New proof that the sum of natural number is -1/12 of zeta function

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Abstract

We prove that the sum of natural number is -1/12 of the value of the zeta function by the new method.

Abel calculated the sum of the divergent series by the Abel summation method. However, we cannot calculate the sum of natural number by the method. In this paper, we calculate the sum of the natural number by the extended Abel summation method.

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1 Introduction

1.1 Issue

The integral representation of the zeta function converges by the analytic continuation. On the other hand, the zeta function also has a series representation. One of the representation is the sum of all natural numbers. The sum does not converge. It diverges. This paper converges the sum by the extended Abel summation method.

$$"1+2+3+\cdots" = -\frac{1}{12} \tag{1.1}$$

1.2 Importance of the issue

It is desirable that both the integral representation and the series representation have a same value for mathematical consistency. One method to converge the divergent series is Abel summation method. The method converges the divergent series by multiplying convergence factor. However, the sum of all natural numbers does not converge by the method. Therefore, it is an important issue to find a new summation method.

1.3 Research trends so far

Leonhard Euler[2] suggested that the sum of all natural numbers is −1/12 in 1749. Bernhard Riemann[3] showed that the integral representation of the zeta function is −1/12 in 1859. Srinivasa Ramanujan[4] proposed that the sum of all natural numbers is −1/12 by Ramanujan summation method in 1913.

Niels Abel[5] introduced Abel summation method in order to converge the divergent series in about 1829. J. Satoh[6] constructed q-analogue of zeta-function in 1989. M. Kaneko, N. Kurokawa, and M. Wakayama[7] derived the sum of the double quoted natural numbers by the q-analogue of zeta-function in 2002.

1.4 New derivation method of this paper

We define a new “sum of all natural numbers” by a new summation method, damped oscillation summation method. The method converges the divergent series by multiplying convergence factor, which is damped and oscillating very slowly. The traditional sum diverges for the infinite terms. On the other hand, the new “sum” is equal to the traditional sum for the finite term. In addition, the “sum” converges on −1/12 for the infinite term.

(Damped oscillation summation method for the sum of all natural numbers)

$$\lim_{x\to 0+}\sum_{k=1}^{\infty} k\exp(-kx)\cos(kx) = -\frac{1}{12} \tag{1.2}$$

1.5 Old method by Abel summation method

Niels Abel introduced Abel summation method in about 1829.

We consider the following sum of the series.

$$S = \sum_{k=1}^{\infty} a_k \tag{1.3}$$

Then we define the following function.
(Abel summation method)

$$F(x) = \sum_{k=1}^{\infty} a_k \exp(-kx) \tag{1.4}$$

$$0 < x \tag{1.5}$$

We define Abel sum as follows.

$$S_A = \lim_{x \to 0+} F(x) \tag{1.6}$$

We consider the following divergent series.

$$S = \sum_{k=1}^{\infty} (-1)^{k-1} k \tag{1.7}$$

Then we define the following function.

$$G(x) = \sum_{k=1}^{\infty} (-1)^{k-1} k \exp(-kx) \tag{1.8}$$

$$0 < x \tag{1.9}$$

Here, we will use the following formula.
(Formula of the geometric series that has coefficients of natural numbers)

$$\sum_{k=1}^{\infty} k r^k = \frac{r}{(1-r)^2} \tag{1.10}$$

We obtain the following equation by the above formula.

$$F(x) = \frac{-e^{-x}}{(1+e^{-x})^2} \tag{1.11}$$

We calculate the Abel sum as follows.

$$S_A = \lim_{x \to 0+} F(x) = \frac{-e^{-0}}{(1+e^{-0})^2} = -\frac{1}{4} \tag{1.12}$$

Abel calculated the sum of the divergent series by the Abel summation method as above stated.
However, we cannot calculate the sum of natural number by the method.

We consider the following sum of all natural numbers.

$$S = \sum_{k=1}^{\infty} k \qquad (1.13)$$

Then we define the following function.

$$G(x) = \sum_{k=1}^{\infty} k \exp(-kx) \qquad (1.14)$$

$$0 < x \qquad (1.15)$$

Here, we will use the following formula.
(Formula of the geometric series that has coefficients of natural numbers)

$$\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2} \qquad (1.16)$$

We obtain the following equation by the above formula.

$$G(x) = \frac{e^{-x}}{(1-e^{-x})^2} \qquad (1.17)$$

We calculate the Abel sum as follows.

$$S_A = \lim_{x \to 0+} G(x) = \frac{-e^{-0}}{(1-e^{-0})^2} = \infty \qquad (1.18)$$

The Abel sum diverges as above stated. Therefore, we cannot caluculate the sum of the natural number by the method.

In order to examine the mechanism of the divergence, we use the following definitional formula of Bernoulli numbers.

(Definitional formula of Bernoulli numbers)

$$\frac{ze^z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \qquad (1.19)$$

In this paper, Bernoulli numbers B_n is Bernoulli polynomial B_n (1).

We will square the both sides of the above formula. In addition, we will divide the both sides by z^2. Then we obtain the following equation.

$$\frac{e^{2z}}{(1-e^z)^2} = \frac{1}{z^2}\left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n\right)^2 \qquad (1.20)$$

We will multiply the left side of the above formula by e^{-z}. In addition, we will multiply the right side by Maclaurin series of e^{-z}. Then we obtain the following equation.

$$\frac{e^z}{(1-e^z)^2} = \frac{1}{z^2}\left(\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n\right)^2 \sum_{n=0}^{\infty}\frac{1}{n!}(-z)^n \qquad (1.21)$$

Therefore, we express the function $G\,(x)$ as follows.

$$G(x) = \frac{1}{x^2}\left(\sum_{n=0}^{\infty}\frac{B_n}{n!}(-x)^n\right)^2 \sum_{n=0}^{\infty}\frac{1}{n!}x^n \qquad (1.22)$$

We obtain the following equation by calculating the above formula.

$$G(x) = \frac{1}{x^2}\left(1-\frac{x}{2}+\frac{x^2}{12}+O(x^3)\right)^2\left(1+x+\frac{x^2}{2}+O(x^3)\right) \qquad (1.23)$$

$$G(x) = \frac{1}{x^2}\left(1-x+\frac{5}{12}x^2+O(x^3)\right)\left(1+x+\frac{x^2}{2}+O(x^3)\right) \qquad (1.24)$$

$$G(x) = \frac{1}{x^2}\left(1+(1-1)x+\left(\frac{1}{2}-1+\frac{5}{12}\right)x^2+O(x^3)\right) \qquad (1.25)$$

$$G(x) = \frac{1}{x^2}\left(1-\frac{1}{12}x^2+O(x^3)\right) \qquad (1.26)$$

$$G(x) = \frac{1}{x^2}-\frac{1}{12}+O(x) \qquad (1.27)$$

Here the symbols $O\,(x)$ are Landau symbols. The symbols mean that the error has the order of the variable x.
The first term diverges. The first term is called singular term.

Therefore, the following Abel sum diverges.

$$S_A = \lim_{x\to 0+} G(x) = \infty \qquad (1.28)$$

It is the purpose of this paper to remove this divergence.

2 New method

2.1 New method by new proposition

We have the following proposition. We will prove the proposition in the next sections.

Proposition 1.

$$\lim_{x\to 0+}\sum_{k=1}^{n} k\exp(-kx)\cos(kx) = \sum_{k=1}^{n} k \tag{2.1}$$

We have the following proposition. We will prove the proposition in the next sections, too.

Proposition 2.

$$\lim_{x\to 0+}\sum_{k=1}^{\infty} k\exp(-kx)\cos(kx) = -\frac{1}{12} \tag{2.2}$$

We define the function S_n and $H_n(x)$ as follows.

$$S_n = \sum_{k=1}^{n} k \tag{2.3}$$

$$H_n(x) = \sum_{k=1}^{n} k\exp(-kx)\cos(kx) \tag{2.4}$$

Then we define the following symbols.

$$1+2+3+\cdots+n := S_n \tag{2.5}$$

$$1+2+3+\cdots := \lim_{n\to\infty} S_n \tag{2.6}$$

$$"1+2+3+\cdots+n" := \lim_{x\to 0+} H_n(x) \tag{2.7}$$

$$"1+2+3+\cdots" := \lim_{x\to 0+}\left(\lim_{n\to\infty} H_n(x)\right) \tag{2.8}$$

Then we have the following propositions.

$$1+2+3+\cdots = \infty \tag{2.9}$$

$$"1+2+3+\cdots+n" = 1+2+3+\cdots+n \tag{2.10}$$

$$"1+2+3+\cdots" = -\frac{1}{12} \tag{2.11}$$

The double quotes mean the analytic continuation of the sum of natural numbers.

