Lessons of the Isotropic Schwarzschild Metric's Horizon

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Abstract

Notation based on four-by-four time and space projection matrices enables solution from scratch of the isotropic static metric-tensor ansatz for the empty-space Einstein equation. This isotropic Schwarzschild solution's version of the Riemann tensor which contracts directly to the Ricci tensor is finite at the horizon despite the divergence there of the contravariant metric tensor and of the affine connection, but raising that tensor's third index produces a horizon-divergent version of the Riemann tensor as well; the horizon-finitude of the first-mentioned version of the Riemann tensor is nothing more than a fortuitous by-product of the Ricci tensor's being zero. Furthermore, certain curvature scalars calculated from the isotropic Schwarzschild solution become ill-defined at the horizon, which is a departure from the "no drama" horizon curvature behavior of the "standard" Schwarzschild solution. That departure occurs because the Jacobian of the map of the "standard" Schwarzschild solution into the isotropic solution diverges at the horizon, which invalidates tensor-transformation theorems there. Gravitational horizons in empty-space models in fact merely reflect unphysical overstatement of the effective mass of the gravitational source: the internal gravitation of any extended source which can be inscribed in a sphere reduces the effective mass of that source below the value needed to produce a horizon in the empty space outside the sphere.

Introduction

Experience with the "standard" form of the Schwarzschild metric [1] has suggested that curvature behaves smoothly and unremarkably at the horizon [2], which has given impetus to the widely accepted idea that such a horizon can physically exist. One subtle issue in that regard is that the *vanishing* of the Ricci tensor mandated for empty-space Schwarzschild solutions *strongly predisposes* the most closely related version of the Riemann curvature tensor—namely the one *which contracts directly to the Ricci tensor*, whose first index is contravariant and whose remaining three indices are covariant—to indeed be *finite* and unremarkable at the horizon. One is therefore motivated to *check* whether *other versions* of the Riemann curvature tensor *that have different covariant/contravariant placements of the four indices* are *also* well-behaved at the horizon. Another subtle issue is that the *determinant* of the "standard" form of the Schwarzschild metric tensor turns out to everywhere have the flat-space constant value -1. One of course wonders whether that feature *contributes to smooth curvature behavior at the horizon*, which is motivation to check the curvature behavior at the horizon of *alternate forms* of the Schwarzschild metric.

Here we concentrate on the maximally symmetric *isotropic* form of the Schwarzschild metric [3], whose determinant *definitely isn't constant*, with emphasis on covariant/contravariant-index versions of the Riemann curvature tensor which don't contract *directly* to the vanishing Ricci tensor, and are therefore not so strongly predisposed to be finite and well-behaved at the horizon.

The strong symmetry of the *isotropic* static metric-tensor ansatz for the empty-space Einstein equation (which itself entails nothing more than the vanishing of the Ricci tensor) greatly facilitates development of an extremely helpful notational formalism which makes calculation of the corresponding affine connection and the Riemann and Ricci tensors straightforwardly mechanical, if still very tedious. The vanishing of the Ricci tensor implies three (redundant) coupled nonlinear second-order ordinary differential equations in two dependent variables. Remarkably, pondering these equations reveals a path to reducing them to first order, decoupling the result and transforming that to fully solvable linear form. Thus one can do *everything* in the context of the isotropic static metric-tensor ansatz which is customarily done in the context of the "standard" spherically-symmetric static metric-tensor ansatz through the use of symmetry-related notational tools that we now proceed to lay out.

Isotropic tensor calculus based on time and space projectors

The isotropic static ansatz for the empty-space Einstein equation takes the space-time coordinate system to be such that the metric tensor's components have the spatially-symmetric static form [3],

$$g_{00} = b(r), \quad g_{ii} = -a(r) \text{ for } i = 1, 2, 3, \quad g_{\mu\nu} = 0 \text{ for } \mu \neq \nu,\ \mu, \nu = 0, 1, 2, 3, \tag{1a}$$

where, of course,

$$r \stackrel{\text{def}}{=} ((x^1)^2 + (x^2)^2 + (x^3)^2)^{\frac{1}{2}}, \tag{1b}$$

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and the functions $b(r)$ and $a(r)$ are to be determined from the empty-space Einstein-equation requirement that this metric's resulting Ricci tensor vanishes. It is useful to introduce the components of the 4×4 identity matrix as the flexible second-rank "Kronecker tensor",

$$\delta_{\alpha\beta} = \delta_{\beta\alpha} = \delta^{\alpha\beta} = \delta^{\beta\alpha} = \delta^{\alpha}_{\beta} = \delta^{\beta}_{\alpha} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases} \text{ for } \alpha, \beta = 0, 1, 2, 3, \tag{2a}$$

because the components of the 4×4 time-projection matrix as an equally flexible second-rank tensor are conveniently defined in terms of it, i.e.,

$$Q_{\alpha\beta} = Q_{\beta\alpha} = Q^{\alpha\beta} = Q^{\beta\alpha} = Q^{\alpha}_{\beta} = Q^{\beta}_{\alpha} \stackrel{\text{def}}{=} \delta^0_{\alpha}\delta^0_{\beta} \text{ for } \alpha, \beta = 0, 1, 2, 3. \tag{2b}$$

The components of the complementary 4×4 space-projection matrix as a likewise flexible second-rank tensor are then of course,

$$P_{\alpha\beta} = P_{\beta\alpha} = P^{\alpha\beta} = P^{\beta\alpha} = P^{\alpha}_{\beta} = P^{\beta}_{\alpha} \stackrel{\text{def}}{=} (\delta_{\alpha\beta} - Q_{\alpha\beta}) \text{ for } \alpha, \beta = 0, 1, 2, 3. \tag{2c}$$

As 4×4 matrices the time and space projectors Q and P satisfy the following simple and very useful identities,

$$Q^2 = Q, \ \ P = I - Q, \ \ P^2 = P, \ \ QP = PQ = 0, \ \ \mathrm{Tr}Q = 1, \ \ \mathrm{Tr}P = 3. \tag{2d}$$

Eqs. (2) permit us to reexpress the isotropic static metric-tensor ansatz of Eq. (1a) in the compact form,

$$g_{\alpha\beta} = b(r)Q_{\alpha\beta} - a(r)P_{\alpha\beta}. \tag{3a}$$

This metric tensor's contravariant version, which is its matrix inverse, is given by,

$$g^{\alpha\beta} = (1/b(r))Q^{\alpha\beta} - (1/a(r))P^{\alpha\beta}, \tag{3b}$$

as can be readily verified by applying the subset of the matrix projector identities of Eq. (2d) that don't involve traces.

The space projector P plays a crucial role in *isotropic static tensor partial differential calculus* because,

$$\partial f(r)/\partial x^{\mu} = \begin{cases} 0 & \text{if } \mu = 0 \\ (f'(r)/r)x^{\mu} & \text{if } \mu = 1, 2, 3 \end{cases} = (f'(r)/r)P_{\mu\sigma}x^{\sigma} = (f'(r)/r)(Px)_{\mu} \text{ for } \mu = 0, 1, 2, 3. \tag{4a}$$

Note that because of the flexible second-rank tensor character of $P_{\alpha\beta}$,

$$(Px)_{\mu} = (Px)^{\mu}. \tag{4b}$$

From the index-symmetry and space projection properties of $P_{\alpha\beta}$ it also follows that,

$$(Px)_{\sigma}(Px)^{\sigma} = x^T P^T P x = x^T P^2 x = x^T P x = r^2. \tag{4c}$$

Our immediate current goal is to evaluate $b(r)$ and $a(r)$ by solving the empty-space Einstein equation, i.e., by setting the Ricci tensor to zero. The Ricci tensor is a contracted form of the Riemann curvature tensor, which is conveniently expressed in terms of the affine connection $\Gamma^{\kappa}_{\lambda\mu}$ [4], an entity that in turn involves the contraction of the contravariant metric tensor with partial derivatives of the covariant metric tensor,

$$\Gamma^{\kappa}_{\lambda\mu} \stackrel{\text{def}}{=} \tfrac{1}{2}g^{\kappa\sigma}(\partial_{\mu}g_{\sigma\lambda} + \partial_{\lambda}g_{\sigma\mu} - \partial_{\sigma}g_{\lambda\mu}), \tag{5a}$$

where we have used the conveniently abbreviated notation ∂_{μ} to denote partial differentiation with respect to x^{μ}, namely $\partial/\partial x^{\mu}$.

