Fibonacci Quarternions

By John Frederick Sweeney

$M_{16}(R)$	$M_{16}(C)$	$M_{16}(H)$	$M_{16}(H) \oplus M_{16}(H)$	$M_{32}(H)$	$M_{64}(C)$	$M_{128}(R)$	$M_{128}(R) \oplus M_{128}(R)$	$M_{256}(R)$
$M_{8}(C)$	$M_{8}(H)$	$M_{8}(H) \oplus M_{8}(H)$	$M_{16}(H)$	$M_{32}(C)$	$M_{64}(R)$	$M_{64}(R) \oplus M_{64}(R)$	$M_{128}(R)$	$M_{128}(C)$
$M_{4}(H)$	$M_{4}(H) \oplus M_{4}(H)$	$M_{8}(H)$	$M_{16}(C)$	$M_{32}(R)$	$M_{32}(R) \oplus M_{32}(R)$	$M_{64}(R)$	$M_{64}(C)$	$M_{64}(H)$
$M_{2}(H) \oplus M_{2}(H)$	$M_{4}(H)$	$M_{8}(C)$	$M_{16}(R)$	$M_{16}(R) \oplus M_{16}(R)$	$M_{32}(R)$	$M_{32}(C)$	$M_{32}(H)$	$M_{32}(H) \oplus M_{32}(H)$
$M_{2}(H)$	$M_{4}(C)$	$M_{8}(R)$	$M_{8}(R) \oplus M_{8}(R)$	$M_{16}(R)$	$M_{16}(C)$	$M_{16}(H)$	$M_{16}(H) \oplus M_{16}(H)$	$M_{32}(H)$
$M_{2}(C)$	$M_{4}(R)$	$M_{4}(R) \oplus M_{4}(R)$	$M_{8}(R)$	$M_{8}(C)$	$M_{8}(H)$	$M_{8}(H) \oplus M_{8}(H)$	$M_{16}(H)$	$M_{32}(C)$
$M_{2}(R)$	$M_{2}(R) \oplus M_{2}(R)$	$M_{4}(R)$	$M_{4}(C)$	$M_{4}(H)$	$M_{4}(H) \oplus M_{4}(H)$	$M_{8}(H)$	$M_{16}(C)$	$M_{32}(R)$
$R \oplus R$	$M_{2}(R)$	$M_{2}(C)$	$M_{2}(H)$	$M_{2}(H) \oplus M_{2}(H)$	$M_{4}(H)$	$M_{8}(C)$	$M_{16}(R)$	$M_{16}(R) \oplus M_{16}(R)$
R	C	H	$H \oplus H$	$M_{2}(H)$	$M_{4}(C)$	$M_{8}(R)$	$M_{8}(R) \oplus M_{8}(R)$	$M_{16}(R)$

Abstract

Pascal’s Triangle, originally Mount Meru of Vedic Physics, provides the perfect format for a combinatorial Universe, with its binomial coefficients, as well as its ease of determining Fibonacci Numbers. Matrix and Clifford algebras, in the form of the chart above, can be shaped into a form identical with Pascal’s Triangle. At the same time, a Romanian researcher has devised an algorithm for determining a Fibonacci Number as a quarternion. This paper poses the question as to whether the properties of Pascal's Triangle hold for a similar triangle constructed of Clifford Algebras.

Table of Contents

Introduction

Fibonacci Numbers came into Europe via Italy and were made known by Leonardo Fibonacci, who had translated them from Arabic into Italian. In the opinion of the author, the discovery of these numbers must be attributed to extremely ancient Hindu civilization, which is the source of most advanced mathematics in the world. Over the millennia, this Hindu math has slowly leaked from India and spread throughout the world via translation into the languages of neighboring societies.

In its desire to atomize everything, western civilization has cut off the Fibonacci Numbers from Pisano Periodicity, thereby giving two misleading names to mathematical concepts which should have never been separated in the first place. Vedic Physics, the superior science of the remotely ancient past of 14,000 years ago, needs to be taught and learned in whole form, not chopped up piecemeal and lent out bit by bit to surrounding neighbors, but alas, such as proven the case.