The traditional sum diverges for the infinite terms. On the other hand, the new "sum" is equal to the traditional sum for the finite term. In addition, the "sum" converges on −1/12 for the infinite term.

2.2 Proof of the proposition 1

Proposition 1.

$$\lim_{x \to 0+} \sum_{k=1}^{n} k \exp(-kx) \cos(kx) = \sum_{k=1}^{n} k \tag{2.12}$$

Proof. We define the function $H_n(x)$ as follows.

$$H_n(x) = \sum_{k=1}^{n} k \exp(-kx) \cos(kx) \tag{2.13}$$

On the other hand, we have the following equation.

$$\lim_{x \to 0+} H_n(x) = \sum_{k=1}^{n} k \exp(-k0) \cos(k0) = \sum_{k=1}^{n} k \tag{2.14}$$

Therefore, we have the following equation.

$$\lim_{x \to 0+} \sum_{k=1}^{n} k \exp(-kx) \cos(kx) = \sum_{k=1}^{n} k \tag{2.15}$$

This completes the proof.

2.3 Proof of the proposition 2

Proposition 2.

$$\lim_{x \to 0+} \sum_{k=1}^{\infty} k \exp(-kx) \cos(kx) = -\frac{1}{12} \tag{2.16}$$

Proof. We consider the following sum of all natural numbers.

$$S = \sum_{k=1}^{\infty} k \tag{2.17}$$

Then we define the following function.

$$\boxed{H(x) := \sum_{k=1}^{\infty} k \exp(-kx) \cos(kx)} \tag{2.18}$$

$$0 < x \tag{2.19}$$

Here, we use the following formula.
(Euler's formula)

$$\exp(i\theta) = \cos(\theta) + i\ \sin(\theta) \tag{2.20}$$

We obtain the following formula from the above formula.

$$\cos(\theta) = \frac{1}{2}(\exp(i\theta) + \exp(-i\theta)) \tag{2.21}$$

We obtain the following formula from the above formula by putting $\theta = k\,x$.

$$\cos(kx) = \frac{1}{2}(\exp(ikx) + \exp(-ikx)) \tag{2.22}$$

Hence, we express the function $H\,(x)$ as follows.

$$H(x) = \sum_{k=1}^{\infty} k \exp(-kx) \frac{1}{2}(\exp(ikx) + \exp(-ikx)) \tag{2.23}$$

$$H(x) = \frac{1}{2}\sum_{k=1}^{\infty} k \exp(-kx) \exp(ikx) + \frac{1}{2}\sum_{k=1}^{\infty} k \exp(-kx) \exp(-ikx) \tag{2.24}$$

$$H(x) = \frac{1}{2}\sum_{k=1}^{\infty} k \exp(-kx + ikx) + \frac{1}{2}\sum_{k=1}^{\infty} k \exp(-kx - ikx) \tag{2.25}$$

$$H(x) = \frac{1}{2}\sum_{k=1}^{\infty} k \exp(-k(x - ix)) + \frac{1}{2}\sum_{k=1}^{\infty} k \exp(-k(x + ix)) \tag{2.26}$$

In addition, we define the following function.

$$G(\mathrm{z}) = \sum_{k=1}^{\infty} k \exp(-kz) \tag{2.27}$$

$$z \in \mathbb{C} \tag{2.28}$$

We express the function $G\,(z)$ from the formula (1.27).

$$G(z) = \frac{1}{z^2} - \frac{1}{12} + O(z) \tag{2.29}$$

Hence, we express the function $G\,(x - ix)$ as follows.

$$G(x - ix) = \sum_{k=1}^{\infty} k \exp(-k(x - ix)) \tag{2.30}$$

$$G(x - ix) = \frac{1}{(x - ix)^2} - \frac{1}{12} + O(x) \tag{2.31}$$

Then, we express the function $H\,(x)$.

$$H(x) = \frac{1}{2} G(x - ix) + \frac{1}{2} G(x + ix) \tag{2.32}$$

$$H(x) = \frac{1}{2}\left(\frac{1}{(x - ix)^2} - \frac{1}{12} + O(x)\right) + \frac{1}{2}\left(\frac{1}{(x + ix)^2} - \frac{1}{12} + O(x)\right) \tag{2.33}$$

$$H(x) = \frac{1}{2(x - ix)^2} + \frac{1}{2(x + ix)^2} - \frac{1}{12} + O(x) \tag{2.34}$$

The first term is shown below.

$$\frac{1}{2(x - ix)^2} = \frac{1}{2(x^2 - 2ix^2 - x^2)} = \frac{1}{-4ix^2} \tag{2.35}$$

On the other hand, the second term is shown below.

$$\frac{1}{2(x + ix)^2} = \frac{1}{2(x^2 + 2ix^2 - x^2)} = \frac{1}{4ix^2} \tag{2.36}$$

Therefore, the sum of the first term and the second term vanishes.

$$\frac{1}{2(x - ix)^2} + \frac{1}{2(x + ix)^2} = 0 \tag{2.37}$$

Since the singular term vanished, we have the following equation.

$$H(x) = -\frac{1}{12} + O(x) \tag{2.38}$$

We express the function $H_n\,(x)$ as follows.

$$H_n(x) = \sum_{k=1}^{n} k \exp(-kx) \cos(kx) \tag{2.39}$$

On the other hand, from the formula (2.38) we have the following equation.

$$\lim_{n\to\infty} H_n(x) = \sum_{k=1}^{\infty} k\exp(-kx)\cos(kx) = H(x) = -\frac{1}{12} + O(x) \tag{2.40}$$

Therefore, we have the following equation.

$$\lim_{x\to 0+}\left(\lim_{n\to\infty} H_n(x)\right) = \lim_{x\to 0+}\left(-\frac{1}{12} + O(x)\right) = -\frac{1}{12} \tag{2.41}$$

This completes the proof.

3 Conclusion

We obtained the following results in this paper.

- We proved that the sum of natural number is -1/12 of the value of the zeta function by the new method.

4 Future issues

The future issues are shown below.

- To study the relation between the damped oscillation summation and the *q*-analog.

5 Supplement

5.1 Numerical calculation

We will calculate the following value numerically.

$$H = \sum_{k=1}^{3000} k\exp(-k0.01)\cos(k0.01) \tag{5.1}$$

The result is shown below.

$$H = -0.0833333498\cdots \tag{5.2}$$

This value is very close to the following $-1/12$.

$$-\frac{1}{12} = -0.0833333333\cdots \tag{5.3}$$

The graph is shown in the Figure 5.1.