If we apply the partial derivative result of Eq. (4a) to the isotropic static covariant metric-tensor ansatz of Eq. (3a), we obtain,

$$\partial_{\gamma}g_{\alpha\beta} = (1/r)[b'(r)Q_{\alpha\beta}(Px)_{\gamma} - a'(r)P_{\alpha\beta}(Px)_{\gamma}]. \tag{5b}$$

Insertion of Eq. (5b) and the contravariant isotropic static metric of Eq. (3b) into the Eq. (5a) definition of the affine connection yields,

$$\begin{gathered} \Gamma^{\kappa}_{\lambda\mu} = \tfrac{1}{2}((1/b)Q^{\kappa\sigma} - (1/a)P^{\kappa\sigma})[(b'/r)Q_{\sigma\lambda}(Px)_{\mu} + Q_{\sigma\mu}(Px)_{\lambda} - Q_{\lambda\mu}(Px)_{\sigma}) \\ -(a'/r)(P_{\sigma\lambda}(Px)_{\mu} + P_{\sigma\mu}(Px)_{\lambda} - P_{\lambda\mu}(Px)_{\sigma})] = \\ \tfrac{1}{2}(1/r)[(b'/b)(Q^{\kappa}_{\lambda}(Px)_{\mu} + Q^{\kappa}_{\mu}(Px)_{\lambda}) + (b'/a)((Px)^{\kappa}Q_{\lambda\mu}) + (a'/a)(P^{\kappa}_{\lambda}(Px)_{\mu} + P^{\kappa}_{\mu}(Px)_{\lambda} - (Px)^{\kappa}P_{\lambda\mu})], \end{gathered} \tag{5c}$$

where we have used the matrix projector identities $Q^2 = Q$, $P^2 = P$ and $QP = PQ = 0$ that are given in Eq. (2d).

Now the version of the Riemann curvature tensor which contracts directly to the Ricci tensor is given in terms of the affine connection by [5],

$$R^{\kappa}_{\lambda\mu\nu} \stackrel{\text{def}}{=} (\partial_\nu \Gamma^{\kappa}_{\lambda\mu} - \partial_\mu \Gamma^{\kappa}_{\lambda\nu}) - (\Gamma^{\kappa}_{\sigma\mu}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\kappa}_{\sigma\nu}\Gamma^{\sigma}_{\lambda\mu}). \tag{6}$$

To facilitate the calculation of $\partial_\nu \Gamma^{\kappa}_{\lambda\mu}$ from the result for $\Gamma^{\kappa}_{\lambda\mu}$ which is given in Eq. (5c), we note that combining the fact that $\partial_\nu (Px)_\alpha = P_{\alpha\nu}$ with the basic partial derivative lemma given by Eq. (4a) permits one to deduce that,

$$\partial_\nu[\tfrac{1}{2}(1/r)(f'(r)/g(r))(Px)_\alpha] = \tfrac{1}{2}(1/r)\left(\tfrac{f'(r)}{g(r)}\right)P_{\alpha\nu} + \tfrac{1}{2}\left(\tfrac{f''(r) - f'(r)((1/r)+(g'(r)/g(r)))}{r^2 g(r)}\right)(Px)_\alpha (Px)_\nu. \tag{7a}$$

Application of the lemma of Eq. (7a) to the six-term result for $\Gamma^{\kappa}_{\lambda\mu}$ that is given in Eq. (5c) in order to calculate $\partial_\nu \Gamma^{\kappa}_{\lambda\mu}$ produces twelve terms,

$$\begin{aligned} \partial_\nu \Gamma^{\kappa}_{\lambda\mu} = {} & \tfrac{1}{2}(1/r)[(b'/b)(Q^{\kappa}_{\lambda}P_{\mu\nu} + Q^{\kappa}_{\mu}P_{\lambda\nu}) + (b'/a)(P^{\kappa}_{\nu}Q_{\lambda\mu}) + (a'/a)(P^{\kappa}_{\lambda}P_{\mu\nu} + P^{\kappa}_{\mu}P_{\lambda\nu} - P^{\kappa}_{\nu}P_{\lambda\mu})] + \\ & \tfrac{1}{2}\left(\tfrac{b'' - b'((1/r)+(b'/b))}{r^2 b}\right)(Q^{\kappa}_{\lambda}(Px)_\mu (Px)_\nu + Q^{\kappa}_{\mu}(Px)_\lambda (Px)_\nu) + \tfrac{1}{2}\left(\tfrac{b'' - b'((1/r)+(a'/a))}{r^2 a}\right)((Px)^{\kappa} Q_{\lambda\mu}(Px)_\nu) \\ & + \tfrac{1}{2}\left(\tfrac{a'' - a'((1/r)+(a'/a))}{r^2 a}\right)(P^{\kappa}_{\lambda}(Px)_\mu (Px)_\nu + P^{\kappa}_{\mu}(Px)_\lambda (Px)_\nu - (Px)^{\kappa} P_{\lambda\mu}(Px)_\nu). \end{aligned} \tag{7b}$$

Now twice the part of Eq. (7b) which is antisymmetric in the index pair (μ, ν) can readily be extracted to yield the fourteen-term result,

$$\begin{aligned} (\partial_\nu \Gamma^{\kappa}_{\lambda\mu} - \partial_\mu \Gamma^{\kappa}_{\lambda\nu}) = {} & \tfrac{1}{2}(1/r)[(b'/b)(1/r)(Q^{\kappa}_{\mu}P_{\lambda\nu} - Q^{\kappa}_{\nu}P_{\lambda\mu}) \\ & + (b'/a)(P^{\kappa}_{\nu}Q_{\lambda\mu} - P^{\kappa}_{\mu}Q_{\lambda\nu}) + 2(a'/a)(P^{\kappa}_{\mu}P_{\lambda\nu} - P^{\kappa}_{\nu}P_{\lambda\mu})] + \\ & \tfrac{1}{2}\left(\tfrac{b'' - b'((1/r)+(b'/b))}{r^2 b}\right)(Q^{\kappa}_{\mu}(Px)_\lambda (Px)_\nu - Q^{\kappa}_{\nu}(Px)_\lambda (Px)_\mu) \\ & + \tfrac{1}{2}\left(\tfrac{b'' - b'((1/r)+(a'/a))}{r^2 a}\right)((Px)^{\kappa} Q_{\lambda\mu}(Px)_\nu - (Px)^{\kappa} Q_{\lambda\nu}(Px)_\mu) \\ & + \tfrac{1}{2}\left(\tfrac{a'' - a'((1/r)+(a'/a))}{r^2 a}\right)(P^{\kappa}_{\mu}(Px)_\lambda (Px)_\nu - P^{\kappa}_{\nu}(Px)_\lambda (Px)_\mu - (Px)^{\kappa} P_{\lambda\mu}(Px)_\nu + (Px)^{\kappa} P_{\lambda\nu}(Px)_\mu). \end{aligned} \tag{7c}$$

Referring to the Eq. (6) expression for the $R^{\kappa}_{\lambda\mu\nu}$ version of the Riemann curvature tensor, we can see that, having obtained the result of Eq. (7c), our next task is to calculate,

$$\begin{aligned} \Gamma^{\kappa}_{\sigma\mu}\Gamma^{\sigma}_{\lambda\nu} = {} & \tfrac{1}{4}(1/r^2)\times \\ & [(b'/b)(Q^{\kappa}_{\sigma}(Px)_\mu + Q^{\kappa}_{\mu}(Px)_\sigma) + (b'/a)((Px)^{\kappa} Q_{\sigma\mu}) + (a'/a)(P^{\kappa}_{\sigma}(Px)_\mu + P^{\kappa}_{\mu}(Px)_\sigma - (Px)^{\kappa} P_{\sigma\mu})]\times \\ & [(b'/b)(Q^{\sigma}_{\lambda}(Px)_\nu + Q^{\sigma}_{\nu}(Px)_\lambda) + (b'/a)((Px)^{\sigma} Q_{\lambda\nu}) + (a'/a)(P^{\sigma}_{\lambda}(Px)_\nu + P^{\sigma}_{\nu}(Px)_\lambda - (Px)^{\sigma} P_{\lambda\nu})], \end{aligned} \tag{8a}$$

which on its face could generate 36 terms. About half of these terms vanish, however, because of the projector identity $QP = PQ = 0$ that is given in Eq. (2d). We also make use of the projector identities $P^2 = P$ and $Q^2 = Q$ of Eq. (2d) and the identity $(Px)_\sigma (Px)^\sigma = r^2$ of Eq. (4c) to obtain,

$$\begin{aligned} \Gamma^{\kappa}_{\sigma\mu}\Gamma^{\sigma}_{\lambda\nu} = {} & \tfrac{1}{4}(b'/b)(b'/a)(Q^{\kappa}_{\mu}Q_{\lambda\nu}) - \tfrac{1}{4}(b'/b)(a'/a)(Q^{\kappa}_{\mu}P_{\lambda\nu}) + \tfrac{1}{4}(b'/a)(a'/a)(P^{\kappa}_{\mu}Q_{\lambda\nu}) - \tfrac{1}{4}(a'/a)(a'/a)(P^{\kappa}_{\mu}P_{\lambda\nu}) \\ & + (1/r^2)[\tfrac{1}{4}(b'/b)^2(Q^{\kappa}_{\lambda}(Px)_\mu (Px)_\nu + Q^{\kappa}_{\nu}(Px)_\lambda (Px)_\mu) + \tfrac{1}{2}(b'/b)(a'/a)(Q^{\kappa}_{\mu}(Px)_\lambda (Px)_\nu) \\ & + \tfrac{1}{4}(b'/a)(b'/b)((Px)^{\kappa} Q_{\lambda\mu}(Px)_\nu + (Px)^{\kappa} Q_{\mu\nu}(Px)_\lambda) \\ & + \tfrac{1}{4}(a'/a)^2(P^{\kappa}_{\lambda}(Px)_\mu (Px)_\nu - (Px)^{\kappa}(Px)_\lambda P_{\mu\nu} \\ & + P^{\kappa}_{\mu}(Px)_\lambda (Px)_\nu + P^{\kappa}_{\nu}(Px)_\lambda (Px)_\mu + P^{\kappa}_{\mu}(Px)_\lambda (Px)_\nu - (Px)^{\kappa} P_{\lambda\mu}(Px)_\nu)]. \end{aligned} \tag{8b}$$