Now is the time to re – unite the various bits and pieces of Vedic Physics which have leaked through to the west and combine them into a uniform whole which makes explicit sense as a complete theory. By so doing, it may prove possible to gain great advances in mathematical physics.

This paper, based on the work of physicist Frank “Tony” Smith, takes the heuristic construct of Pascal’s Triangle, to align matrix and Clifford Algebras, as Smith has done and as appears on a Wikipedia page. The paper then suggests that the same seemingly “magical” properties of Pascal’s Triangle hold true for the Clifford Algebra triangle.

As events would transpire, Christina Flaut published a paper on the Archiv server a week before this idea arose with the author. This paper gives her version of Fibonacci Quarternions, and this paper adopts her introduction to demonstrate her algorithm.

Clifford Algebras

In mathematics, **Clifford algebras** are a type of associative algebra. As K-algebras, they generalize the real numbers, complex numbers, quaternions and several other hypercomplex number systems.[1][2] The theory of Clifford algebras is intimately connected with the theory of quadratic forms and orthogonal transformations. Clifford algebras have important applications in a variety of fields including geometry, theoretical physics and digital image processing. They are named after the English geometer William Kingdon Clifford.

A Clifford algebra is a unital associative algebra that contains and is generated by a vector space V over a field K, where V is equipped with a quadratic form Q. The Clifford algebra $C\ell(V, Q)$ is the "freest" algebra generated by V subject to the condition[4]

$$v^2 = Q(v)1 \quad \text{for all } v \in V,$$

where the product on the left is that of the algebra, and the 1 is its multiplicative identity.

The definition of a Clifford algebra endows it with more structure than a "bare" K-algebra: specifically it has a designated or privileged subspace that is isomorphic to V. Such a subspace cannot in general be uniquely determined given only a K-algebra isomorphic to the Clifford algebra.

If the characteristic of the ground field K is not 2, then one can rewrite this fundamental identity in the form

$$uv + vu = 2\langle u, v\rangle 1 \quad \text{for all } u, v \in V,$$

where

$$\langle u, v\rangle = \frac{1}{2}\left(Q(u+v) - Q(u) - Q(v)\right)$$

is the symmetric bilinear form associated with Q, via the polarization identity. The idea of being the "freest" or "most general" algebra subject to this identity can be formally expressed through the notion of a universal property, as done below.

Quadratic forms and Clifford algebras in characteristic 2 form an exceptional case. In particular, if char(K) = 2 it is not true that a quadratic form determines a symmetric bilinear form, or that every quadratic form admits an orthogonal basis. Many of the statements in this article include the condition that the characteristic is not 2, and are false if this condition is removed.

As a quantization of the exterior algebra[edit]

Clifford algebras are closely related to exterior algebras. In fact, if $Q = 0$ then the Clifford algebra $C\ell(V, Q)$ is just the exterior algebra $\Lambda(V)$. For nonzero Q there exists a canonical *linear* isomorphism between $\Lambda(V)$ and $C\ell(V, Q)$ whenever the ground field K does not have characteristic two. That is, they are naturally isomorphic as vector spaces, but with different multiplications (in the case of characteristic two, they are still isomorphic as vector spaces, just not naturally). Clifford multiplication together with the privileged subspace is strictly richer than the exterior product since it makes use of the extra information provided by Q.

More precisely, Clifford algebras may be thought of as *quantizations* (cf. Quantum group) of the exterior algebra, in the same way that the Weyl algebra is a quantization of the symmetric algebra.

Weyl algebras and Clifford algebras admit a further structure of a *-algebra, and can be unified as even and odd terms of a superalgebra, as discussed in CCR and CAR algebras.

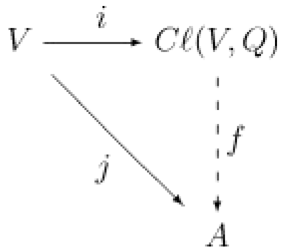
Universal property and construction[edit]

Let V be a vector space over a field K, and let $Q: V \to K$ be a quadratic form on V. In most cases of interest the field K is either the field of real numbers **R**, or the field of complex numbers **C**, or a finite field.