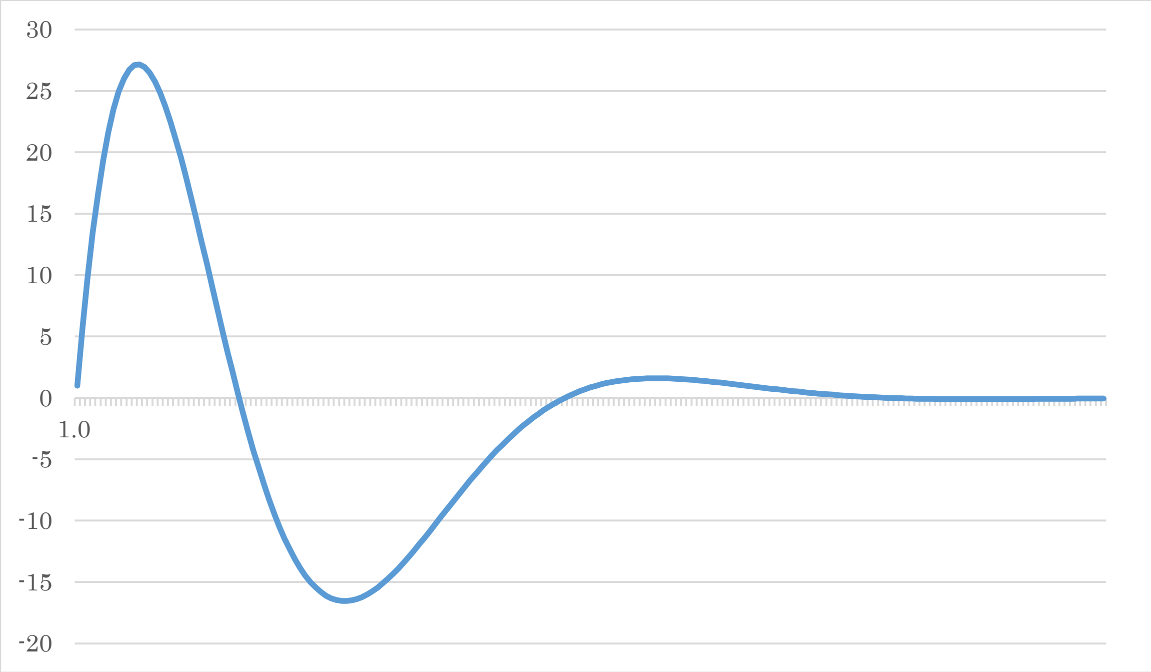

Figure 5.1: Damped oscillation of natural number

Here, we will double the attenuation factor as follows. We do not change the vibration period.

$$H = \sum_{k=1}^{3000} k \exp(-k0.02) \cos(k0.01) \tag{5.4}$$

The result is shown below.

$$H = 1199.9166679 \cdots \tag{5.5}$$

Therefore, the series does not converge if the attenuation factor and the vibration period do not have a special relation. We will consider the relation in the next section.

5.2 Singular equation

We consider the following series.

$$S = \sum_{k=1}^{\infty} a_k \tag{5.6}$$

Then we define the following function.

$$H(x) = \sum_{k=1}^{\infty} a_k \exp(-k\phi(x)) \cos(kx) \tag{5.7}$$

$$0 < x \tag{5.8}$$

$$0 < \phi(x) \tag{5.9}$$

In addition, we define the following function.

$$G(\mathrm{z}) = \sum_{k=1}^{\infty} k \exp(-kz) \tag{5.10}$$

$$z \in \mathbb{C} \tag{5.11}$$

We express the function *G* (*z*) by using singular term *A* (*z*), and the constant *C*.

$$G(z) = A(z) + C + O(z) \tag{5.12}$$

Here, we use the following formula.
(Euler's formula)

$$\exp(i\theta) = \cos(\theta) + i\ \sin(\theta) \tag{5.13}$$

We express the function *H* (*x*) as follows by using the above formula.

$$H(x) = \frac{1}{2}(G(z) + G(\bar{z})) \tag{5.14}$$

$$z = \phi(x) + ix \tag{5.15}$$

$$\bar{z} = \phi(x) - ix \tag{5.16}$$

We obtain the following equation by using the formula (5.12) of the function *G* (*z*).

$$H(x) = \frac{1}{2}\left(A(z) + A(\bar{\mathrm{z}})\right) + C + O(z) \tag{5.17}$$

We determine the function ϕ (*x*) by the following singular equation in order to remove the singular term of the above equation.
(Singular equation)

$$\boxed{\frac{1}{2}\left(A(z) + A(\bar{z})\right) = O(z)} \tag{5.18}$$

The series a_k and the function *G* (*z*) and ϕ (*x*) are shown below.

Series a_k	Function $G(z)$	Function $\phi(x)$
1	$G(z) = -\frac{1}{z} - \frac{1}{2} + O(z)$	$\phi(x) = \frac{x}{\tan(\pi/2)} + O(x^3)$
k	$G(z) = \frac{1}{z^2} - \frac{1}{12} + O(z)$	$\phi(x) = \frac{x}{\tan(\pi/4)} + O(x^4)$
k^2	$G(z) = -\frac{2}{z^3} + O(z)$	$\phi(x) = \frac{x}{\tan(\pi/6)} + O(x^5)$
k^3	$G(z) = \frac{6}{z^4} + \frac{1}{120} + O(z)$	$\phi(x) = \frac{x}{\tan(\pi/8)} + O(x^6)$

The series and the example of the damped oscillation summation method are shown below.

Series a_k	Example of the damped oscillation summation method
1	$H(x) = \sum_{k=1}^{\infty} 1 \exp(-kx^3) \cos(kx)$
k	$H(x) = \sum_{k=1}^{\infty} k \exp(-kx) \cos(kx)$
k^2	$H(x) = \sum_{k=1}^{\infty} k^2 \exp\left(-kx\sqrt{3}\right) \cos(kx)$
k^3	$H(x) = \sum_{k=1}^{\infty} k^3 \exp\left(-kx\left(1 + \sqrt{2}\right)\right) \cos(kx)$

Therefore, we have the following equations.

$$"1+1+1+\cdots" = -\frac{1}{2} \tag{5.19}$$

$$"1+2+3+\cdots" = -\frac{1}{12} \tag{5.20}$$

$$"1^2+2^2+3^2+\cdots" = 0 \tag{5.21}$$

$$"1^3+2^3+3^3+\cdots" = \frac{1}{120} \tag{5.22}$$

We calculate the above equations as follows.

$$\lim_{x\to 0+} \sum_{k=1}^{\infty} 1 \exp(-kx^3)\cos(kx) = -\frac{1}{2} \tag{5.23}$$

$$\lim_{x\to 0+} \sum_{k=1}^{\infty} k \exp(-kx)\cos(kx) = -\frac{1}{12} \tag{5.24}$$

$$\lim_{x\to 0+} \sum_{k=1}^{\infty} k^2 \exp\left(-kx\sqrt{3}\right)\cos(kx) = 0 \tag{5.25}$$

$$\lim_{x\to 0+} \sum_{k=1}^{\infty} k^3 \exp\left(-kx\left(1+\sqrt{2}\right)\right)\cos(kx) = \frac{1}{120} \tag{5.26}$$

5.3 General damped oscillation summation method

We consider the following sum of all natural numbers.

$$S := \sum_{k=1}^{\infty} k \tag{5.27}$$

Then we define the following function.

$$H(x) = \sum_{k=1}^{\infty} k \exp(-kx)\cos(kx) \tag{5.28}$$

$$G(z) = \sum_{k=1}^{\infty} k \exp(-kz) \tag{5.29}$$

We express the function $H(x)$ as follows.

$$H(x) = \frac{1}{2}(G(x - ix) + G(x + ix)) \quad (5.30)$$

We interpret the above function as a sum of the function $G(z_1)$ and $G(z_2)$ like the following Figure 5.2.

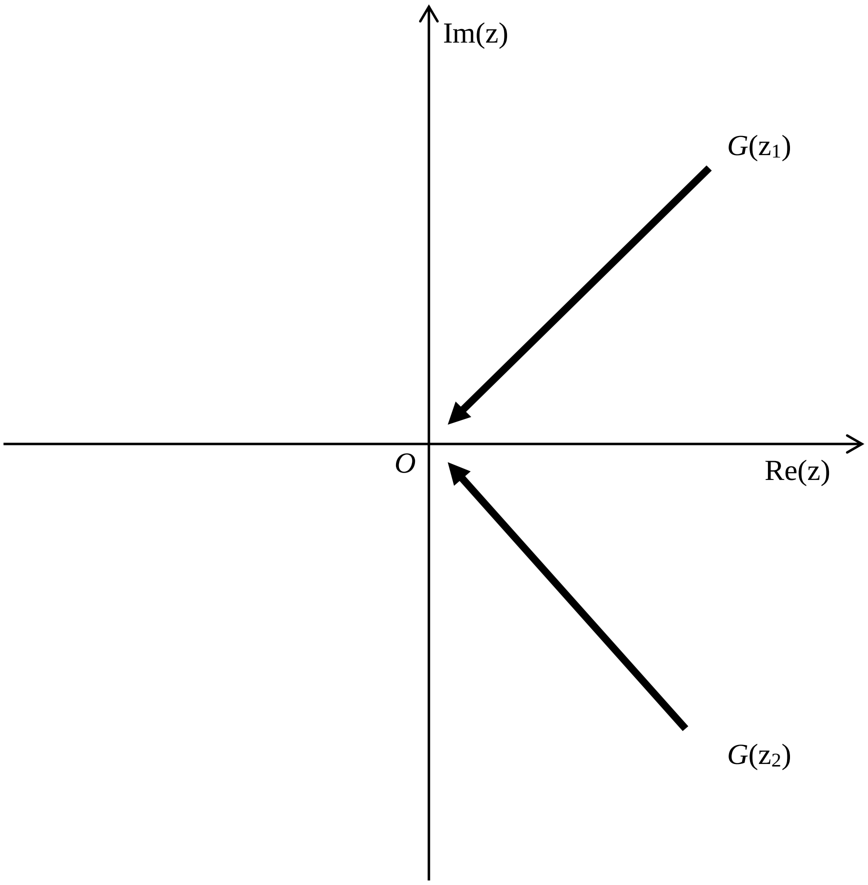

Figure 5.2: Special damped oscillation summation method

In the above Figure 5.2, the two functions $G(z_1)$ and $G(z_2)$ approach to the origin O at the two angles. This is special condition. Is it possible to make the condition more general?