The negative of twice the part of Eq. (8b) which is antisymmetric in the index pair (μ, ν) yields,

$$\begin{aligned} -(\Gamma^{\kappa}_{\sigma\mu}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\kappa}_{\sigma\nu}\Gamma^{\sigma}_{\lambda\mu}) = {} & \tfrac{1}{4}\{(b'/b)(a'/a)(Q^{\kappa}_{\mu}P_{\lambda\nu} - Q^{\kappa}_{\nu}P_{\lambda\mu}) \\ & + (b'/a)(a'/a)(P^{\kappa}_{\nu}Q_{\lambda\mu} - P^{\kappa}_{\mu}Q_{\lambda\nu}) + (a'/a)^2(P^{\kappa}_{\mu}P_{\lambda\nu} - P^{\kappa}_{\nu}P_{\lambda\mu}) \\ & - (1/r^2)[(b'/b)(2(a'/a) - (b'/b))(Q^{\kappa}_{\mu}(Px)_\lambda (Px)_\nu - Q^{\kappa}_{\nu}(Px)_\lambda (Px)_\mu) \\ & + (b'/a)(b'/b))((Px)^{\kappa} Q_{\lambda\mu}(Px)_\nu - (Px)^{\kappa} Q_{\lambda\nu}(Px)_\mu) \\ & + (a'/a)^2(P^{\kappa}_{\mu}(Px)_\lambda (Px)_\nu - P^{\kappa}_{\nu}(Px)_\lambda (Px)_\mu - (Px)^{\kappa} P_{\lambda\mu}(Px)_\nu + (Px)^{\kappa} P_{\lambda\nu}(Px)_\mu)]\}, \end{aligned} \tag{8c}$$

where we have dropped the contribution which has the factor,

$$(Q^{\kappa}_{\mu}Q_{\lambda\nu} - Q^{\kappa}_{\nu}Q_{\lambda\mu}) = (\delta^{\kappa}_{0}\delta^{0}_{\mu}\delta^{0}_{\lambda}\delta^{0}_{\nu} - \delta^{\kappa}_{0}\delta^{0}_{\nu}\delta^{0}_{\lambda}\delta^{0}_{\mu}) = 0.$$

To obtain the version of the Riemann curvature tensor given in Eq. (6) we add Eq. (8c) to Eq. (7c),

$$\begin{aligned}R^{\kappa}_{\lambda\mu\nu} = \tfrac{1}{2}(b'/b)((1/r)+\tfrac{1}{2}(a'/a))(Q^{\kappa}_{\mu}P_{\lambda\nu}-Q^{\kappa}_{\nu}P_{\lambda\mu})\\ +\tfrac{1}{2}(b'/a)((1/r)+\tfrac{1}{2}(a'/a))(P^{\kappa}_{\nu}Q_{\lambda\mu}-P^{\kappa}_{\mu}Q_{\lambda\nu})\\ +(a'/a)((1/r)+\tfrac{1}{4}(a'/a))(P^{\kappa}_{\mu}P_{\lambda\nu}-P^{\kappa}_{\nu}P_{\lambda\mu})\\ +\tfrac{1}{2}\left(\tfrac{b''-b'((1/r)+(a'/a)+\frac{1}{2}(b'/b))}{r^2 b}\right)(Q^{\kappa}_{\mu}(Px)_{\lambda}(Px)_{\nu}-Q^{\kappa}_{\nu}(Px)_{\lambda}(Px)_{\mu})\\ +\tfrac{1}{2}\left(\tfrac{b''-b'((1/r)+(a'/a)+\frac{1}{2}(b'/b))}{r^2 a}\right)((Px)^{\kappa}Q_{\lambda\mu}(Px)_{\nu}-(Px)^{\kappa}Q_{\lambda\nu}(Px)_{\mu})\\ +\tfrac{1}{2}\left(\tfrac{a''-a'((1/r)+\frac{3}{2}(a'/a))}{r^2 a}\right)(P^{\kappa}_{\mu}(Px)_{\lambda}(Px)_{\nu}-P^{\kappa}_{\nu}(Px)_{\lambda}(Px)_{\mu}-(Px)^{\kappa}P_{\lambda\mu}(Px)_{\nu}+(Px)^{\kappa}P_{\lambda\nu}(Px)_{\mu}).\end{aligned} \tag{9a}$$

The Ricci tensor is obtained from the above version of the Riemann curvature tensor by the direct contraction [6],

$$\begin{aligned}R_{\lambda\nu} \stackrel{\text{def}}{=} R^{\sigma}_{\lambda\sigma\nu} = \tfrac{1}{2}(b'/b)((1/r)+\tfrac{1}{2}(a'/a))(Q^{\sigma}_{\sigma}P_{\lambda\nu}-Q^{\sigma}_{\nu}P_{\lambda\sigma})+\tfrac{1}{2}(b'/a)((1/r)+\tfrac{1}{2}(a'/a))(P^{\sigma}_{\nu}Q_{\lambda\sigma}-P^{\sigma}_{\sigma}Q_{\lambda\nu})\\ +(a'/a)((1/r)+\tfrac{1}{4}(a'/a))(P^{\sigma}_{\sigma}P_{\lambda\nu}-P^{\sigma}_{\nu}P_{\lambda\sigma})+\\ \tfrac{1}{2}\left(\tfrac{b''-b'((1/r)+(a'/a)+\frac{1}{2}(b'/b))}{r^2 b}\right)(Q^{\sigma}_{\sigma}(Px)_{\lambda}(Px)_{\nu}-Q^{\sigma}_{\nu}(Px)_{\lambda}(Px)_{\sigma})\\ +\tfrac{1}{2}\left(\tfrac{b''-b'((1/r)+(a'/a)+\frac{1}{2}(b'/b))}{r^2 a}\right)((Px)^{\sigma}Q_{\lambda\sigma}(Px)_{\nu}-(Px)^{\sigma}Q_{\lambda\nu}(Px)_{\sigma})\\ +\tfrac{1}{2}\left(\tfrac{a''-a'((1/r)+\frac{3}{2}(a'/a))}{r^2 a}\right)(P^{\sigma}_{\sigma}(Px)_{\lambda}(Px)_{\nu}-P^{\sigma}_{\nu}(Px)_{\lambda}(Px)_{\sigma}-(Px)^{\sigma}P_{\lambda\sigma}(Px)_{\nu}+(Px)^{\sigma}P_{\lambda\nu}(Px)_{\sigma}).\end{aligned} \tag{9b}$$

Eq. (9b) is readily evaluated by using the identities given in Eqs. (2d) and (4c),

$$\begin{aligned}R_{\lambda\nu} = \tfrac{1}{2}(b'/b)((1/r)+\tfrac{1}{2}(a'/a))P_{\lambda\nu}+\tfrac{1}{2}(b'/a)((1/r)+\tfrac{1}{2}(a'/a))(-3Q_{\lambda\nu})+(a'/a)((1/r)+\tfrac{1}{4}(a'/a))(2P_{\lambda\nu})+\\ \tfrac{1}{2}\left(\tfrac{b''-b'((1/r)+(a'/a)+\frac{1}{2}(b'/b))}{r^2 b}\right)(Px)_{\lambda}(Px)_{\nu}+\tfrac{1}{2}\left(\tfrac{b''-b'((1/r)+(a'/a)+\frac{1}{2}(b'/b))}{a}\right)(-Q_{\lambda\nu})\\ +\tfrac{1}{2}\left(\tfrac{a''-a'((1/r)+\frac{3}{2}(a'/a))}{r^2 a}\right)((Px)_{\lambda}(Px)_{\nu}+r^2P_{\lambda\nu}) =\\ \tfrac{1}{2}\left(\left(\tfrac{b''-b'((1/r)+(a'/a)+\frac{1}{2}(b'/b))}{b}\right)+\left(\tfrac{a''-a'((1/r)+\frac{3}{2}(a'/a))}{a}\right)\right)(((Px)_{\lambda}(Px)_{\nu})/r^2)\\ +\tfrac{1}{2}\left(-\left(\tfrac{b''-b'((1/r)+(a'/a)+\frac{1}{2}(b'/b))}{a}\right)-3(b'/a)((1/r)+\tfrac{1}{2}(a'/a))\right)Q_{\lambda\nu}\\ +\tfrac{1}{2}\left(\left(\tfrac{a''-a'((1/r)+\frac{3}{2}(a'/a))}{a}\right)+(b'/b)((1/r)+\tfrac{1}{2}(a'/a))+(a'/a)((4/r)+(a'/a))\right)P_{\lambda\nu}.\end{aligned} \tag{9c}$$

The empty-space Einstein equation is equivalent to the vanishing of the Ricci tensor $R_{\lambda\nu}$. That can *only* occur, however, when the *coefficients* of the three linearly independent two-index entities $Q_{\lambda\nu}$, $P_{\lambda\nu}$, and $(((Px)_{\lambda}(Px)_{\nu})/r^2)$ *all vanish*, which produces *three* (hopefully redundant) coupled nonlinear second-order ordinary differential equations for the *two* isotropic static ansatz functions $b(r)$ and $a(r)$.

