Generalized Determinant

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The report suggests an approach to extend a concept of determinant to the systems of any order.

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Introduction

Conventionally, determinants are defined for the *square matrices* only. Square matrices are graphical representations of the second order systems, the systems that depend on *two* parameters. For example, second order system a_s^r can be presented as a collection of all its components arranged in a square, the *square matrix*:

$$a_s^r \equiv \begin{pmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{pmatrix} \tag{1}$$

In the case the upper index denotes the row and the lower index denotes the column [1]. The matrix is **3** x **3** because in this particular case indices range from **1** to **3**. Determinant of the third order (*i.e.* for the **3** x **3** square matrix) is defined as:

$$|a_s^r| \equiv \underset{(A)}{\sum_{i,j,k}^{3} \pm a_1^i a_2^j a_3^k} \equiv \underset{(B)}{\sum_{i,j,k}^{3} \pm a_i^1 a_j^2 a_k^3} \tag{2}$$

where ***i***, ***j***, ***k*** is a permutation of **1**, **2**, **3**. Plus sign is given to every even permutation. Minus sign is given to every odd permutation. Determinant can be expanded in full either by columns (A) or by rows (B) [1].

In general, a second order system can be presented as a **w** x **w** square matrix provided the indices range from **1** to **w**. Determinant of such a matrix will be the determinant of **w**-th order:

$$|a^r_s| \equiv \sum_{r1,\, r2,\, r3,\, \ldots\, rw}^{w} \pm a^{(r1)}_1 a^{(r2)}_2 a^{(r3)}_3 \ldots a^{(rw)}_w \equiv \sum_{s1,\, s2,\, s3,\, \ldots\, sw}^{w} \pm a^1_{(s1)} a^2_{(s2)} a^3_{(s3)} \ldots a^w_{(sw)} \qquad (3)$$

where ***r1***, ***r2***, ***r3***, …, ***rw*** or ***s1***, ***s2***, ***s3***, …, ***sw*** is a permutation of **1**, **2**, **3**, … , ***w***. Plus sign is given to every even permutation. Minus sign is given to every odd permutation.

It should be stressed here, that even though there exists an approach to define and calculate the determinant of any order (see eq. (3) above), the determinant of **w-**th order is conventionally defined only for the second order systems.

This report suggests an approach to extend a concept of determinant to the systems of any order.

3^{rd} order systems' determinants

In the spirit with the conventional definition of the term *determinant*, which was developed for the 2^{nd} order systems, it seems reasonable if one would suggest a following equation as definition and the calculating procedure of determinant of the **w**-th order for the 3^{rd} order systems:

$$\left| a_{s}^{r}{}_{t} \right| \equiv \sum_{\substack{r1,\, r2,\, r3,\, \dots\, rw \\ s1,\, s2,\, s3,\, \dots\, sw}}^{w} \pm a_{(s1)\,1}^{(r1)}\, a_{(s2)\,2}^{(r2)}\, a_{(s3)\,3}^{(r3)} \cdots a_{(sw)\,w}^{(rw)}$$

$$\equiv \sum_{\substack{r1,\, r2,\, r3,\, \dots\, rw \\ t1,\, t2,\, t3,\, \dots\, tw}}^{w} \pm a_{1\,(t1)}^{(r1)}\, a_{2\,(t2)}^{(r2)}\, a_{3\,(t3)}^{(r3)} \cdots a_{w\,(tw)}^{(rw)} \qquad (4)$$

$$\equiv \sum_{\substack{s1,\, s2,\, s3,\, \dots\, sw \\ t1,\, t2,\, t3,\, \dots\, tw}}^{w} \pm a_{(s1)\,(t1)}^{1}\, a_{(s2)\,(t2)}^{2}\, a_{(s3)\,(t3)}^{3} \cdots a_{(sw)\,(tw)}^{w}$$

where ***r1***, ***r2***, ***r3***, …, ***rw*** and ***s1***, ***s2***, ***s3***, …, ***sw*** or ***r1***, ***r2***, ***r3***, …, ***rw*** and ***t1***, ***t2***, ***t3***, …, ***tw*** or ***s1***, ***s2***, ***s3***, …, ***sw*** and ***t1***, ***t2***, ***t3***, …, ***tw*** are the separate permutations of **1**, **2**, **3**, … , ***w*** within each of the pairs of index sequences.