Then we consider the case that the many functions $G(z)$ approach to the origin from the all points on the circle of the radius x like the following Figure 5.3

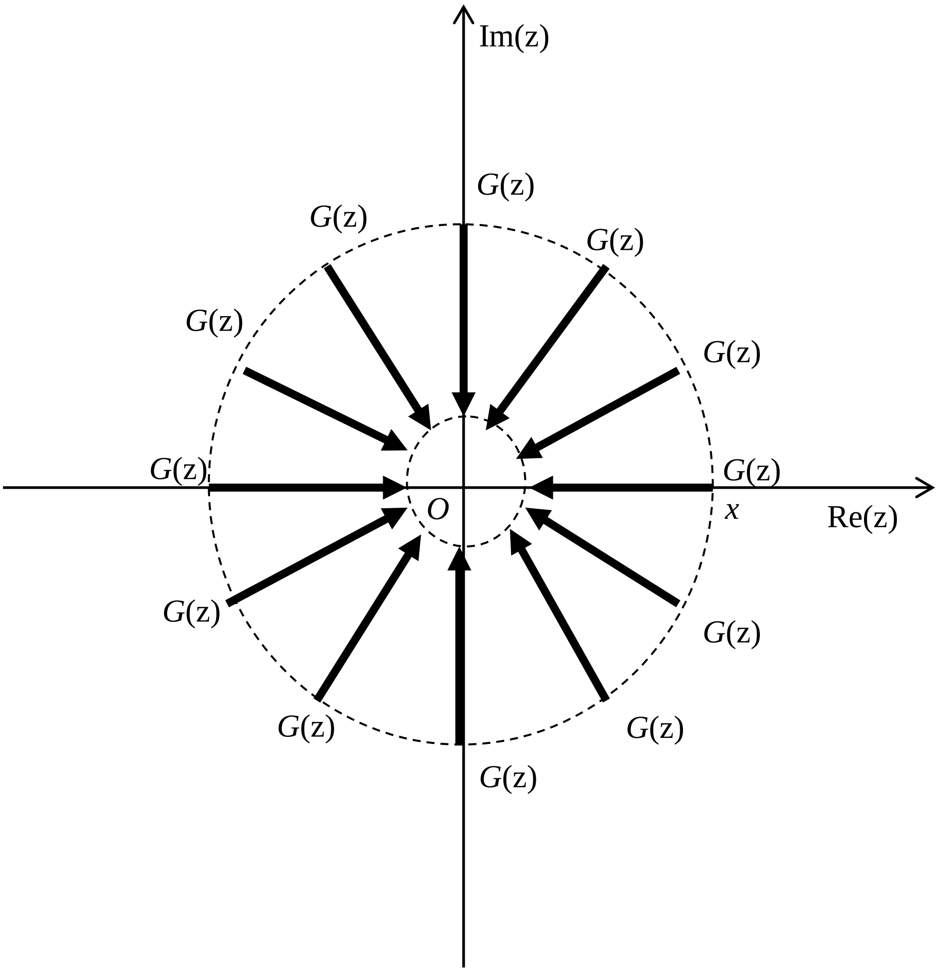

Figure 5.3: General damped oscillation summation method

We express the above consideration in the following equations.

$$H(x) = \frac{1}{2}(G(z_1) + G(z_2)) \tag{5.31}$$

$$H(x) = \frac{1}{3}(G(z_1) + G(z_2) + G(z_3)) \tag{5.32}$$

$$H(x) = \frac{1}{4}(G(z_1) + G(z_2) + G(z_3) + G(z_4)) \tag{5.33}$$

$$H(x) = \frac{1}{n}(G(z_1) + G(z_2) + G(z_3) + \cdots + G(z_n)) \tag{5.34}$$

$$H(x) = \sum_{k=1}^{n} \frac{1}{n} G(z_k) \tag{5.35}$$

The function *H* (*x*) is the mean value of the function *G* (*z*).

We suppose that the circumference of the circle of the radius *x* is *L*. We express the circumference *L* of the circle as follows.

$$L = 2\pi x \tag{5.36}$$

On the other hand, we express the circumference L as follows.

$$L = n|z_{k+1} - z_k| \tag{5.37}$$

Therefore, we have the following equation.

$$2\pi x = n|z_{k+1} - z_k| \tag{5.38}$$

Then, we express the normalized constant $1/n$ as follows.

$$\frac{1}{n} = \frac{|z_{k+1} - z_k|}{2\pi|z_k|} \tag{5.39}$$

$$|z_k| = x \tag{5.40}$$

On the other hand, the direction of $-i\,(z_{k+1} - z_k)$ is same as the direction of z_k as shown in the Figure 5.4

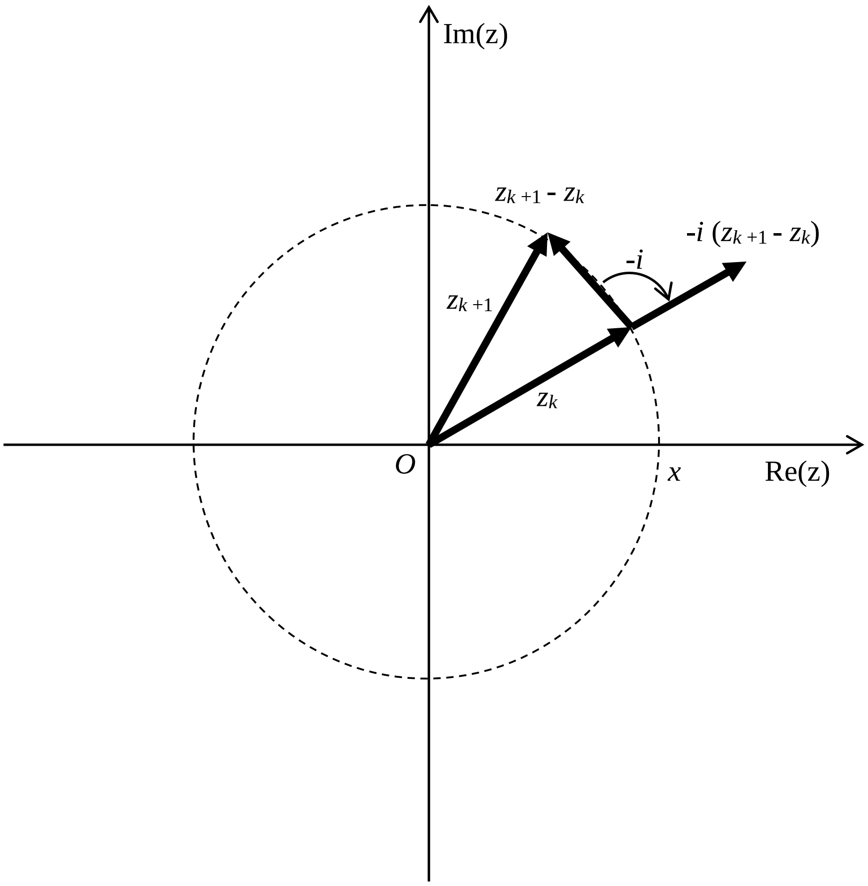

Figure 5.4: General damped oscillation summation method

Therefore, we express $1/n$ as follows.

$$\frac{1}{n} = \frac{|z_{k+1} - z_k|}{2\pi|z_k|} = \frac{-i(z_{k+1} - z_k\ )}{2\pi z_k} = \frac{-i\delta z_k}{2\pi z_k} \tag{5.41}$$

$$\delta z_k := z_{k+1} - z_k \tag{5.42}$$

Therefore, we express the function *H* (*x*) as follows.

$$H(x) = \sum_{k=1}^{n} \frac{-i\delta z_k}{2\pi z_k} G(z_k) \tag{5.43}$$

We express the above sum as the following integration.

$$H(x) = \oint_{|z|=x} \frac{-idz}{2\pi z} G(z) \tag{5.44}$$

$$H(x) = \oint_{|z|=x} \frac{dz}{2\pi iz} G(z) \tag{5.45}$$

The above result is equal to the residue theorem. We can interpret the residue theorem the mean value theorem of contour integration.