These three equations can be linearly combined so as to eliminate the second derivatives of $b(r)$ and $a(r)$, and the resulting equation can then be algebraically solved for (b'/b), the logarithmic derivative of $b(r)$, in terms of (a'/a) and elementary functions of r. This decoupling of $b(r)$ from $a(r)$ then results in a nonlinear first-order differential equation for (a'/a) which can be transformed into an inhomogeneous *linear* first-order differential equation by simple inversion of the dependent variable (a'/a).

We eliminate the second derivatives of $b(r)$ and $a(r)$ by multiplying the coefficient of $Q_{\lambda\mu}$ through by (a/b), adding the resulting product to the coefficient of $(((Px)_{\lambda}(Px)_{\nu})/r^2)$, and then subtracting the coefficient of $P_{\lambda\nu}$ from the resulting sum. The upshot is the equation,

$$\tfrac{1}{2}(-3(b'/b)((1/r)+\tfrac{1}{2}(a'/a))-(b'/b)((1/r)+\tfrac{1}{2}(a'/a))-(a'/a)((4/r)+(a'/a)))=0, \tag{9d}$$

which is readily algebraically solved for (b'/b) in terms of (a'/a) and $(1/r)$,

$$(b'/b) = -(a'/a)((1/r)+\tfrac{1}{4}(a'/a))/((1/r)+\tfrac{1}{2}(a'/a)). \tag{9e}$$

When this result is inserted into the coefficient of $P_{\lambda\nu}$, terms collected and this now modified coefficient set to zero, the resulting equation is,

$$(a''/a)-\tfrac{3}{4}(a'/a)^2+(2/r)(a'/a)=0. \tag{9f}$$

Since $(a'/a)' = (a''/a)-(a'/a)^2$, Eq. (9f) is conveniently reexpressed as,

$$(a'/a)'+\tfrac{1}{4}(a'/a)^2+(2/r)(a'/a)=0, \tag{9g}$$

which is a first-order nonlinear differential equation for (a'/a). The nonlinearity of Eq. (9g) can be *eliminated* by changing its dependent variable to $\varrho(r)$, where $(a'/a) = (1/\varrho)$. Since $(a'/a)' = -(\varrho'/\varrho^2)$, the result of this *inversion* of the dependent variable comes out to be the inhomogeneous linear first-order equation,

$$\varrho' - (2/r)\varrho = \tfrac{1}{4}, \tag{9h}$$

which, after it is multiplied through by the intgrating factor $(1/r^2)$, can be written,

$$(\varrho/r^2)' = \tfrac{1}{4}(1/r^2), \tag{9i}$$

and immediately integrated,

$$(\varrho/r^2) = -\tfrac{1}{4}((1/r) + (1/r_0)) = \left(\tfrac{r+r_0}{-4r_0 r}\right), \tag{9j}$$

where r_0 is a constant of integration with the dimension of length. Since $(\ln a)' = (a'/a) = (1/\varrho)$, we can see from Eq. (9j) that,

$$(\ln a)' = \left(\tfrac{-4r_0}{r(r+r_0)}\right) = 4\left(\tfrac{1}{r+r_0} - \tfrac{1}{r}\right), \tag{9k}$$

which can be immediately integrated,

$$\ln a = 4\ln\left(\tfrac{r+r_0}{r}\right) + C = 4\ln(1 + (r_0/r)) + C. \tag{9l}$$

Eq. (9l) implies that,

$$a(r) = \exp(C)(1 + (r_0/r))^4, \tag{9m}$$

where C is a dimensionless constant of integration. Since our isotropic static metric tensor ansatz $g_{\mu\nu}(r)$ of Eqs. (1a) and (3a) must approach the flat-space Minkowski metric $\eta_{\mu\nu}$ as $r \to \infty$, we must have *both* that $\lim_{r\to\infty} b(r) = 1$ *and* that $\lim_{r\to\infty} a(r) = 1$. The latter implies that $C = 0$, i.e., that,

$$a(r) = (1 + (r_0/r))^4. \tag{9n}$$

Eq. (9n) directly yields the result,

$$(a'/a) = -4(1/r)\left(\tfrac{(r_0/r)}{1+(r_0/r)}\right), \tag{9o}$$

which is, of course, consistent with Eq. (9k), and implies that,

$$((1/r) + \tfrac{1}{4}(a'/a)) = (1/r)\left(\tfrac{1}{1+(r_0/r)}\right) \text{ and } ((1/r) + \tfrac{1}{2}(a'/a)) = (1/r)\left(\tfrac{1-(r_0/r)}{1+(r_0/r)}\right). \tag{9p}$$

Putting together Eqs. (9e), (9o) and (9p), we obtain,

$$(b'/b) = 4(1/r)\left(\tfrac{(r_0/r)}{(1-(r_0/r))(1+(r_0/r))}\right), \tag{9q}$$

which implies that,

$$(\ln b)' = \tfrac{4r_0}{(r-r_0)(r+r_0)} = 2\left(\tfrac{1}{(r-r_0)} - \tfrac{1}{(r+r_0)}\right). \tag{9r}$$

Therefore,

$$(\ln b) = 2\ln\left(\tfrac{r-r_0}{r+r_0}\right) + C' = 2\ln\left(\tfrac{1-(r_0/r)}{1+(r_0/r)}\right) + C', \tag{9s}$$

which yields,

$$b(r) = \exp(C')\left(\tfrac{1-(r_0/r)}{1+(r_0/r)}\right)^2, \tag{9t}$$

where C' is a dimensionless constant of integration. Since $\lim_{r\to\infty} b(r) = 1$, as was pointed out beneath Eq. (9m), $C' = 0$, so,

$$b(r) = \left(\tfrac{1-(r_0/r)}{1+(r_0/r)}\right)^2. \tag{9u}$$

In addition, we note from Eq. (1a) that $b(r) = g_{00}(r)$, and we further note that as $r \to \infty$, the difference $(g_{00}(r) - \eta_{00})$, where $\eta_{\mu\nu}$ is the flat-space Minkowski metric, asymptotically approaches *twice* the Newtonian gravitational potential divided by c^2 [7]. Since from Eq. (9u), $(b(r) - 1)$ is asymptotic to $-4(r_0/r)$ as $r \to \infty$, and the Newtonian gravitational potential for a source of effective mass M is $-GM/r$, the value of r_0 is,

$$r_0 = ((GM)/(2c^2)). \tag{9v}$$

With this result for r_0, the functions $b(r)$ and $a(r)$ of our isotropic static metric ansatz of Eqs. (1a) and (3a) are now fully determined by the vanishing Ricci tensor equation solutions that are given by Eqs. (9u) and (9n) respectively. We still need to check that these results for $b(r)$ and $a(r)$ do indeed cause the coefficients of $Q_{\lambda\nu}$, $P_{\lambda\nu}$ and $(((Px)_\lambda(Px)_\nu)/r^2)$ in the Eq. (9c) expression for the Ricci tensor to *all vanish*. To that end, and to also be able to write down the Einstein-equation solved representation of the Eq. (9a) version of the Riemann curvature tensor, we take note of the Eq. (9q) result for (b'/b), the Eq. (9o) result for (a'/a), the Eq. (9p) results for $((1/r) + \frac{1}{4}(a'/a))$ and $((1/r) + \frac{1}{2}(a'/a))$, and in addition the following *further list of important equalities* which follow from the Eq. (9u) and Eq. (9n) solution results for $b(r)$ and $a(r)$,