A Clifford algebra $C\ell(V, Q)$ is a unital associative algebra over K together with a linear map $i : V \to C\ell(V, Q)$ satisfying $i(v)^2 = Q(v)1$ for all $v \in V$, defined by the following universal property: given any associative algebra A over K and any linear map $j : V \to A$ such that

$$j(v)^2 = Q(v)1_A \text{ for all } v \in V$$

(where 1_A denotes the multiplicative identity of A), there is a unique algebra homomorphism $f : C\ell(V, Q) \to A$ such that the following diagram commutes (i.e. such that $f \circ i = j$):



Working with a symmetric bilinear form $\langle\cdot,\cdot\rangle$ instead of Q (in characteristic not 2), the requirement on j is

$$j(v)j(w) + j(w)j(v) = 2\langle v, w\rangle 1_A \quad \text{for all } v, w \in V \; .$$

A Clifford algebra as described above always exists and can be constructed as follows: start with the most general algebra that contains V, namely the tensor algebra $T(V)$, and then enforce the fundamental identity by taking a suitable quotient. In our case we want to take the two-sided ideal I_Q in $T(V)$ generated by all elements of the form

$$v \otimes v - Q(v)1 \text{for all } v \in V$$

and define $C\ell(V, Q)$ as the quotient algebra

$$C\ell(V, Q) = T(V)/I_Q.$$

The ring product inherited by this quotient is sometimes referred to as the **Clifford product**[5] to differentiate it from the inner and outer products.

It is then straightforward to show that $C\ell(V, Q)$ contains V and satisfies the above universal property, so that $C\ell$ is unique up to a unique isomorphism; thus one speaks of "the" Clifford algebra $C\ell(V, Q)$. It also follows from this construction that i is injective. One usually drops the i and considers V as a linear subspace of $C\ell(V, Q)$.

The universal characterization of the Clifford algebra shows that the construction of $C\ell(V, Q)$ is *functorial* in nature. Namely, $C\ell$ can be considered as a functor from the category of vector spaces with quadratic forms (whose morphisms are linear maps preserving the quadratic form) to the category of associative algebras. The universal property guarantees that linear maps between vector spaces (preserving the quadratic form) extend uniquely to algebra homomorphisms between the associated Clifford algebras.

Basis and dimension

If the dimension of *V* is *n* and $\{e_1, \cdots, e_n\}$ is a basis of *V*, then the set

$$\{e_{i_1}e_{i_2}\cdots e_{i_k} \mid 1 \le i_1 < i_2 < \cdots < i_k \le n \text{ and } 0 \le k \le n\}$$

is a basis for $C\ell(V, Q)$. The empty product ($k = 0$) is defined as the multiplicative identity element. For each value of *k* there are *n* choose *k* basis elements, so the total dimension of the Clifford algebra is

$$\dim C\ell(V,Q) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Since *V* comes equipped with a quadratic form, there is a set of privileged bases for *V*: the orthogonal ones. An orthogonal basis is one such that

$$\langle e_i, e_j \rangle = 0 \qquad i \neq j.$$

where $\langle \cdot, \cdot \rangle$ is the symmetric bilinear form associated to *Q*. The fundamental Clifford identity implies that for an orthogonal basis

$$e_i e_j = -e_j e_i \qquad i \neq j.$$

This makes manipulation of orthogonal basis vectors quite simple. Given a product $e_{i_1}e_{i_2}\cdots e_{i_k}$ of *distinct* orthogonal basis vectors of *V*, one can put them into standard order while including an overall sign determined by the number of pairwise swaps needed to do so (i.e. the signature of the ordering permutation).

Examples: real and complex Clifford algebras

The most important Clifford algebras are those over real and complex vector spaces equipped with nondegenerate quadratic forms.

It turns out that every one of the algebras $C\ell_{p,q}(\mathbf{R})$ and $C\ell_n(\mathbf{C})$ is isomorphic to A or $A \oplus A$, where A is a full matrix ring with entries from **R**, **C**, or **H**. For a complete classification of these algebras see classification of Clifford algebras.

Real numbers

The geometric interpretation of real Clifford algebras is known as geometric algebra.