We obtain the following result by the residue theorem.

$$H(x) = \oint_{|z|=x} \frac{dz}{2\pi iz} \left(\frac{1}{z^2} - \frac{1}{12} + O(z) \right) = -\frac{1}{12} \tag{5.46}$$

Therefore, we obtain the following new general summation method.
(General damped oscillation summation method)

$$S\ = \sum_{k=1}^{\infty} a_k \tag{5.47}$$

$$G(z)\ = \sum_{k=1}^{\infty} a_k \exp(-kz) \tag{5.48}$$

$$H(x)\ = \oint_{|z|=x} \frac{dz}{2\pi iz} G(z) \tag{5.49}$$

$$S_H = \lim_{x \to 0+} H(x) \tag{5.50}$$

This summation method can sum any series by the residue theorem.

We call the damped oscillation summation method with the residue theorem "**general damped oscillation summation method**."

We call the damped oscillation summation method without the residue theorem “**special damped oscillation summation method**.”

5.4 Definition of the propositions by the (*ε, δ*)-definition of limit

The value of the limit depends on the order of the limit as follows.

$$\lim_{x\to 0}\left(\lim_{n\to\infty}\sum_{k=1}^{n} ke^{-kx}\cos(kx)\right) = -\frac{1}{12} \tag{5.51}$$

$$\lim_{n\to\infty}\left(\lim_{x\to 0}\sum_{k=1}^{n} ke^{-kx}\cos(kx)\right) = \infty \tag{5.52}$$

In this paper, we define the order of the limit as follows.

$$\lim_{x\to 0}\sum_{k=1}^{\infty} f_k(x) := \lim_{x\to 0}\left(\lim_{n\to\infty}\sum_{k=1}^{n} f_k(x)\right) \tag{5.53}$$

$$f_k(x) = ke^{-kx}\cos(kx) \tag{5.54}$$

This order of the limit means the following relation.

$$0 \ll \frac{1}{x} \ll n \tag{5.55}$$

We confirm the above relation by the (ε, δ)-definition of limit.

In this section, we define the following proposition by the (ε, δ)-definition of limit.
We define the function S_n and $H_n(x)$, and the constant α as follows.

$$S_n = \sum_{k=1}^{n} k \tag{5.56}$$

$$H_n(x) = \sum_{k=1}^{n} k\exp(-kx)\cos(kx) \tag{5.57}$$

$$\alpha = -\frac{1}{12} \tag{5.58}$$

We have the following proposition.

$$\lim_{n\to\infty} S_n = \infty \tag{5.59}$$

We can express the above proposition by (R, N)-definition of the limit as follows.

Given any number $R > 0$, there exists a natural number N such that for all n satisfying

$$N < n, \tag{5.60}$$

we have the following inequation.

$$R < S_n \tag{5.61}$$

We have the following proposition.

Proposition 1.

$$\lim_{x \to 0+} H_n(x) = S_n \tag{5.62}$$

We can express the above proposition by (ε, δ)-definition of the limit as follows.

Given any number $x > 0$ and any natural number n, there exists a number $\delta > 0$ such that for all $x > 0$ satisfying

$$x < \delta, \tag{5.63}$$

we have the following inequation.

$$|H_n(x) - S_n| < \varepsilon \tag{5.64}$$

We can summarize the above definition as follows.

$$\begin{gathered} \forall \varepsilon > 0, \forall n \in \mathbb{N}, \exists \delta > 0 \text{ s.t.} \\ \forall x > 0, x < \delta \Rightarrow |H_n(x) - S_n| < \varepsilon \end{gathered} \tag{5.65}$$

We have the following proposition.

Proposition 2.

$$\lim_{x \to 0+} \left(\lim_{n \to \infty} H_n(x) \right) = \alpha \tag{5.66}$$

$$0 \ll \frac{1}{x} \ll n \tag{5.67}$$

We can express the above proposition by (ε, δ)-definition of the limit as follows.

Given any number $\varepsilon > 0$, there exist a natural number N and a number $\delta > 0$ such that for all n satisfying

$$N < n, \tag{5.68}$$

we have the following inequation.

$$|H_n(\delta) - \alpha| < \varepsilon \tag{5.69}$$

We can summarize the above definition as follows.

$$\begin{aligned}&\forall\varepsilon>0,\exists N\in\mathbb{N},\exists\delta>0\ \text{s.t.}\\&\forall n\in\mathbb{N},N<n\Rightarrow|H_n(\delta)-\alpha|<\varepsilon\end{aligned}\tag{5.70}$$

The proposition 1 is the region of 1+2+3+…+n and the proposition 2 is the region −1/12 in the following figure.

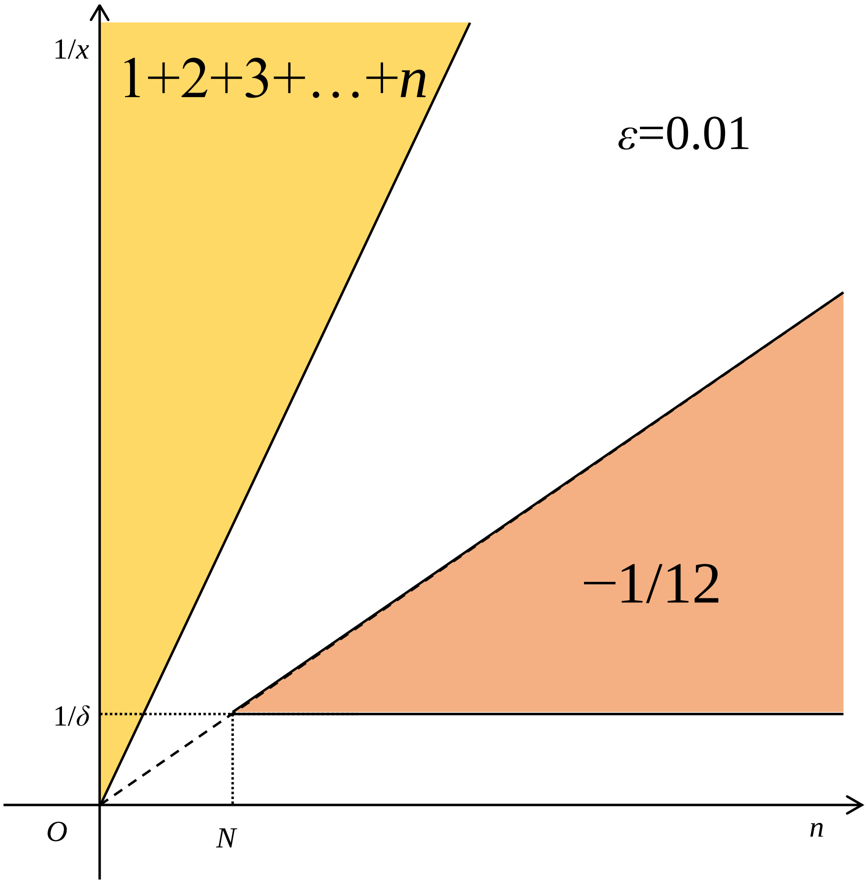

Figure 5.5: (ε, δ)-definition of limit

5.5 Relation with *q*-analog

5.5.1 *q-analog*

F. H. Jackson[8] defined the following quantized new natural number, ***q*-number** in 1904.
(q-number)

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1} \tag{5.71}$$

$$[n]_q = \sum_{k=1}^{n} q^{k-1} \tag{5.72}$$

$$[n]_q = \frac{1-q^n}{1-q} \tag{5.73}$$

We call the quantized mathematical object ***q*-analog** generally. The q-analog of the natural number n is q-number $[n]_q$. We express the natural number by the classical limit of the q-number.