$$\begin{gathered}
\tfrac{1}{2}(b'/b)((1/r) + \tfrac{1}{2}(a'/a)) = 2(1/r)^2(r_0/r)\left(\tfrac{1}{(1+(r_0/r))^2}\right), \\
\tfrac{1}{2}(b'/a)((1/r) + \tfrac{1}{2}(a'/a)) = 2(1/r)^2(r_0/r)\tfrac{(1-(r_0/r))^2}{(1+(r_0/r))^8}, \\
(a'/a)((1/r) + \tfrac{1}{4}(a'/a)) = -4(1/r)^2(r_0/r)\left(\tfrac{1}{(1+(r_0/r))^2}\right), \\
(b''/b) = -4(1/r)^2(r_0/r)\tfrac{(2-4(r_0/r))}{(1-(r_0/r))^2(1+(r_0/r))^2}, \\
(a''/a) = 4(1/r)^2(r_0/r)\tfrac{(2+5(r_0/r))}{(1+(r_0/r))^2}, \\
\tfrac{1}{2}\left(\tfrac{b''-b'((1/r)+(a'/a)+\frac{1}{2}(b'/b))}{b}\right) = -6(1/r)^2(r_0/r)\left(\tfrac{1}{(1+(r_0/r))^2}\right), \\
\tfrac{1}{2}\left(\tfrac{b''-b'((1/r)+(a'/a)+\frac{1}{2}(b'/b))}{a}\right) = -6(1/r)^2(r_0/r)\tfrac{(1-(r_0/r))^2}{(1+(r_0/r))^8}, \\
\tfrac{1}{2}\left(\tfrac{a''-a'((1/r)+\frac{3}{2}(a'/a))}{a}\right) = 6(1/r)^2(r_0/r)\left(\tfrac{1}{(1+(r_0/r))^2}\right).
\end{gathered} \tag{9w}$$

The equalities given in Eq. (9w), together with those given in Eqs. (9o), (9p) and (9q), do indeed show that the three coefficients of respectively $Q_{\lambda\nu}$, $P_{\lambda\nu}$ and $(((Px)_\lambda(Px)_\nu)/r^2)$ in the Eq. (9c) expression for the Ricci tensor all vanish when $b(r)$ is given by Eq. (9u) and $a(r)$ is given by Eq. (9n). This list of equalities *also* enables us to write down the Eq. (9a) version of the Riemann curvature tensor when those same Eq. (9u) and (9n) empty-space Einstein-equation solution results for respectively $b(r)$ and $a(r)$ are taken into account,

$$\begin{gathered}
R^\kappa_{\lambda\mu\nu} = 2(1/r)^2(r_0/r)\left(\tfrac{1}{(1+(r_0/r))^2}\right)\Big[(Q^\kappa_\mu P_{\lambda\nu} - Q^\kappa_\nu P_{\lambda\mu}) + \tfrac{(1-(r_0/r))^2}{(1+(r_0/r))^6}(P^\kappa_\nu Q_{\lambda\mu} - P^\kappa_\mu Q_{\lambda\nu}) \\
-2(P^\kappa_\mu P_{\lambda\nu} - P^\kappa_\nu P_{\lambda\mu}) - 3(1/r^2)(Q^\kappa_\mu (Px)_\lambda (Px)_\nu - Q^\kappa_\nu (Px)_\lambda (Px)_\mu) \\
-3\tfrac{(1-(r_0/r))^2}{(1+(r_0/r))^6}(1/r^2)((Px)^\kappa Q_{\lambda\mu}(Px)_\nu - (Px)^\kappa Q_{\lambda\nu}(Px)_\mu) \\
+3(1/r^2)(P^\kappa_\mu (Px)_\lambda (Px)_\nu - P^\kappa_\nu (Px)_\lambda (Px)_\mu - (Px)^\kappa P_{\lambda\mu}(Px)_\nu + (Px)^\kappa P_{\lambda\nu}(Px)_\mu)\Big].
\end{gathered} \tag{10a}$$

Eq. (10a) becomes a bit more compact if one inserts the Eq. (9u) result that $((1-(r_0/r))/(1+(r_0/r)))^2 = b(r)$ and the Eq. (9n) result that $(1 + (r_0/r))^4 = a(r)$,

$$\begin{gathered}
R^\kappa_{\lambda\mu\nu} = 2(1/r)^2(r_0/r)(1/a(r))^{\frac{1}{2}}\big[(Q^\kappa_\mu P_{\lambda\nu} - Q^\kappa_\nu P_{\lambda\mu}) + (b(r)/a(r))(P^\kappa_\nu Q_{\lambda\mu} - P^\kappa_\mu Q_{\lambda\nu}) \\
-2(P^\kappa_\mu P_{\lambda\nu} - P^\kappa_\nu P_{\lambda\mu}) - 3(1/r^2)(Q^\kappa_\mu (Px)_\lambda (Px)_\nu - Q^\kappa_\nu (Px)_\lambda (Px)_\mu) \\
-3(b(r)/a(r))(1/r^2)((Px)^\kappa Q_{\lambda\mu}(Px)_\nu - (Px)^\kappa Q_{\lambda\nu}(Px)_\mu) \\
+3(1/r^2)(P^\kappa_\mu (Px)_\lambda (Px)_\nu - P^\kappa_\nu (Px)_\lambda (Px)_\mu - (Px)^\kappa P_{\lambda\mu}(Px)_\nu + (Px)^\kappa P_{\lambda\nu}(Px)_\mu)\big].
\end{gathered} \tag{10b}$$

The Riemann curvature tensor version which has *all four* of its indices *covariant* could be expected to have have greater symmetry,

$$\begin{gathered}
R_{\kappa\lambda\mu\nu} = g_{\kappa\sigma}R^\sigma_{\lambda\mu\nu} = (b(r)Q_{\kappa\sigma} - a(r)P_{\kappa\sigma})\times \\
\big(2(1/r)^2(r_0/r)(1/a(r))^{\frac{1}{2}}\big[(Q^\sigma_\mu P_{\lambda\nu} - Q^\sigma_\nu P_{\lambda\mu}) + (b(r)/a(r))(P^\sigma_\nu Q_{\lambda\mu} - P^\sigma_\mu Q_{\lambda\nu}) \\
-2(P^\sigma_\mu P_{\lambda\nu} - P^\sigma_\nu P_{\lambda\mu}) - 3(1/r^2)(Q^\sigma_\mu (Px)_\lambda (Px)_\nu - Q^\sigma_\nu (Px)_\lambda (Px)_\mu) \\
-3(b(r)/a(r))(1/r^2)((Px)^\sigma Q_{\lambda\mu}(Px)_\nu - (Px)^\sigma Q_{\lambda\nu}(Px)_\mu) \\
+3(1/r^2)(P^\sigma_\mu (Px)_\lambda (Px)_\nu - P^\sigma_\nu (Px)_\lambda (Px)_\mu - (Px)^\sigma P_{\lambda\mu}(Px)_\nu + (Px)^\sigma P_{\lambda\nu}(Px)_\mu)\big]\big).
\end{gathered} \tag{10c}$$

Carrying out the Eq. (10c) calculation is a slightly tedious but straightforward application of the time and space matrix projector identities $Q^2 = Q$, $P^2 = P$ and $QP = PQ = 0$ of Eq. (2d),

$$\begin{gathered}
R_{\kappa\lambda\mu\nu} = 2(1/r)^2(r_0/r)(1/a(r))^{\frac{1}{2}}\big[2a(r)(P_{\kappa\mu}P_{\lambda\nu} - P_{\kappa\nu}P_{\lambda\mu}) \\
+b(r)(Q_{\kappa\mu}P_{\lambda\nu} - Q_{\kappa\nu}P_{\lambda\mu} + Q_{\lambda\nu}P_{\kappa\mu} - Q_{\lambda\mu}P_{\kappa\nu}) \\
-3a(r)(1/r^2)(P_{\kappa\mu}(Px)_\lambda(Px)_\nu - P_{\kappa\nu}(Px)_\lambda(Px)_\mu + P_{\lambda\nu}(Px)_\kappa(Px)_\mu - P_{\lambda\mu}(Px)_\kappa(Px)_\nu) \\
-3b(r)(1/r^2)(Q_{\kappa\mu}(Px)_\lambda(Px)_\nu - Q_{\kappa\nu}(Px)_\lambda(Px)_\mu + Q_{\lambda\nu}(Px)_\kappa(Px)_\mu - Q_{\lambda\mu}(Px)_\kappa(Px)_\nu)\big].
\end{gathered} \tag{10d}$$

It is readily seen from Eq. (10d) that the *scalar density* $(\varepsilon^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu})$ vanishes because *each individual term* of the $R_{\kappa\lambda\mu\nu}$ of Eq. (10d) is *symmetric* in each of two *particular* pairs of its four indices. In fact the vanishing of this scalar density follows from a cyclic symmetry that holds for the completely covariant version of *any* Riemann curvature tensor [8].