Every nondegenerate quadratic form on a finite-dimensional real vector space is equivalent to the standard diagonal form:

$$Q(v) = v_1^2 + \cdots + v_p^2 - v_{p+1}^2 - \cdots - v_{p+q}^2$$

where $n = p + q$ is the dimension of the vector space. The pair of integers (p, q) is called the signature of the quadratic form. The real vector space with this quadratic form is often denoted $\mathbf{R}^{p,q}$. The Clifford algebra on $\mathbf{R}^{p,q}$ is denoted $C\ell_{p,q}(\mathbf{R})$. The symbol $C\ell_n(\mathbf{R})$ means either $C\ell_{n,0}(\mathbf{R})$ or $C\ell_{0,n}(\mathbf{R})$ depending on whether the author prefers positive definite or negative definite spaces.

A standard orthonormal basis $\{e_i\}$ for $\mathbf{R}^{p,q}$ consists of $n = p + q$ mutually orthogonal vectors, p of which have norm +1 and q of which have norm −1. The algebra $C\ell_{p,q}(\mathbf{R})$ will therefore have p vectors that square to +1 and q vectors that square to −1.

Note that $C\ell_{0,0}(\mathbf{R})$ is naturally isomorphic to **R** since there are no nonzero vectors. $C\ell_{0,1}(\mathbf{R})$ is a two-dimensional algebra generated by a single vector e_1 that squares to −1, and therefore is isomorphic to **C**, the field of complex numbers. The algebra $C\ell_{0,2}(\mathbf{R})$ is a four-dimensional algebra spanned by $\{1, e_1, e_2, e_1e_2\}$. The latter three elements square to −1 and all anticommute, and so the algebra is isomorphic to the quaternions **H**. $C\ell_{0,3}(\mathbf{R})$ is an 8-dimensional algebra isomorphic to the direct sum $\mathbf{H} \oplus \mathbf{H}$ called split-biquaternions.

Fibonacci Clifford Algebras

A Clifford algebra associated to generalized Fibonacci quaternions

Cristina FLAUT

Abstract. In this paper we find a Clifford algebra associated to generalized Fibonacci quaternions. In this way, we provide a nice algorithm to obtain a division quaternion algebra starting from a quaternion non-division algebra and vice-versa.

2. Preliminaries

Let $\mathbb{H}(\beta_1, \beta_2)$ be the generalized real quaternion algebra, the algebra of the elements of the form $a = a_1 \cdot 1 + a_2 e_2 + a_3 e_3 + a_4 e_4$, where $a_i \in \mathbb{R}, i \in \{1, 2, 3, 4\}$, and the elements of the basis $\{1, e_2, e_3, e_4\}$ satisfy the following multiplication table:

$\cdot$	1	e_2	e_3	e_4
1	1	e_2	e_3	e_4
e_2	e_2	$-\beta_1$	e_4	$-\beta_1 e_3$
e_3	e_3	$-e_4$	$-\beta_2$	$\beta_2 e_2$
e_4	e_4	$\beta_1 e_3$	$-\beta_2 e_2$	$-\beta_1 \beta_2$

We denote by $\boldsymbol{n}(a)$ the norm of a real quaternion a. The norm of a generalized quaternion has the following expression $\boldsymbol{n}(a) = a_1^2 + \beta_1 a_2^2 + \beta_2 a_3^2 + \beta_1 \beta_2 a_4^2$. For $\beta_1 = \beta_2 = 1$, we obtain the real division algebra $\mathbb{H}$, with the basis $\{1, i, j, k\}$, where $i^2 = j^2 = k^2 = -1$ and $ij = -ji, ik = -ki, jk = -kj$.

Proposition 2.1. ([La; 04], Proposition 1.1) *The quaternion algebra* $\mathbb{H}(\beta_1, \beta_2)$ *is isomorphic with quaternion algebra* $\mathbb{H}(x^2\beta_1, y^2\beta_2)$ *, where* $x, y \in K^*$. $\square$

The Fibonacci numbers was introduced by *Leonardo of Pisa (1170-1240)* in his book *Liber abbaci*, book published in 1202 AD (see [Kos; 01], p. 1, 3). This name is attached to the following sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, ...,$$

with the nth term given by the formula:

$$f_n = f_{n-1} + f_{n-2}, \; n \geq 2,$$

where $f_0 = 0, f_1 = 1$.