$$n = \lim_{q\to 1-} [n]_q \tag{5.74}$$

The **zeta function** is shown below.
(Zeta function)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{5.75}$$

The q-analog of the zeta function is ***q*-zeta function**. The function is shown below.[7]
(q-zeta function)

$$\zeta_q(s) := \sum_{n=1}^{\infty} \frac{1}{[n]_q^s} q^{n(s-1)} \tag{5.76}$$

We express the zeta function by the classical limit of the q-zeta function.

$$\zeta(s) = \lim_{q\to 1-} \zeta_q(s) \tag{5.77}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{[n]_1^s} 1^{n(s-1)} \tag{5.78}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{5.79}$$

$$1 < \mathrm{Re}(s) \tag{5.80}$$

5.5.2 Zeta function

This paper defines the following **waved** (quantized) new natural number, **natural function**.
(Natural function)

$$`n`_z := e^z + e^{2z} + e^{3z} + \cdots + e^{nz} \qquad (5.81)$$

$$`n`_z = \sum_{k=1}^{n} e^{kz} \qquad (5.82)$$

$$`n`_z = e^z \frac{1 - e^{nz}}{1 - e^z} \qquad (5.83)$$

$$\nu_{nz} := `\nu_n`_z := `n`_z \qquad (5.84)$$

The relation between the natural function and q-number as follows.

$$\nu_{nz} = q[n]_q \qquad (5.85)$$

$$q = e^z \qquad (5.86)$$

We differentiate the natural function by the variable z.

$$\nu_{nz} = \sum_{k=1}^{n} e^{kz} \qquad (5.87)$$

$$\frac{d}{dz}\nu_{nz} = \sum_{k=1}^{n} k e^{kz} \qquad (5.88)$$

$$\frac{d^2}{dz^2}\nu_{nz} = \sum_{k=1}^{n} k^2 e^{kz} \qquad (5.89)$$

$$\frac{d^3}{dz^3}\nu_{nz} = \sum_{k=1}^{n} k^3 e^{kz} \qquad (5.90)$$

Therefore, we obtain the following function by differentiating the natural function by the variable z m times.

$$\frac{d^m}{dz^m}\nu_{nz} = \sum_{k=1}^{n} k^m e^{kz} \qquad (5.91)$$

We define the following function, **natural derivative** by replacing the natural number *m* by the complex *s*-1.
(Natural derivative)

$`\nu_n`_z(s) := \sum_{k=1}^{n} k^{s-1} e^{kz}$	(5.92)
$\nu_{nz}(s) := `\nu_n`_z(s)$	(5.93)

On the other hand, we define the **natural number** as follows.
(Natural number)

$$`n` := 1 + 1 + 1 + \cdots + 1 \tag{5.94}$$

$$`n` = \sum_{k=1}^{n} 1 \tag{5.95}$$

$$n = `n` \tag{5.96}$$

We express the natural number by a limit of the natural derivative.

$$n = \lim_{s \to 1} \left(\lim_{z \to 0} \nu_{nz}(s) \right) \tag{5.97}$$

$$n = \sum_{k=1}^{n} \frac{1}{k^0} e^{k0} \tag{5.98}$$

$$n = \sum_{k=1}^{n} 1 \tag{5.99}$$

We define the **zeta function** as follows.
(Zeta function)

$$`\zeta`(s) := \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \tag{5.100}$$

$$`\zeta`(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \tag{5.101}$$

$$\zeta(s) := `\zeta`(s) \tag{5.102}$$

We express the zeta function by a limit of the natural derivative.

$$\zeta(s) = \lim_{z\to 0}\left(\lim_{n\to 0} \nu_{nz}(1-s)\right) \qquad (5.103)$$

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} e^{k0} \qquad (5.104)$$

$$\zeta(s) \ = \sum_{k=1}^{\infty} \frac{1}{k^s} \qquad (5.105)$$

We define the **partition function** as follows.
(Partition function)

$$`Z`(z) := e^{z} + e^{2z} + e^{3z} + \cdots \qquad (5.106)$$

$$`Z`(z) = \sum_{k=1}^{\infty} e^{kz} \qquad (5.107)$$

$$Z(z) := `Z`(z) \qquad (5.108)$$

We express the partition function by a limit of the natural derivative.

$$Z(z) = \lim_{s\to 1}\left(\lim_{n\to\infty} \nu_{nz}(s)\right) \qquad (5.109)$$

$$Z(z) = \sum_{k=1}^{\infty} k^{1-1} e^{kz} \qquad (5.110)$$

$$Z(z) = \sum_{k=1}^{\infty} e^{kz} \qquad (5.111)$$

We summarize the above results as follows.
• The natural number is a limit of the natural derivative.
• The zeta function is a limit of the natural derivative.
• The partition function is a limit of the natural derivative.

We express these relations in the following figure.

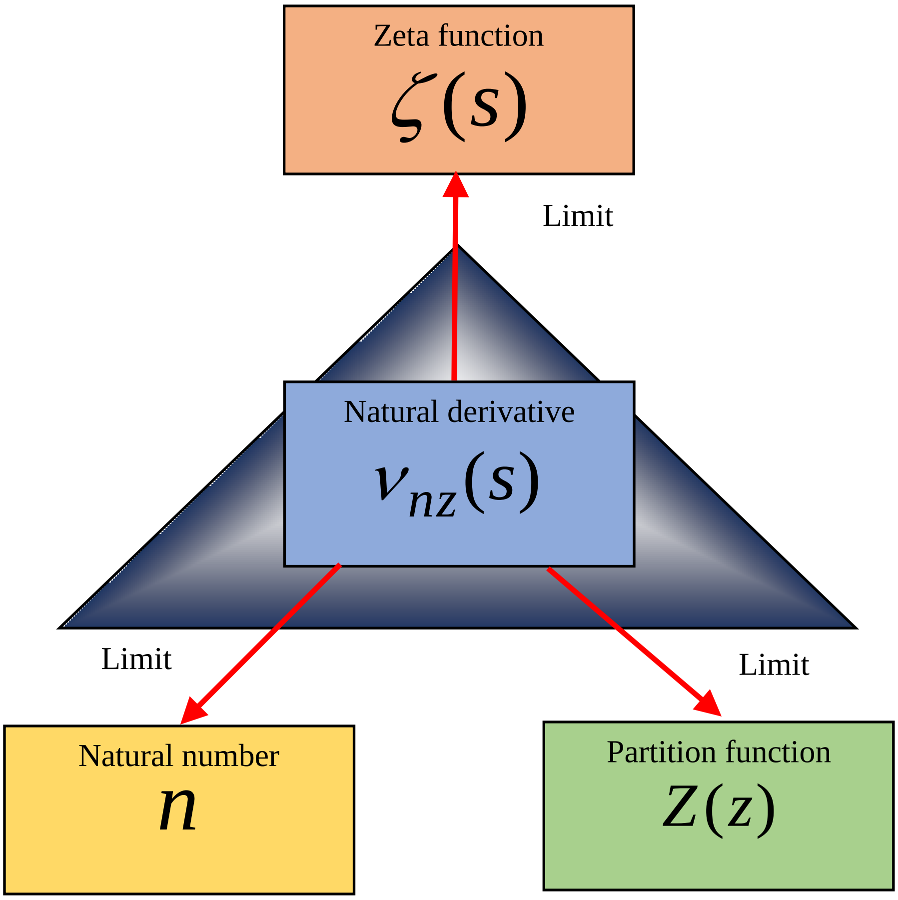

Figure 5.6: Zeta function

5.5.3 Analytic zeta function

We define the **harmonic natural derivative** as follows by averaging (uncertainizing) the natural derivative.
(Harmonic natural derivative)

$$"n"_\varepsilon(s) := \oint_{|z|=\varepsilon} \frac{dz}{2\pi i z}\, \nu_{nz}(s) \qquad (5.112)$$

$$\nu_{n\varepsilon}(s) := "\,\nu_n"_\varepsilon\,(s) := "n"_\varepsilon(s) \qquad (5.113)$$

We define the **analytic zeta function** as follows.
(Analytic zeta function)

$$"\zeta"(s) := \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx \qquad (5.114)$$

$$\zeta(s) := "\zeta"(s) \qquad (5.115)$$

The analytic zeta function is the analytic continuation of the zeta function.