From the foregoing Eqs. (10) we see that the Riemann curature tensor versions $R^{\kappa}_{\lambda\mu\nu}$ and $R_{\kappa\lambda\mu\nu}$ are *finite* at $r = r_0$, i.e. *at the horizon*, even *notwithstanding* that the Eq. (3b) *contravariant metric tensor* with the $b(r)$ of Eq. (9u) and the $a(r)$ of Eq. (9n), namely,

$$g^{\alpha\beta} = \left(\tfrac{1+(r_0/r)}{1-(r_0/r)}\right)^2 Q^{\alpha\beta} - \left(\tfrac{1}{1+(r_0/r)}\right)^4 P^{\alpha\beta}, \tag{11a}$$

is clearly *divergent* at the $r = r_0$ horizon, and horizon divergence *is also clearly the case* for the Eq. (5c) *affine connection* with the $b(r)$ of Eq. (9u) and the $a(r)$ of Eq. (9n), namely for,

$$\begin{aligned}\Gamma^{\kappa}_{\lambda\mu} = 2(1/r)^2(r_0/r)\left(\tfrac{1}{1+(r_0/r)}\right)\Big[\left(\tfrac{1}{1-(r_0/r)}\right)(Q^{\kappa}_{\lambda}(Px)_\mu + Q^{\kappa}_{\mu}(Px)_\lambda)\\ + \left(\tfrac{1-(r_0/r)}{(1+(r_0/r))^6}\right)((Px)^\kappa Q_{\lambda\mu}) - (P^{\kappa}_{\lambda}(Px)_\mu + P^{\kappa}_{\mu}(Px)_\lambda - (Px)^\kappa P_{\lambda\mu})\Big].\end{aligned} \tag{11b}$$

In this regard it is *essential* to note, however, that the Riemann curvature tensor version $R^{\kappa}_{\lambda\mu\nu}$ contracts *directly* to the *vanishing* Ricci tensor, i.e., $R^{\sigma}_{\lambda\sigma\nu} = 0$, which certainly *strongly biases* $R^{\kappa}_{\lambda\mu\nu}$ *against being divergent at any point*. Of course the Riemann curvature tensor version $R_{\kappa\lambda\mu\nu}$ is merely $R^{\kappa}_{\lambda\mu\nu}$ contracted with the Eq. (3a) *covariant metric tensor* that has the $b(r)$ of Eq. (9u) and the $a(r)$ of Eq. (9n), namely,

$$g_{\alpha\beta} = \left(\tfrac{1-(r_0/r)}{1+(r_0/r)}\right)^2 Q_{\alpha\beta} - (1 + (r_0/r))^4 P_{\alpha\beta},$$

which is clearly *finite* at the $r = r_0$ horizon.

As a matter of fact, we need but *raise a single suitable additional index of* $R^{\kappa}_{\lambda\mu\nu}$ to produce a version of the Riemann curvature tensor *that is indeed divergent at the horizon* $r = r_0$. Such a suitable additional index turns out to be either one of the two indices *which are not contiguous to this tensor's already-raised first index*. Consider, for example, the Riemann curvature tensor version ${R^{\kappa}}_{\lambda}{}^{\mu}{}_{\nu}$,

$$\begin{aligned}{R^{\kappa}}_{\lambda}{}^{\mu}{}_{\nu} = g^{\mu\sigma}R^{\kappa}_{\lambda\sigma\nu} = \left(\left(\tfrac{1+(r_0/r)}{1-(r_0/r)}\right)^2 Q^{\mu\sigma} - \left(\tfrac{1}{1+(r_0/r)}\right)^4 P^{\mu\sigma}\right)\times\\ \Big\{2(1/r)^2(r_0/r)\left(\tfrac{1}{1+(r_0/r)}\right)^2\Big[(Q^{\kappa}_{\sigma}P_{\lambda\nu} - Q^{\kappa}_{\nu}P_{\lambda\sigma})\\ + \left(\tfrac{(1-(r_0/r))^2}{(1+(r_0/r))^6}\right)(P^{\kappa}_{\nu}Q_{\lambda\sigma} - P^{\kappa}_{\sigma}Q_{\lambda\nu}) - 2(P^{\kappa}_{\sigma}P_{\lambda\nu} - P^{\kappa}_{\nu}P_{\lambda\sigma})\\ -3(1/r^2)(Q^{\kappa}_{\sigma}(Px)_\lambda(Px)_\nu - Q^{\kappa}_{\nu}(Px)_\lambda(Px)_\sigma) - 3\left(\tfrac{(1-(r_0/r))^2}{(1+(r_0/r))^6}\right)(1/r^2)((Px)^\kappa Q_{\lambda\sigma}(Px)_\nu - (Px)^\kappa Q_{\lambda\nu}(Px)_\sigma)\\ +3(1/r^2)(P^{\kappa}_{\sigma}(Px)_\lambda(Px)_\nu - P^{\kappa}_{\nu}(Px)_\lambda(Px)_\sigma - (Px)^\kappa P_{\lambda\sigma}(Px)_\nu + (Px)^\kappa P_{\lambda\nu}(Px)_\sigma)\Big]\Big\}.\end{aligned} \tag{12a}$$

The terms of ${R^{\kappa}}_{\lambda}{}^{\mu}{}_{\nu}$ in Eq. (12a) which *diverge at the horizon* $r = r_0$ are readily *picked out*. (We use the by now familiar matrix projector identities $Q^2 = Q$, $P^2 = P$ and $QP = PQ = 0$.) There are *two* such horizon-divergent terms, which are given by the expression,

$$\begin{aligned}\text{Horizon-divergent terms of } {R^{\kappa}}_{\lambda}{}^{\mu}{}_{\nu} =\\ 2(1/r)^2(r_0/r)\left(\tfrac{1}{1-(r_0/r)}\right)^2\left[Q^{\kappa\mu}P_{\lambda\nu} - 3(1/r^2)Q^{\kappa\mu}(Px)_\lambda(Px)_\nu\right].\end{aligned} \tag{12b}$$

Although it is completely feasible to do so, there is no apparent *reason* to take the trouble to explicitly display *the remaining dozen terms* of the ${R^{\kappa}}_{\lambda}{}^{\mu}{}_{\nu}$ version of the Riemann curvature tensor.

The *important* lesson of this exercise is that one *naturally* tends to have contact *only* with versions of Schwarzschild-solution Riemann curvature tensors *that don't diverge at the horizon* because the *vanishing* of the Ricci tensor exerts *a strong mathematical bias* in that regard. But *raising enough of the indices* of the Riemann curvature tensor with the horizon-divergent contravariant metric tensor *will* finally reveal that the horizon *does* in fact *deleteriously affect curvature*.

In a somewhat similar way a certain *subset* of the curvature *scalars* becomes *ill-defined* at the horizon. We note that the covariant metric tensor $g_{\mu\nu}$ of Eqs. (1a) and (3a) and the $b(r)$ and $a(r)$ of respectively Eqs. (9u) and (9n) yields,

$$(-det g_{\mu\nu}(r))^{\frac{1}{2}} = (b(r)(a(r))^3)^{\frac{1}{2}} = ((1 - (r_0/r))^2(1 + (r_0/r))^{10})^{\frac{1}{2}} = |1 - (r_0/r)|(1 + (r_0/r))^5. \tag{13a}$$

Therefore $(-detg_{\mu\nu})^{\frac{1}{2}}$ *vanishes* at the horizon $r = r_0$. This obviously causes curvature scalars such as [9],

$$\left(\frac{\varepsilon^{\alpha\beta\gamma\kappa} R_{\alpha\beta}{}^{\sigma\tau} R_{\gamma\kappa\sigma\tau}}{(-detg_{\mu\nu})^{\frac{1}{2}}} \right) \quad \text{and} \quad \left(\frac{\varepsilon^{\alpha\beta\gamma\kappa} R_{\lambda\xi\sigma\tau} R^{\sigma\tau}{}_{\alpha\beta} R_{\gamma\kappa}{}^{\lambda\xi}}{(-detg_{\mu\nu})^{\frac{1}{2}}} \right), \tag{13b}$$

to become *ill-defined* at the horizon. As a matter of fact, even though the *scalar density* $(\varepsilon^{\alpha\beta\gamma\kappa} R_{\alpha\beta\gamma\kappa})$ is well-defined at the horizon and vanishes everywhere, as has been pointed out underneath Eq. (10d), the closely related *scalar*,

$$\left(\frac{\varepsilon^{\alpha\beta\gamma\kappa} R_{\alpha\beta\gamma\kappa}}{(-detg_{\mu\nu})^{\frac{1}{2}}} \right), \tag{13c}$$

albeit vanishing elsewhere obviously becomes an *ill-defined* "zero over zero" entity at the horizon!