In [Ho; 61], the author generalized Fibonacci numbers and gave many properties of them:

$$h_n = h_{n-1} + h_{n-2}, \quad n \geq 2,$$

where $h_0 = p, h_1 = q$, with p, q being arbitrary integers. In the same paper [Ho; 61], relation (7), the following relation between Fibonacci numbers and generalized Fibonacci numbers was obtained:

$$h_{n+1} = pf_n + qf_{n+1}. \tag{2.1}$$

For the generalized real quaternion algebra, the Fibonacci quaternions and generalized Fibonacci quaternions are defined in the same way:

$$F_n = f_n \cdot 1 + f_{n+1}e_2 + f_{n+2}e_3 + f_{n+3}e_4,$$

for the nth Fibonacci quaternions, and

$$H_n = h_n \cdot 1 + h_{n+1}e_2 + h_{n+2}e_3 + h_{n+3}e_4 = pF_n + qF_{n+1}, \tag{2.2}$$

for the nth generalized Fibonacci quaternions.

In the following, we will denote the nth generalized Fibonacci number and a nth generalized Fibonacci quaternion element with $h_n^{p,q}$, respectively $H_n^{p,q}$. In this way, we emphasis the starting integers p and q.

It is known that the expression for the nth term of a Fibonacci element is

$$f_n = \frac{1}{\sqrt{5}}[\alpha^n - \beta^n] = \frac{\alpha^n}{\sqrt{5}}[1 - \frac{\beta^n}{\alpha^n}], \tag{2.3}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

From the above, we obtain the following limit:

$\lim\limits_{n\to\infty} \boldsymbol{n}(F_n) = \lim\limits_{n\to\infty}(f_n^2 + \beta_1 f_{n+1}^2 + \beta_2 f_{n+2}^2 + \beta_1\beta_2 f_{n+3}^2) =$

$= \lim\limits_{n\to\infty}(\frac{\alpha^{2n}}{5} + \beta_1 \frac{\alpha^{2n+2}}{5} + \beta_2 \frac{\alpha^{2n+4}}{5} + \beta_1\beta_2 \frac{\alpha^{2n+6}}{5}) =$

$= sgnE(\beta_1, \beta_2) \cdot \infty$, where $E(\beta_1, \beta_2) = \frac{1}{5}[1 + \beta_1 + 2\beta_2 + 5\beta_1\beta_2 + \alpha(\beta_1 + 3\beta_2 + 8\beta_1\beta_2)]$, since $\alpha^2 = \alpha + 1$.(see [Fl, Sh; 13])

If $E(\beta_1, \beta_2) > 0$, there exist a number $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have $\boldsymbol{n}(F_n) > 0$. In the same way, if $E(\beta_1, \beta_2) < 0$, there exist a number $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$ we have $\boldsymbol{n}(F_n) < 0$. Therefore for all $\beta_1, \beta_2 \in \mathbb{R}$ with $E(\beta_1, \beta_2) \neq 0$, in the algebra $\mathbb{H}(\beta_1, \beta_2)$ there is a natural number $n_0 = \max\{n_1, n_2\}$ such that $\boldsymbol{n}(F_n) \neq 0$, hence F_n is an invertible element for all $n \geq n_0$. Using the same arguments, we can compute the following limit: $\lim\limits_{n\to\infty} (\boldsymbol{n}(H_n^{p,q})) = \lim\limits_{n\to\infty} \left(h_n^2 + \beta_1 h_{n+1}^2 + \beta_2 h_{n+2}^2 + \beta_1\beta_2 h_{n+3}^2\right) = sgnE'(\beta_1, \beta_2) \cdot \infty$, where $E'(\beta_1, \beta_2) = \frac{1}{5}(p + \alpha q)^2 E(\beta_1, \beta_2)$, if $E'(\beta_1, \beta_2) \neq 0$.(see [Fl, Sh; 13])

Therefore, for all $\beta_1, \beta_2 \in \mathbb{R}$ with $E'(\beta_1, \beta_2) \neq 0$, in the algebra $\mathbb{H}(\beta_1, \beta_2)$ there exist a natural number n_0' such that $\boldsymbol{n}(H_n^{p,q}) \neq 0$, hence $H_n^{p,q}$ is an invertible element for all $n \geq n_0'$.