We express the analytic zeta function by a limit of the harmonic natural derivative.

$$\zeta(s) = \lim_{z\to 0}\left(\lim_{n\to\infty} \nu_{n\varepsilon}(1-s)\right) \qquad (5.116)$$

$$\zeta(s) = \lim_{x\to 0}\oint_{|z|=\varepsilon}\frac{dz}{2\pi iz}\ \nu_{\infty\varepsilon}(1-s) \qquad (5.117)$$

$$\zeta(s) = \lim_{\varepsilon\to 0}\oint_{|z|=\varepsilon}\frac{dz}{2\pi iz}\sum_{k=1}^{\infty}\frac{1}{k^s}e^{kz} \qquad (5.118)$$

$$\zeta(s) = \lim_{\varepsilon\to 0}\oint_{|z|=\varepsilon}\frac{dz}{2\pi iz}\mathrm{Li}_s(e^z) \qquad (5.119)$$

Here, the function $\mathrm{Li}_s(w)$ is the following **polylogarithm function**.
(Polylogarithm function)

$$\mathrm{Li}_s(w) = \sum_{k=1}^{\infty}\frac{1}{k^s}w^k \qquad (5.120)$$

We obtain the following equation by using Taylor expansion of the $\mathrm{Li}_s(e^z)$.

$$\mathrm{Li}_s(e^z) = O\left(\frac{1}{z^{1-s}}\right) + \zeta(s) + \zeta(s-1)z + O(z^2) \qquad (5.121)$$

The constant term of the Taylor expansion is the analytic zeta function. Therefore, we have the following equation by using the residue theorem.

$$\zeta(s) = \lim_{\varepsilon\to 0}\oint_{|z|=\varepsilon}\frac{dz}{2\pi iz}\zeta(s) \qquad (5.122)$$

We express these relations in the following figure.

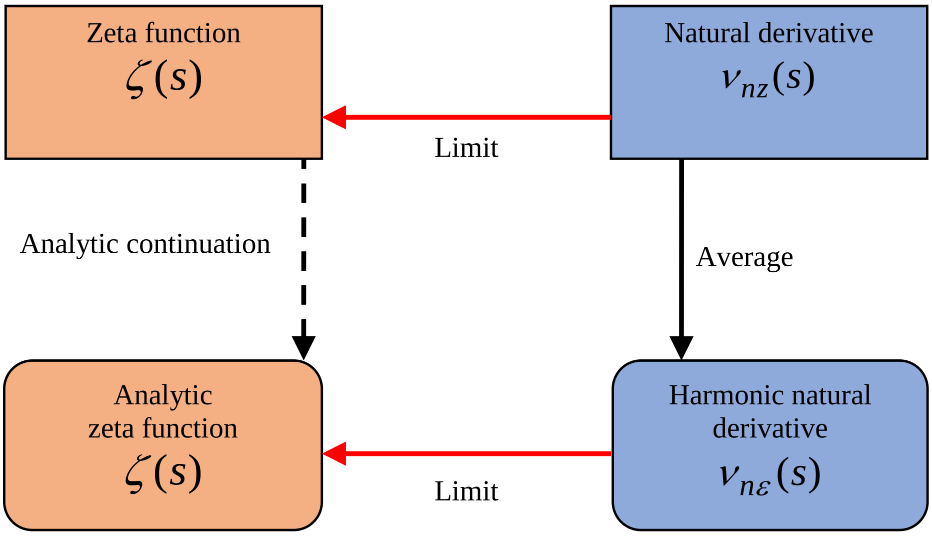

Figure 5.7: Analytic zeta function

5.6 Reflection integral equation

We shoe the reflection integral equation of the natural derivative.

We define the **factorial function** by the gamma function as follows.
(Factorial function)

$$\Pi(s) = \Gamma(s)s \tag{5.123}$$

We have the following equation for the natural number *n*.

$$\Pi(n) = n! \tag{5.124}$$

We define the **falling factorial** as follows.
(Falling factorial)

$$s^{\underline{t}} = \frac{\Pi(s)}{\Pi(s-t)} \tag{5.125}$$

We have the following equation for the natural number *n*.

$$n^{\underline{1}} = (n-0) \tag{5.126}$$

$$n^{\underline{2}} = (n-0)(n-1) \tag{5.127}$$

$$n^{\underline{3}} = (n-0)(n-1)(n-2) \tag{5.128}$$

We define the **combination function** by the beta function as follows.
(Combination function)

$$\mathrm{C}(s,t) = \frac{1}{B(s,t)}\frac{s+t}{st} \tag{5.129}$$

We have the following equation for the complex *s*, *t*.

$$\mathrm{C}(s,t) = \frac{\Pi(s+t)}{\Pi(s)\Pi(t)} \tag{5.130}$$

We have the following equation for the natural number *m*, *n*.

$$\mathrm{C}(m,n) = \frac{(m+n)!}{m!\,n!} = \binom{m+n}{n} \tag{5.131}$$

We show the **reflection integral equation** of the complex as follows. [9]
(Reflection integral equation of the complex)

$$\nu_{nz}(s+1) = \oint_{S^1} \frac{idt}{2\pi} \mathrm{B}(s,t)\nu_{nz}(-t) \tag{5.132}$$

We express the equation by using the combination function as follows.

$$\nu_{nz}(s+1) = \oint_{S^1} \frac{idt}{2\pi} \frac{1}{\mathrm{C}(s,t)} \frac{s+t}{st} \nu_{nz}(-t) \tag{5.133}$$

We deform the equation as follows.

$$\nu_{nz}(s+1) = \oint_{S^1} \frac{idt}{2\pi} \frac{\nu_{nz}(-t)}{\mathrm{C}(s+1,t)t} \tag{5.134}$$

We replace the variable s+1 to s.

$$\nu_{nz}(s) = \oint_{S^1} \frac{idt}{2\pi} \frac{\nu_{nz}(-t)}{\mathrm{C}(s,t)t} \tag{5.135}$$

We show the reflection integral equation of the quaternion as follows. [10]
(Reflection integral equation of the quaternion)

$$\nu_{nz}(s+1) = \oint_{S^3} \frac{dt^3}{2\pi^2} \frac{\mathrm{B}(s,t+2)}{(t+1)t} \nu_{nz}(-t) \tag{5.136}$$

We express the equation by using the combination function as follows.

$$\nu_{nz}(s+1) = \oint_{S^3} \frac{dt^3}{2\pi^2} \frac{1}{\mathrm{C}(s,t+2)} \frac{s+t+2}{s(t+2)} \frac{\nu_{nz}(-t)}{(t+1)t} \tag{5.137}$$

We deform the equation as follows.

$$\nu_{nz}(s+1) = \oint_{S^3} \frac{dt^3}{2\pi^2} \frac{\nu_{nz}(-t)}{\mathrm{C}(s+1,t+2)(t+2)^{3}} \tag{5.138}$$

We replace the variable t to t-2.

$$\nu_{nz}(s+1) = \oint_{S^3} \frac{dt^3}{2\pi^2} \frac{\nu_{nz}(2-t)}{\mathrm{C}(s+1,t)t^{3}} \tag{5.139}$$

We replace the variable s+1 to s.

$$\nu_{nz}(s) = \oint_{S^3} \frac{dt^3}{2\pi^2} \frac{\nu_{nz}(2-t)}{\mathrm{C}(s,t)t^{3}} \tag{5.140}$$

We have the reflection integral equation of the complex and the quaternion as follows.
(Reflection integral equations)

$$\nu_{nz}(s) = \oint_{S^1} \frac{idt}{2\pi t} \frac{\nu_{nz}(-t)}{\mathrm{C}(s,t)} \tag{5.141}$$

$$\nu_{nz}(s) = \oint_{S^3} \frac{dt^3}{2\pi^2 t^{3}} \frac{\nu_{nz}(2-t)}{\mathrm{C}(s,t)} \tag{5.142}$$

6 Appendix

6.1 Hurwitz zeta function

We calculate the sum of natural numbers by the Hurwitz zeta function.