For instance, a **w**-th order determinant for the 3rd order system can be expanded in full by one of the index sequences (t-index sequence in this example) as:

$$\left| a^{r}_{\substack{s\\t}} \right| \equiv \sum_{\substack{r1,\, r2,\, r3\\ s1,\, s2,\, s3}}^{3} \pm\, a^{(r1)}_{\substack{(s1)\\1}}\, a^{(r2)}_{\substack{(s2)\\2}}\, a^{(r3)}_{\substack{(s3)\\3}} =$$

$$\begin{aligned}
= \Big[& \big(a^{1}_{\substack{1\\1}} a^{2}_{\substack{2\\2}} a^{3}_{\substack{3\\3}}\big) + \big(a^{2}_{\substack{1\\1}} a^{3}_{\substack{2\\2}} a^{1}_{\substack{3\\3}}\big) + \big(a^{3}_{\substack{1\\1}} a^{1}_{\substack{2\\2}} a^{2}_{\substack{3\\3}}\big) + \\
+ & \big(a^{1}_{\substack{1\\1}} a^{3}_{\substack{3\\2}} a^{2}_{\substack{2\\3}}\big) + \big(a^{2}_{\substack{1\\1}} a^{1}_{\substack{3\\2}} a^{3}_{\substack{2\\3}}\big) + \big(a^{3}_{\substack{1\\1}} a^{2}_{\substack{3\\2}} a^{1}_{\substack{2\\3}}\big) + \\
+ & \big(a^{1}_{\substack{2\\1}} a^{3}_{\substack{1\\2}} a^{2}_{\substack{3\\3}}\big) + \big(a^{2}_{\substack{2\\1}} a^{1}_{\substack{1\\2}} a^{3}_{\substack{3\\3}}\big) + \big(a^{3}_{\substack{2\\1}} a^{2}_{\substack{1\\2}} a^{1}_{\substack{3\\3}}\big) + \\
+ & \big(a^{1}_{\substack{2\\1}} a^{2}_{\substack{3\\2}} a^{3}_{\substack{1\\3}}\big) + \big(a^{2}_{\substack{2\\1}} a^{3}_{\substack{3\\2}} a^{1}_{\substack{1\\3}}\big) + \big(a^{3}_{\substack{2\\1}} a^{1}_{\substack{3\\2}} a^{2}_{\substack{1\\3}}\big) + \\
+ & \big(a^{1}_{\substack{3\\1}} a^{2}_{\substack{1\\2}} a^{3}_{\substack{2\\3}}\big) + \big(a^{2}_{\substack{3\\1}} a^{3}_{\substack{1\\2}} a^{1}_{\substack{2\\3}}\big) + \big(a^{3}_{\substack{3\\1}} a^{1}_{\substack{1\\2}} a^{2}_{\substack{2\\3}}\big) + \\
+ & \big(a^{1}_{\substack{3\\1}} a^{3}_{\substack{2\\2}} a^{2}_{\substack{1\\3}}\big) + \big(a^{2}_{\substack{3\\1}} a^{1}_{\substack{2\\2}} a^{3}_{\substack{1\\3}}\big) + \big(a^{3}_{\substack{3\\1}} a^{2}_{\substack{2\\2}} a^{1}_{\substack{1\\3}}\big) \Big] -
\end{aligned} \tag{5}$$