Theorem 2.2. ([Fl, Sh; 13], Theorem 2.6) *For all $\beta_1, \beta_2 \in \mathbb{R}$ with $E'(\beta_1, \beta_2) \neq 0$, there exist a natural number n' such that for all $n \geq n'$ Fibonacci elements F_n and generalized Fibonacci elements $H_n^{p,q}$ are invertible elements in the algebra $\mathbb{H}(\beta_1, \beta_2)$.*$\Box$

Theorem 2.3. ([Fl, Sh; 13], Theorem 2.1) *The set $\mathcal{H}_n = \{H_n^{p,q} \ / \ p, q \in \mathbb{Z}, n \geq m, m \in \mathbb{N}\} \cup \{0\}$ is a $\mathbb{Z}-$module.*$\Box$

3. Main results

Remark 3.1. We remark that the $\mathbb{Z}-$module from Theorem 2.3 is a free $\mathbb{Z}-$module of rank 2. Indeed, $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathcal{H}_n$, $\varphi((p,q)) = H_n^{p,q}$ is a $\mathbb{Z}-$module isomorphism and $\{\varphi(1,0) = F_n, \varphi(0,1) = F_{n+1}\}$ is a basis in $\mathcal{H}_n$.

Remark 3.2. By extension of scalars, we obtain that $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{H}_n$ is a $\mathbb{R}-$vector space of dimension two. A basis is $\{\overline{e}_1 = 1 \otimes F_n, \overline{e}_2 = 1 \otimes F_{n+1}\}$. We have that $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{H}_n$ is isomorphic with the $\mathbb{R}-$vector space $\mathcal{H}_n^{\mathbb{R}} = \{H_n^{p,q} \ / \ p, q \in \mathbb{R}\} \cup \{0\}$. A basis in $\mathcal{H}_n^{\mathbb{R}}$ is $\{F_n, F_{n+1}\}$.

Let $T\left(\mathcal{H}_n^{\mathbb{R}}\right)$ be the tensor algebra associated to the $\mathbb{R}-$vector space $\mathcal{H}_n^{\mathbb{R}}$ and let $C\left(\mathcal{H}_n^{\mathbb{R}}\right)$ be the Clifford algebra associated to tensor algebra $T\left(\mathcal{H}_n^{\mathbb{R}}\right)$. From Theorem 1.1, it results that this algebra has dimension four.

North

M_{14}(R) CL (0,8)				M_2(H)	**H** **H**	**H**	**C**	**R**
M_{32}(H)				M_{16}(R)				M_2(H)
M_{256}(R)				M_{32}(H)				M_{16}(C)

Pascal Triangle of Clifford Algebras

(Wikipedia)

Of all matrix ring types mentioned, there is only one type shared between complex and real algebras: the type $\mathbf{C}(2^m)$.

For example, $C\ell_2(\mathbf{C})$ and $C\ell_{3,0}(\mathbf{R})$ are determined as $\mathbf{C}(2)$. There is a difference in the classifying isomorphisms used.

Since $C\ell_2(\mathbf{C})$ algebra is isomorphic via a **C**-linear map (which is necessarily **R**-linear), and $C\ell_{3,0}(\mathbf{R})$ algebra is isomorphic via an **R**-linear map,

Then: $C\ell_2(\mathbf{C})$ and $C\ell_{3,0}(\mathbf{R})$ are **R**-algebra isomorphic.

A table of this classification for $p + q \leqslant 8$ follows: Here $p + q$ runs vertically and $p - q$ runs horizontally (e.g. the algebra $C\ell_{1,3}(\mathrm{R}) \cong \mathrm{M}_2(\mathrm{H})$ is found in row 4, column −2).