The issue at the horizon with the curvature scalars of Eqs. (13b) and (13c) doesn't occur for the "standard form" of the Schwarzschild solution metric [2], which has the property that $detg_{\mu\nu} = -1$ (the Minkowski flat-space value) instead of tending toward *zero* as the horizon is approached, which is the case for the isotropic static form of the Schwarzschild solution that is treated in detail above. In the remainder of this article we shall endeavor to elucidate *the manifold intertwined reasons* for this *discrepancy* between these two forms of the Schwarzschild solution at the horizon, *both* deep-lying technical-mathematical ones *and* astonishly elementary gravitational-physics ones.

But *first* let us for just a few moments pull ourselves *entirely away* from these highly technically involved considerations of *curvature* and focus instead on *equally fundamental*, but vastly less technically complex, considerations of *signature*. The *signature* of a finite real number has *one* of *three possible values* -1, 0 or $+1$ in the obvious way, and it has the value 0 only if the real number itself is zero. It is a theorem that *the unordered set of the signatures of the four eigenvalues of the metric is an invariant* [10], obviously *a vastly less technically involved invariant* than the curvature scalars of Eqs. (13b) and (13c). Away from the horizon, the signature set of the four eigenvalues of *both* the "standard" and isotropic static forms of the Schwarzschild metric is $\{1, -1, -1, -1\}$, which is identical to *the Minkowski flat-space value* of this unordered signature set. At the $r = r_0$ horizon radius the isotropic Schwarzschild metric's eigenvalue signature set *changes* to $\{0, -1, -1, -1\}$, as we clearly see from Eq. (1a) with the empty-space Einstein-equation functions $b(r) = ((1 - (r_0/r))/(1 + (r_0/r)))^2$ and $a(r) = (1 + (r_0/r))^4$. That *change* of an invariant entity at the horizon *patently does not tally* with the widely accepted narrative of "no dramas at the horizon". The situation with the signature set of the "standard" Schwarzschild metric at the horizon is more difficult to decipher, since while *one* eigenvalue of the "standard" Schwarzschild metric *definitely does go to zero there*, another eigenvalue tends toward *infinity* at the horizon, *which is not in the domain of the signature function.* Still, if we lean toward treating the problematic infinite eigenvalue in the physically sensible seeming $r \to r_0+$ one-sided limit, we will get the *same change* in the signature set from $\{1, -1, -1, -1\}$ to $\{0, -1, -1, -1\}$ at the horizon as occurs for the isotropic Schwarzschild metric. Any *other* possibile treatment of the infinite eigenvalue *can only produce an even more drastic change* in the signature set at the horizon!

The *lesson* of this *signature exercise* at the horizon is that the widely accepted "no dramas at the horizon" narrative *doesn't hold water* for *either* the "standard" *or* the isotropic form of the Schwarzschild metric because *there definitely is a change* at the horizon in the invariant eigenvalue-signature set—*a change in this invariant set at the horizon which has no counterpart whatsoever elsewhere.*

Before we *return* to thinking about the horizon curvature-scalar discrepancy between the "standard" and isotropic Schwarzschild metric forms, let us take a few *more* moments to ponder the consequences of the *divergence in the affine connection at the horizon* of the Schwarzschild metric, which is clearly exhibited in the first two terms on the right-hand side of Eq. (11b) for the isotropic form of the Schwarzschild metric, and which for the "standard" form of that metric occurs at the $R = ((2GM)/c^2) = 4r_0$ horizon in three of the affine connection's spherical polar coordinate components [11],

$$\Gamma^0_{\;0R}(R) = \Gamma^0_{\;R0}(R) = -\Gamma^R_{\;RR}(R) = \tfrac{2r_0}{R^2(1-4(r_0/R))}.$$

Momentarily skirting the issue of geodesic trajectories, we focus instead on *the absolutely central role* of the affine connection in *the covariant derivatives of arbitrary tensor fields*. All manner of nongravitational physical fields have differential equations of motion; *their coupling to gravitation* therefore involves *covariant derivatives*, in which the affine connection is patently *the central player*. Given the well-nigh endless possibilities for forms of equations of motion of nongravitational physical fields, many of which contemporary physics could well still be unaware, it is hardly conceivable that *divergences in the affine connection* are by any stretch of the imagination *acceptable on general theoretical physics grounds*. That said, there are *as well* persuasive arguments that *geodesic trajectories* become light-like and thereby cease to make physical sense at the Schwarzschild horizon [12].

The *thrust* of the slight detour undertaken in the preceding three paragraphs away from the narrow issue of the horizon curvature-scalar discrepancy between the "standard" and the isotropic forms of the Schwarzschild metric is that *that* issue is obviously *only a particular tiny aspect of the physical nonexistence of horizons*, notwithstanding that horizons occur in empty-space Einstein equation solutions. In regard to the physical acceptability of solutions of fundamental differential equations of theoretical physics such as the Einstein equation, the maxim of "garbage in, garbage out" definitely applies: such equations are *never* to be approached thoughtlessly. Maxwell's differential equations support a *source-free* constant static electric field of *arbitrary strength*. Bound-state Schrödinger equations happily swallow *unphysical energy inputs* and then produce perfectly well-defined "wave functions" *which can't be normalized*. There is certainly no reason to suppose that solutions of the Einstein equation are untouchable sacred writ in regions where they produce *manifest pathologies*. What is *very worthwhile* is to establish *whether* a problematic solution of a credible physical equation in fact *crosses a physical red line*, and if so to zero in on the *nature* of the *subtle misuse* of that equation which *causes* it to produce that unphysical solution.

The great *value* of the narrow issue of the horizon curvature-scalar discrepancy between the "standard" and isotropic forms of the Schwarzschild metric is that it leads step by suggestive step to insight into *exactly which intrinsic red line* of tensor-transformation based metric gravity theory horizons in fact *cross*. A *further* insight, rooted in the gravitational interchangeability of mass and energy *including gravitational potential energy itself*, is needed to understand *how rigid specification of an effective mass M* makes the *empty-space* Einstein equation *gravitationally inapplicable* in a sufficiently small neighborhood of that mass.

Pathological tensor transforms and gravitational horizons: two sides of a coin

The horizon curvature-scalar *discrepancy* between the "standard" and isotropic forms of the Schwarzschild metric *is supposed to be impossible* because these metrics differ only by a coordinate transformation, and scalars are invariant under general coordinate transformations.

This type of smug assertion is usually "good enough" for physicists, but mathematicians know very well that *valid theorems* are typically *riddled with caveats*. Let's take a close look at the actual transformation from the "standard" to the isotropic Schwarzschild metric. It's just a radius map which follows from equating the $g_{00}(r)$ component of the isotropic Schwarzschild metric to the $g_{00}(R)$ component of the "standard" Schwarzschild metric. From Eqs. (1a) and (9u), $g_{00}(r) = b(r) = ((1 - (r_0/r))/(1 + (r_0/r)))^2$, and it is well-known that $g_{00}(R) = (1 - 4(r_0/R))$ [1], where the corresponding "standard" Schwarzschild horizon is at $R = 4r_0 = ((2GM)/c^2)$. Equating $g_{00}(r)$ to $g_{00}(R)$ yields a quadratic equation for the radius map $r(R)$ that has the solution [13],

$$r(R) = \tfrac{1}{2}[R - 2r_0 + (R^2 - 4r_0R)^{\frac{1}{2}}], \tag{14a}$$

which *itself* is perfectly well-defined at the $R = 4r_0$ horizon, but it has *a branch point* there. Thus it happens that the *derivative* of this map,

$$dr(R)/dR = \tfrac{1}{2}\left[1 + \left(\tfrac{R-2r_0}{(R^2-4r_0R)^{\frac{1}{2}}}\right)\right], \tag{14b}$$

is *divergent* at the $R = 4r_0$ horizon!

If we recall that *tensors* transform multi-linearly *with the Jacobian matrix and its inverse that pertain to a given map*, e.g., given a map $x^\mu \to \bar{x}^\mu(x)$ and an arbitrary mixed second-rank tensor T^μ_ν, that tensor's transformation by that map is,

$$T^\mu_\nu \to \tfrac{\partial x^\lambda}{\partial \bar{x}^\nu}\tfrac{\partial \bar{x}^\mu}{\partial x^\kappa}T^\kappa_\lambda, \tag{15a}$$

we *immediately realize* that maps—such as the one of Eqs. (14)—which have divergent components in their Jacobian matrix or in its inverse at certain coordinate points *produce completely unphysical tensor transformations at those coordinate points*.

In classical theoretical physics, tensors typically represent *physical fields*: there can be *no physical sense* in the transformation of a perfectly well-behaved classical physical field *into one afflicted with divergences*.

In other words, the map of Eqs. (14) from the "standard" to the isotropic Schwarzschild metric *is patently unphysical at the $R = 4r_0$ horizon*. It is, of course, well-nigh inconceivable that this could be the case if the horizon *itself* was physical. In fact, we shall see below that any local transformation from a horizon point *to a freely-falling inertial frame* is necessarily *itself unphysical as a tensor transformation*. Thus maintaining *both* the Principle of Equivalence *and* physical sense *wouldn't be possible if horizons actually existed*.

