < K,$$

for some constant  $K$ , implying in conjunction with the Conjecture 3 that  $D(S(n)) = \pi(x, S(n))$  will converge. i.e. will satisfy

$$(8) \quad D(S(n)) = 1.$$

This implies, for instance, that denoting  $F_n$  the  $n$ -th Fermat number, and  $\mathbf{F}(x)$  the number of Fermat primes below  $x$ , then for all  $x$  we have

$$F(x) < 5 + \lim_{N \rightarrow \infty} \sum_{n=5}^N (\log 2^{2^n})^{-1} < 5 + \frac{1}{16 \log 2} < 5.1,$$

showing that it is rather unlikely that any new Fermat prime will ever be discovered<sup>2</sup>.

Now, back to the old conjecture of P. Erdős [7], introduction of the ideas from the beginning of this paper shows that an analogy of a parallel between twins and Goldbach ([5]) can be obtained for the Erdős and Fermat problem here. There is nothing to it, really, by considering a generalization to an arbitrary function  $f(x)$ , the questions about distribution of primes in "sequences"  $f(x)$  and  $A - f(x)$  are complementary. Indeed, all we are interested in are integers  $A$  such that  $f(x)$  is a prime for

$$x = 1, 2, 3, \dots, f^{-1}(A).$$

Obviously, by the above, as long as  $f(x)$  increases to infinity *sufficiently* slowly, the number of wanted  $A$ s has to be finite. What does sufficiently mean here? Evidently as soon as  $\log f = 1$  we should be alright.

This means that, for instance, the number of integers  $A$  such that the difference  $A - n!$  is a prime for all  $n$  for which it is positive is going to be greater than what it was in the similar Erdős problem due to nothing but tendencies of growth of inverses of the functions concerned.

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<sup>2</sup>A conjecture of Selfridge [18].

For Mersenne primes the situation is different. By definition  $S(i) = 2^i - 1$ , giving by above something like

$$(9) \quad \int_{i=1}^{\infty} \frac{1}{\log S(t)} dt \sim \frac{1}{\log 2} \int_{i=1}^{\infty} \frac{1}{\log t} dt.$$

Now, as far as the conditions on primality of  $2^n - 1$  are concerned, nothing explicit<sup>3</sup> can be said beyond the fact that  $n$  must be a prime itself. So, a uniformity property of potential divisors of  $n$  that give  $S(n)$  prime is expected, and in fact can be show to strongly suggests a direct connection to the simple Eratosthenes sieve result. The only effect of this we care about is that except for the necessity of a factor 2, needed due to exclusion of even exponents  $n$  used in the sieve, everything stays unchanged, giving the conjecture  $f(C) = 2$ .

Denoting  $M(x)$  the number of Mersenne primes below  $x$ , one now immediately sees that

$$(10) \quad M(x) = \pi(x, 2^n - 1) \sim 2 \cdot \frac{e^\gamma}{2} \sum_{p < x} \frac{1}{\log p} \sim \frac{e^\gamma \cdot \log \log x}{\log 2},$$

This last asymptotic relation (10) is known as S. S. Wagstaff's conjecture [22], correcting the previous 1964 heuristical result of D. B. Gillies [8]. Therefore in sorbent theory notation we may conclude that

$$F(x) = 1, \quad M(x) = \log \log x.$$

#### 4. Fibonacci and Lucas Primes.

As far as Fibonacci ( $F_n^*$ ) and Lucas ( $L_n$ ) numbers are concerned, we have famous formulas like

$$(11) \quad F_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)^n,$$

where

$$\left|\frac{1 - \sqrt{5}}{2}\right|^n < 1^n = 1.$$

so, denoting  $[x]$  the integer part function, (11) gives

$$F_n = [c^n] + 1 + (-1)^n,$$

where  $c$  is the real number  $(1 + \sqrt{5})/2$ , and an obvious connection to the Mersenne prime case is clearly visible. In fact, the existence of connection based strictly on the fashion of increase of the terms of these two sequences is what the main idea of sorbents, and their quantitative characteristics - Conjecture 3, is all about. Same thing happens for arbitrary Smarandache sequences ( $S_n$ ) based on properties of digital

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<sup>3</sup>this is again just a conjecture we know very little about ...

patterns of integers, although to treat exact behaviour of distribution functions of a particular sequence always needs an additional care. But in general

$$(12) \quad \log M_n = \log F_n^* = \log L_n = \log S_n = n,$$

and due to existence of bounds on conditional divisibility properties of terms of all these cases we also must have:

$$F(x) \ll \log \log x, \quad L(x) \ll \log \log x.$$

and we can conjecture that

$$(13) \quad D(F^*(x)) = D(L(x)) = D(S(x)) = M(x) = \log \log x.$$

For certain special cases this could be made more precise through a discussion concerning the corresponding condition sets  $C$ , although we'll stay content with the illustration of this idea we gave in the case of Mersenne primes distribution.

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