	8	7	6	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8
0									$\mathbf{R}$								
1								$\mathbf{R}^2$		$\mathbf{C}$							
2							$M_2(\mathbf{R})$		$M_2(\mathbf{R})$		$\mathbf{H}$						
3						$M_2(\mathbf{C})$		$M_2^2(\mathbf{R})$		$M_2(\mathbf{C})$		$\mathbf{H}^2$					
4					$M_2(\mathbf{H})$		$M_4(\mathbf{R})$		$M_4(\mathbf{R})$		$M_2(\mathbf{H})$		$M_2(\mathbf{H})$				
5				$M_2^2(\mathbf{H})$		$M_4(\mathbf{C})$		$M_4^2(\mathbf{R})$		$M_4(\mathbf{C})$		$M_2^2(\mathbf{H})$		$M_4(\mathbf{C})$			
6			$M_4(\mathbf{H})$		$M_4(\mathbf{H})$		$M_8(\mathbf{R})$		$M_8(\mathbf{R})$		$M_4(\mathbf{H})$		$M_4(\mathbf{H})$		$M_8(\mathbf{R})$		
7		$M_8(\mathbf{C})$		$M_4^2(\mathbf{H})$		$M_8(\mathbf{C})$		$M_8^2(\mathbf{R})$		$M_8(\mathbf{C})$		$M_4^2(\mathbf{H})$		$M_8(\mathbf{C})$		$M_8^2(\mathbf{R})$	
8	$M_{16}(\mathbf{R})$		$M_8(\mathbf{H})$		$M_8(\mathbf{H})$		$M_{16}(\mathbf{R})$		$M_{16}(\mathbf{R})$		$M_8(\mathbf{H})$		$M_8(\mathbf{H})$		$M_{16}(\mathbf{R})$		$M_{16}(\mathbf{R})$
ω^2	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+	+

Symmetries

There is a tangled web of symmetries and relationships in the above table.

Going over 4 spots in any row yields an identical algebra.

From these Bott periodicity follows:

If the signature satisfies $p - q \equiv 1 \pmod 4$ then

(The table is symmetric about columns with signature 1, 5, −3, −7, and so forth.) Thus if the signature satisfies $p - q \equiv 1 \pmod 4$,

The Fibonacci Series is found in Pascal's Triangle

Pascal's Triangle, developed by the French Mathematician Blaise Pascal, is formed by starting with an apex of 1. Every number below in the triangle is the sum of the two numbers diagonally above it to the left and the right, with positions outside the triangle counting as zero.

The numbers on diagonals of the triangle add to the Fibonacci series, as shown below.

Pascal's triangle has many unusual properties and a variety of uses:

- Horizontal rows add to powers of 2 (i.e., 1, 2, 4, 8, 16, etc.)
- The horizontal rows represent powers of 11 (1, 11, 121, 1331, etc.)
- Adding any two successive numbers in the diagonal 1-3-6-10-15-21-28… results in a perfect square (1, 4, 9, 16, etc.)

Conclusion

Christina Flauta has found a single Clifford Algebra with a relation to the Fibonacci Numbers. In an email to Ms. Flauta, the author attempted to draw links between her work and the Clifford Algebra triangle, to no avail.

The Wiki description of the Clifford Pyramid indicates that the author was aware of special properties therein. This paper suggests that there may exist many more such special properties, which require further research.

It lies beyond the scope of this paper to perform the calculations necessary to prove that the properties of Pascal’s Triangle pertain to the Clifford Algebra triangle. We leave that as a task for the reader, in the hope that this heuristic suggestion will lead to further insights into Clifford Algebras.

Appendix

where $M(n, \mathbf{C})$ denotes the algebra of $n \times n$ matrices over **C**.