In *addition* to a map being *unphysical on its face* as a tensor transformation at coordinate points where its Jacobian matrix or inverse thereof has divergent components, *standard tensor-contraction theorem demonstrations fall away* for a map at such coordinate points. For example, the demonstration that the contraction T^σ_σ of the arbitrary mixed second-rank tensor T^μ_ν of Eq. (15a) is invariant under the map $x^\mu \to \bar{x}^\mu(x)$ of course applies the standard calculus chain-rule lemma,

$$\frac{\partial x^\lambda}{\partial \bar{x}^\sigma}\frac{\partial \bar{x}^\sigma}{\partial x^\kappa} = \delta^\lambda_\kappa. \tag{15b}$$

But at coordinate points where the Jacobian matrix $\partial\bar{x}^\mu/\partial x^\kappa$ or its inverse $\partial x^\lambda/\partial\bar{x}^\nu$ *have divergent components*, the left-hand side of Eq. (15b) *is ill-defined in terms of the real numbers* whereas the right-hand side of Eq. (15b) *is well-defined in terms of the real numbers*. Obviously there is no way to demonstrate such a self-inconsistent lemma.

It is precisely *this falling away of the demonstrations of standard tensor-contraction theorems* at coordinate points where a map's Jacobian matrix or its inverse have *divergent components* that, in light of the *divergence* at the $R = 4r_0$ horizon of the *derivative* $dr(R)/dR$ of the map $r(R)$ from the "standard" to the isotropic form of the Schwarzschild metric, *permits* the horizon curvature-scalar *discrepancy* between the "standard" and isotropic forms of the Schwarzschild metric which was pointed out in the paragraph below Eq. (13c), and was the motivation for this section.

Before wrapping up this section, there is one further important matter to discuss, namely that the local map from a horizon point to a freely falling inertial frame is necessarily unphysical as a tensor transformation at that point, as was mentioned in the paragraph which is the third before this one. At a horizon point the metric tensor or its inverse *will have divergent components*, but *after* its tensor transformation into the local freely falling frame it will be *equal* to the Minkowski metric *at the corresponding point* of the changed coordinate system. Of course the Minkowski metric is comprised of *well-defined real numbers*, as is its inverse (which is itself). Thus the local *divergence* in the metric tensor or in its inverse *must* have been sloughed off onto a local *divergence* in at least one component of Jacobian matrix of the map or onto a local *divergence* in at least one component of the inverse of that Jacobian matrix of the map. Therefore any local map from a horizon point of a metric to a local freely falling inertial frame will be *unphysical* at that point as a tensor transformation. Thus maintaining *both* physical sense *and* the Principle of Equivalence wouldn't be possible if horizons actually existed.

It is interesting that the two Eddington-Finkelstein and the two Kruskal-Szekeres maps [14, 15] were contrived for the express purpose of causing the horizon of the "standard" form of the Schwarzschild metric to *disappear*. The two Eddington-Finkelstein maps both achieve this by sending *every* horizon space-time point to *infinite time*, while the two Kruskal-Szekeres maps both do so by sending the entire horizon set of space-time points, $-\infty < t < +\infty$, $R = 4r_0$, to the *single* fleeting space-time point $\bar{t} = 0$, $\bar{R} = 0$. It is obvious that the Eddington-Finkelstein maps have divergent Jacobian matrix components everywhere on the "standard" Schwarzschild metric's horizon, while the Kruskal-Szekeres maps have divergent components in the *inverses* of their Jacobian matrices everywhere on that horizon, i.e., both Eddington-Finkelstein maps and both Kruskal-Szekeres maps *are unphysical as tensor transformations on that horizon*, just as the map from the "standard" form of the Schwarzschild metric to its isotropic form is unphysical as a tensor transformation there. Is is obviously *no more possible* to make a horizon *disappear* with a *physically legitimate tensor transformation on that horizon* than it is possible to pass from a horizon point to a freely falling inertial frame with a tensor transformation which is physically legitimate at that horizon point.

Having established that assuming the existence of gravitational horizons crosses a physical red line, we are left with the issue of understanding the *nature* of the subtle *misuse* of the Einstein equation which causes that equation to *produce* unphysical horizons.

We have imposed *two* conditions on the Einstein equation, namely that it is valid for *for empty space only* while simultaneously an effective mass M *is definitely present*. At radius $r = r_0 = ((GM)/(2c^2))$ this *empty-space* equation then proceeds to confront us with an *unphysical* horizon. The only *logical* possibility would seem to be that *it isn't gravitationally possible for there to be empty space at* $r = ((GM)/(2c^2)) = r_0$, given our insistence that the effective mass M is definitely present.

The gravitational upper bound on localized effective mass

To go further, we need to stop dealing with the effective mass M which enters into the empty-space Einstein equation outside of all context: *the effect of its size on its internal gravitational potential energy must be modeled*. The easiest way to do that is to break this effective mass into *just two* equal compact parts which gravitationally attract each other. In the limit of infinite separation of the two parts, each one has a mass which we call M_0. When the two parts are separated by a finite distance which we call $2r_s$ (r_s being the *radius* of the system), the system's net effective mass is,

$$M_{\text{eff}}(r_s; M_0) = 2M_0 - ((GM_0^2)/(2c^2 r_s)). \tag{16a}$$

Crucially, we *can't* make $M_{\text{eff}}(r_s; M_0)$ arbitrarily large by driving M_0 to infinity; its *upper bound* for fixed r_s is,

$$M_{\text{eff}}^{\max}(r_s) = (2c^2 r_s)/G, \tag{16b}$$

which is attained when $M_0 = (2c^2 r_s)/G$.

Thus according to this simple model the effective mass M of a system of radius r_s obeys,

$$M \le (2c^2 r_s)/G. \tag{16c}$$

or equivalently, the radius r_s of a system of effective mass M *must satisfy*,

$$r_s \ge ((GM)/(2c^2)) = r_0. \tag{16d}$$

So if we insist on the presence of the effective mass M, we can *under no circumstance* extend untrammeled use of the *empty-space* Einstein equation all the way down to $r = r_0$. To *try* to do so is *misuse* of the empty-space Einstein equation, which brutally *warns* us of this *ultimate limit to its legitimate use* by confronting us *with a completely unphysical horizon* if we dare to go that far. In this respect the empty-space Einstein equation emulates the behavior of the stationary-state Schrödinger equation for the simple harmonic oscillator of natural frequency ω, which dyspeptically responds to any energy input that is *not* an element of the *physical* energy set $\{E | E = (n + \frac{1}{2})\hbar\omega, n = 0, 1, 2, \ldots\}$ with two linearly independent parabolic cylinder "wave functions", *no linear combination of which is normalizable*.

It is worth noting that Eq. (16c) implies that point masses effectively don't exist, that they have *zero* effective mass. This fact is echoed in Abhas Mitra's line-by-line revision of the Oppenheimer-Snyder gravitationally-collapsing spherical dust cloud calculation [16, 17]. Mitra of course finds that no horizon ever forms, nor are there ever any "trapped surfaces" [18]. He further finds that the ultimate limiting state of the gravitationally shrinking dust cloud is one of effective mass zero, exactly the result that one would expect from Eq. (16c). Christoph Schiller has shown that under circumstances that the Principle of Equivalence effectively holds, the Einstein equation is equivalent to his Principle of Maximum Force or Power [19]. It is part and parcel of the latter Principle that horizons are *never* physically attained, and Schiller has as well laid great emphasis on the fact that point masses effectively don't exist.

Conclusion

The narrative that gravitational horizons physically exist, that the Eddington-Finkelstein and Kruskal-Szekeres maps make physical sense everywhere as tensor transformations, that gravitationally collapsing objects form "trapped surfaces" which then proceed to horizons, and that "no dramas" occur at horizons gained enormous cachet in the wake of Stephen Hawking's imaginative 1974 ideas of spontaneous pair production effects across horizons [20]. These proved to be a winner with other researchers, and the thermodynamics of horizons [21] became very popular as well. Much more recently, however, the cornucopia of horizon-based research activity received a mildly painful jolt [22]: the physical consistency of Hawking's classical/quantum horizon picture was strongly questioned. It was suggested, inter alia, that notwithstanding "no dramas" at the horizon, a conceivable scenario could be a "firewall" of highly energetic radiation just behind the horizon. Even more recently, Hawking himself has weighed in [23]; he is astonishingly prepared to jettison horizons themselves, the indispenable foundation of his effort of greatest fame. But disappointingly, his new thesis is that horizons can't properly form due to quantum fluctuations, instead of the elementary fact that *classical* gravity is intrinsically *incapable* of supporting them, that they are naught but artifacts of subtle *misuse* of the empty-space Einstein equation. Still, after so many decades of horizons as indubitable, there is a bit of flux and movement at long last—for the sake of the most elementary proper understanding of gravitation the appropiate culmination of that ferment cannot come too soon.

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