Adolf Hurwitz[11] published the following zeta function in 1882.
(Definitional series of Hurwitz zeta function)

$$\zeta(s,q)=\frac{1}{(q+0)^s}+\frac{1}{(q+1)^s}+\frac{1}{(q+2)^s}+\cdots \tag{6.1}$$

$$\zeta(s,q)=\sum_{k=0}^{\infty}\frac{1}{(q+k)^s} \tag{6.2}$$

Hurwitz zeta function and Riemann zeta function have the following relationship.
(Relation between Hurwitz zeta function and Riemann zeta function)

$$\zeta(s)=\left\{\frac{1}{1^s}+\frac{1}{2^s}+\cdots+\frac{1}{(q-1)^s}\right\}+\left\{\frac{1}{(q+0)^s}+\frac{1}{(q+1)^s}+\cdots\right\} \tag{6.3}$$

$$\zeta(s)=\sum_{k=1}^{q-1}\frac{1}{k^s}+\zeta(s,q) \tag{6.4}$$

Therefore, we can observe the value of the sum to the middle of the sum of the natural numbers.
We calculate the Hurwitz zeta function by the following asymptotic expansion. Here, B_k is the Bernoulli numbers.

(Asymptotic expansion of the Hurwitz zeta function)

$$\zeta(s,q)=\sum_{k=0}^{r}\frac{B_k}{k!}\frac{\Gamma(s+k-1)}{\Gamma(s)q^{s+k-1}} \tag{6.5}$$

For example, we think the sum of one until three and the sum of four until infinity.

$$\zeta(-1)=\sum_{k=1}^{4-1}\frac{1}{k^{-1}}+\zeta(-1,4) \tag{6.6}$$

We can express the value of the Hurwitz zeta function as follows.

$$\zeta(-1,4)=\sum_{k=0}^{r}\frac{B_k}{k!}\frac{\Gamma(k-2)}{\Gamma(-1)4^{k-2}} \tag{6.7}$$

Here, we use the following reflection formula of the gamma function.

$$\Gamma(1-s)=\frac{\pi}{\sin(\pi s)}\frac{1}{\Gamma(s)} \tag{6.8}$$

We can modified the Hurwitz zeta function as follows by the above formula.

$$\zeta(-1,4)=\sum_{k=0}^{r}\frac{B_k}{k!}\frac{\Gamma(2)}{\Gamma(3-k)4^{k-2}}\frac{\sin(2\pi)}{\sin\big((3-k)\pi\big)} \tag{6.9}$$

We have the following equation for integer *k*.

$$\frac{\sin(2\pi)}{\sin\big((3-k)\pi\big)}=(-1)^{k+1} \tag{6.10}$$

Therefore, we can obtain the following equation.

$$\zeta(-1,4)=\sum_{k=0}^{r}\frac{B_k}{k!}\frac{\Gamma(2)}{\Gamma(3-k)4^{k-2}}(-1)^{k+1} \tag{6.11}$$

We have the following equation for integer $k \geq 3$.

$$\frac{1}{\Gamma(3-k)}=0 \tag{6.12}$$

Therefore, we can obtain the following equation.

$$\zeta(-1,4)=\frac{B_0 1!\,(-1)^{0+1}}{0!\,(2-0)!\,4^{0-2}}+\frac{B_1 1!\,(-1)^{1+1}}{1!\,(2-1)!\,4^{1-2}}+\frac{B_2 1!\,(-1)^{2+1}}{2!\,(2-2)!\,4^{2-2}} \tag{6.13}$$

$$\zeta(-1,4)=(-\frac{1}{2}4^2)+(\frac{1}{2}4^1)+(-\frac{1}{12}4^0) \tag{6.14}$$

$$\zeta(-1,4)=-\frac{16}{2}+\frac{4}{2}-\frac{1}{12} \tag{6.15}$$

$$\zeta(-1,4)=-6-\frac{1}{12} \tag{6.16}$$

Therefore, we can obtain the following result.

$$\text{"}(1+2+3)+(4+5+6+\cdots)\text{"}=6+\left(-6-\frac{1}{12}\right)=-\frac{1}{12} \tag{6.17}$$

We can see that the sum of the natural numbers is increasing certainly halfway according the above equation. Then, on the way leading to the infinity, it reduced.

The series of the zeta function seems to correct for negative integers.
However, why will it decrease on the way leading to the infinity?

I think that it oscillates in an infinite time on the way to infinity.
I think that it is damped exponentially on the way to infinity.
This idea became the origin of the damped oscillation summation method.

6.2 Lerch zeta function

This paper proposed that we might interpret Riemann zeta function as the Lerch zeta function in the case of the variable λ is approximately zero.

The Lerch zeta function was published by Lerch[12] in 1887.
(Definitional series of Lerch zeta function)

$$L(\lambda, \alpha, s) = \frac{\exp(2\pi i\lambda 0)}{(0+\alpha)^s} + \frac{\exp(2\pi i\lambda 1)}{(1+\alpha)^s} + \frac{\exp(2\pi i\lambda 2)}{(2+\alpha)^s} + \cdots \tag{6.18}$$

$$L(\lambda, \alpha, s) = \sum_{k=0}^{\infty} \frac{\exp(2\pi i\lambda k)}{(k+\alpha)^s} \tag{6.19}$$

6.3 Goursat's theorem

We derive the following theorem from the Cauchy's integral formula.
(Goursat's theorem of complex)

$$f^{(m)}(a) = \frac{m!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{m+1}} dz \tag{6.20}$$

Here, we consider the following partition function.
(Partition function)

$$Z(z) = \sum_{k=1}^{\infty} e^{kz} = \frac{e^z}{1-e^z} \tag{6.21}$$

We express the above formla by Goursat's theorem.

$$Z^{(m)}(0) = \frac{m!}{2\pi i} \int_C \frac{Z(z)}{z^{m+1}} dz \tag{6.22}$$

Then, we replace the natural number *m* to the complex -*s*.

$$Z^{(-s)}(0) = \frac{\Gamma(1-s)}{2\pi i} \int_C z^{s-1}\ Z(z) dz \tag{6.23}$$

This formula is same as the contour integral definition of the zeta function.
(Contour integral definition of the zeta function)

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_C z^{s-1} \frac{e^z}{1-e^z} dz \tag{6.24}$$

The contour integral definition of the natural derivative is shown below.
(Contour integral definition of the natural derivative)

$$\nu_{n\varepsilon}(s) = \frac{\Gamma(s+1)}{2\pi i} \int_{|z|=\varepsilon} z^{-s-1} \frac{e^z - e^{(n+1)z}}{1-e^z} dz \tag{6.25}$$

On the other hand, Goursat's theorem of quaternion is shown below.
(Goursat's theorem of quaternion)

$$f^{(m)}(a) = \frac{-m!}{2\pi^2} \int_C \frac{f(z)}{(z-a)^{m+3}} dz^3 \tag{6.26}$$

Here, we consider the following partition function.
(Partition function)

$$Z(z) = \sum_{k=1}^{\infty} e^{kz} = \frac{e^z}{1-e^z} \tag{6.27}$$

We express the above formla by Goursat's theorem.

$$Z^{(m)}(0) = \frac{-m!}{2\pi^2} \int_C \frac{Z(z)}{z^{m+3}} dz \tag{6.28}$$

Then, we replace the natural number *m* to the complex *-s*.

$$Z^{(-s)}(0) = \frac{-\Gamma(1-s)}{2\pi^2} \int_C z^{s-3}\ Z(z) dz \tag{6.29}$$

This formula is same as the contour integral definition of the zeta function.
(Contour integral definition of the zeta function)

$$\zeta(s) = \frac{-\Gamma(1-s)}{2\pi^2} \int_C z^{s-3} \frac{e^z}{1-e^z} dz \tag{6.30}$$

The contour integral definition of the natural derivative is shown below.
(Contour integral definition of the natural derivative)

$$\nu_{n\varepsilon}(s) = \frac{-\Gamma(s+1)}{2\pi^2} \int_{|z|=\varepsilon} z^{-s-3} \frac{e^z - e^{(n+1)z}}{1-e^z} dz \tag{6.31}$$

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