$C\ell_0(\mathbf{R})$		**R**	**Real Numbers**	$C\ell_{0,0}(\mathbf{R})$ is naturally isomorphic to **R** since there are no nonzero vectors.
$C\ell_0(\mathbf{C})$	≅	**C**	**Complex Numbers**	$C\ell_{0,1}(\mathbf{R})$ is a two-dimensional algebra generated by a single vector e_1 that squares to −1, and therefore is isomorphic to **C**, the field of complex numbers.
$C\ell_1(\mathbf{C})$	≅	**C** ⊕ **C**	Bi-complex numbers	
$C\ell_2(\mathbf{C})$	≅	$M(2, \mathbf{C})$	Bi Quarternions	Pauli Matrices
$C\ell_2(\mathbf{R})$	≅		Quarternions	$C\ell_{0,2}(\mathbf{R})$ is a four-dimensional algebra spanned by {1, e_1, e_2, e_1e_2}. The latter three elements square to −1 and all anticommute, and so the algebra is isomorphic to the quaternions **H**
$C\ell_3(\mathbf{R})$	≅		Split Bi - Quarternions	$C\ell_{0,3}(\mathbf{R})$ is an 8-dimensional algebra isomorphic to the direct sum **H** ⊕ **H** called split-biquaternions.
$C\ell_5(\mathbf{C})$	≅			
$C\ell_6(\mathbf{C})$	≅			
$C\ell_7(\mathbf{C})$	≅			

In abstract algebra, the **biquaternions** are the numbers *w* + *x* i + *y* j + *z* k, where *w*, *x*, *y*, and *z* are complex numbers and the elements of {1, i, j, k} multiply as in the quaternion group. As there are three types of complex number, so there are three types of biquaternion:

- (Ordinary) biquaternions when the coefficients are (ordinary) complex numbers
- Split-biquaternions when *w*, *x*, *y*, and *z* are split-complex numbers
- Dual quaternions when *w*, *x*, *y*, and *z* are dual numbers.

This article is about the ordinary biquaternions named by William Rowan Hamilton in 1844 (see Proceedings of Royal Irish Academy 1844 & 1850 page 388). Some of the more prominent proponents of these biquaternions include

Alexander Macfarlane, Arthur W. Conway, Ludwik Silberstein, and Cornelius Lanczos. As developed below, the unit quasi-sphere of the biquaternions provides a presentation of the Lorentz group, which is the foundation of special relativity.

The algebra of biquaternions can be considered as a tensor product **C** ⊗ **H** (taken over the reals) where **C** is the field of complex numbers and **H** is the algebra of (real) quaternions. In other words, the biquaternions are just the complexification of the (real) quaternions. Viewed as a complex algebra, the biquaternions are isomorphic to the algebra of 2×2 complex matrices $M_2(\mathbf{C})$. They can be classified as the Clifford algebra $C\ell_2(\mathbf{C}) = C\ell^0_3(\mathbf{C})$. This is also isomorphic to the Pauli algebra $C\ell_{3,0}(\mathbf{R})$, and the even part of the spacetime algebra $C\ell^0_{1,3}(\mathbf{R})$.

Subalgebras

Considering the biquaternion algebra over the scalar field of real numbers **R**, the set {1, h, i, hi, j, hj, k, hk } forms a basis so the algebra has eight real dimensions. Note the squares of the elements hi, hj, and hk are all plus one, for example,

$$(hi)^2 = h^2 i^2 = (-1)(-1) = +1.$$

Then the subalgebra given by $\{x + y(hi) : x, y \in R\}$is ring isomorphic to the plane of split-complex numbers, which has an algebraic structure built upon the unit hyperbola. The elements *hj* and *hk* also determine such subalgebras. Furthermore, $\{x + yj : x, y \in C\}$is a subalgebra isomorphic to the tessarines.

A third subalgebra called coquaternions is generated by hj and hk. First note that (hj)(hk) = (−1) i, and that the square of this element is −1. These elements generate the dihedral group of the square. The linear subspace with basis {1, i, hj, hk} thus is closed under multiplication, and forms the coquaternion algebra.

In the context of quantum mechanics and spinor algebra, the biquaternions hi, hj, and hk (or their negatives), viewed in the M(2,C) representation, are called Pauli matrices.

Contact

The author may be contacted at jaq 2013 at outlook dot com

Some men see things as they are and say *why?* I dream things that never were and say *why not?*

Let's dedicate ourselves to what the Greeks wrote so many years ago:

to tame the savageness of man and make gentle the life of this world.

Robert Francis Kennedy