$$\begin{aligned}
- \Big[& \big(a^{1}_{\substack{1\\1}} a^{3}_{\substack{2\\2}} a^{2}_{\substack{3\\3}}\big) + \big(a^{2}_{\substack{1\\1}} a^{1}_{\substack{2\\2}} a^{3}_{\substack{3\\3}}\big) + \big(a^{3}_{\substack{1\\1}} a^{2}_{\substack{2\\2}} a^{1}_{\substack{3\\3}}\big) + \\
+ & \big(a^{1}_{\substack{1\\1}} a^{2}_{\substack{3\\2}} a^{3}_{\substack{2\\3}}\big) + \big(a^{2}_{\substack{1\\1}} a^{3}_{\substack{3\\2}} a^{1}_{\substack{2\\3}}\big) + \big(a^{3}_{\substack{1\\1}} a^{1}_{\substack{3\\2}} a^{2}_{\substack{2\\3}}\big) + \\
+ & \big(a^{1}_{\substack{2\\1}} a^{2}_{\substack{1\\2}} a^{3}_{\substack{3\\3}}\big) + \big(a^{2}_{\substack{2\\1}} a^{3}_{\substack{1\\2}} a^{1}_{\substack{3\\3}}\big) + \big(a^{3}_{\substack{2\\1}} a^{1}_{\substack{1\\2}} a^{2}_{\substack{3\\3}}\big) + \\
+ & \big(a^{1}_{\substack{2\\1}} a^{3}_{\substack{3\\2}} a^{2}_{\substack{1\\3}}\big) + \big(a^{2}_{\substack{2\\1}} a^{1}_{\substack{3\\2}} a^{3}_{\substack{1\\3}}\big) + \big(a^{3}_{\substack{2\\1}} a^{2}_{\substack{3\\2}} a^{1}_{\substack{1\\3}}\big) + \\
+ & \big(a^{1}_{\substack{3\\1}} a^{3}_{\substack{1\\2}} a^{2}_{\substack{2\\3}}\big) + \big(a^{2}_{\substack{3\\1}} a^{1}_{\substack{1\\2}} a^{3}_{\substack{2\\3}}\big) + \big(a^{3}_{\substack{3\\1}} a^{2}_{\substack{1\\2}} a^{1}_{\substack{2\\3}}\big) + \\
+ & \big(a^{1}_{\substack{3\\1}} a^{2}_{\substack{2\\2}} a^{3}_{\substack{1\\3}}\big) + \big(a^{2}_{\substack{3\\1}} a^{3}_{\substack{2\\2}} a^{1}_{\substack{1\\3}}\big) + \big(a^{3}_{\substack{3\\1}} a^{1}_{\substack{2\\2}} a^{2}_{\substack{1\\3}}\big) \Big]
\end{aligned}$$

n^{th} order systems' determinants

One could notice that the indices of the components of the 3^{rd} order systems (see eq.(4)) form an *indices* matrix:

$$\begin{pmatrix} r1 & r2 & r3 & \dots & rw \\ s1 & s2 & s3 & \dots & sw \\ t1 & t2 & t3 & \dots & tw \end{pmatrix} \tag{6}$$

Notation of the components in the matrix can be unified as:

$$\begin{pmatrix} 3\alpha 1 & 3\alpha 2 & 3\alpha 3 & \cdots & 3\alpha w \\ 2\alpha 1 & 2\alpha 2 & 2\alpha 3 & \cdots & 2\alpha w \\ 1\alpha 1 & 1\alpha 2 & 1\alpha 3 & \cdots & 1\alpha w \end{pmatrix} \tag{7}$$

where every number before α denotes a particular index sequence; every number right after the α denotes a particular member in the index sequence; α acts as a separator.

Thus, following the notation of eq.(7), indices matrix for the components of the system of n^{th} order should look like:

$$\begin{pmatrix} n\alpha 1 & n\alpha 2 & n\alpha 3 & \cdots & n\alpha w \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 3\alpha 1 & 3\alpha 2 & 3\alpha 3 & \cdots & 3\alpha w \\ 2\alpha 1 & 2\alpha 2 & 2\alpha 3 & \cdots & 2\alpha w \\ 1\alpha 1 & 1\alpha 2 & 1\alpha 3 & \cdots & 1\alpha w \end{pmatrix} \tag{8}$$

Thus, according to the sequence of the arguments, developed so far, the **generalized determinant** should be defined as:

$$
\begin{aligned}
\left| a^{\substack{n\alpha \\ \cdots \\ 3\alpha}}_{\substack{2\alpha \\ 1\alpha}} \right|
&\equiv \sum^{w}_{\substack{n\alpha 1,\, n\alpha 2,\, n\alpha 3,\, \ldots,\, n\alpha w \\ \cdots\cdots \\ 3\alpha 1,\, 3\alpha 2,\, 3\alpha 3,\, \ldots,\, 3\alpha w \\ 2\alpha 1,\, 2\alpha 2,\, 2\alpha 3,\, \ldots,\, 2\alpha w}} \pm a^{\substack{(n\alpha 1) \\ \cdots \\ (3\alpha 1)}}_{\substack{(2\alpha 1) \\ 1}} \, a^{\substack{(n\alpha 2) \\ \cdots \\ (3\alpha 2)}}_{\substack{(2\alpha 2) \\ 2}} \, a^{\substack{(n\alpha 3) \\ \cdots \\ (3\alpha 3)}}_{\substack{(2\alpha 3) \\ 3}} \cdots a^{\substack{(n\alpha w) \\ \cdots \\ (3\alpha w)}}_{\substack{(2\alpha w) \\ w}} \\
&\equiv \sum^{w}_{\substack{n\alpha 1,\, n\alpha 2,\, n\alpha 3,\, \ldots,\, n\alpha w \\ \cdots\cdots \\ 3\alpha 1,\, 3\alpha 2,\, 3\alpha 3,\, \ldots,\, 3\alpha w \\ 1\alpha 1,\, 1\alpha 2,\, 1\alpha 3,\, \ldots,\, 1\alpha w}} \pm a^{\substack{(n\alpha 1) \\ \cdots \\ (3\alpha 1)}}_{\substack{1 \\ (1\alpha 1)}} \, a^{\substack{(n\alpha 2) \\ \cdots \\ (3\alpha 2)}}_{\substack{2 \\ (1\alpha 2)}} \, a^{\substack{(n\alpha 3) \\ \cdots \\ (3\alpha 3)}}_{\substack{3 \\ (1\alpha 3)}} \cdots a^{\substack{(n\alpha w) \\ \cdots \\ (3\alpha w)}}_{\substack{w \\ (1\alpha w)}} \\
&\equiv \sum^{w}_{\substack{n\alpha 1,\, n\alpha 2,\, n\alpha 3,\, \ldots,\, n\alpha w \\ \cdots\cdots \\ 2\alpha 1,\, 2\alpha 2,\, 2\alpha 3,\, \ldots,\, 2\alpha w \\ 1\alpha 1,\, 1\alpha 2,\, 1\alpha 3,\, \ldots,\, 1\alpha w}} \pm a^{\substack{(n\alpha 1) \\ \cdots \\ 1}}_{\substack{(2\alpha 1) \\ (1\alpha 1)}} \, a^{\substack{(n\alpha 2) \\ \cdots \\ 2}}_{\substack{(2\alpha 2) \\ (1\alpha 2)}} \, a^{\substack{(n\alpha 3) \\ \cdots \\ 3}}_{\substack{(2\alpha 3) \\ (1\alpha 3)}} \cdots a^{\substack{(n\alpha w) \\ \cdots \\ w}}_{\substack{(2\alpha w) \\ (1\alpha w)}} \qquad (9) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\equiv \sum^{w}_{\substack{[n-1]\alpha 1,\, [n-1]\alpha 2,\, [n-1]\alpha 3,\, \ldots,\, [n-1]\alpha w \\ \cdots\cdots \\ 2\alpha 1,\, 2\alpha 2,\, 2\alpha 3,\, \ldots,\, 2\alpha w \\ 1\alpha 1,\, 1\alpha 2,\, 1\alpha 3,\, \ldots,\, 1\alpha w}} \pm a^{\substack{1 \\ ([n-1]\alpha 1) \\ \cdots}}_{\substack{(2\alpha 1) \\ (1\alpha 1)}} \, a^{\substack{2 \\ ([n-1]\alpha 2) \\ \cdots}}_{\substack{(2\alpha 2) \\ (1\alpha 2)}} \, a^{\substack{3 \\ ([n-1]\alpha 3) \\ \cdots}}_{\substack{(2\alpha 3) \\ (1\alpha 3)}} \cdots a^{\substack{w \\ ([n-1]\alpha w) \\ \cdots}}_{\substack{(2\alpha w) \\ (1\alpha w)}}
\end{aligned}
$$

where **n** denotes order of the system; **w** denotes order of the determinant.

Concluding remarks

It seems reasonable to anticipate a further development of the approach described here via the application of *e-systems* and *Kronecker deltas* to eq.(9) in the same fashion as it was previously outlined for the second order systems [1]. That would allow one, for instance, to derive a rule of multiplication of two generalized determinants.

Generalized determinants may also call for a revision of math apparatus in quantum mechanics. Indeed, it is well known, that physical quantities in quantum mechanics are represented by *square matrices*, which are the 2nd order systems [2]. This means that quantum mechanics in a present form is intrinsically devised to deal with the 2nd order systems only. Meanwhile, Special and General Theories of Relativity (STR&GTR) are intrinsically devised to deal with the systems, which orders are higher than 2. One might hope that extension of the math of quantum mechanics to the higher order systems would eliminate the *order mismatch* between quantum mechanics and STR&GTR, thus making the theories mutually compatible, therefore providing the basis for their further unification.

References

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