Oppenheimer-Snyder Dust: Spatially Localized or Uniform?

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Abstract

Over a period spanning at least the past fifteen years it has been repeatedly pointed out that the singularities and event horizons which arise from the Oppenheimer-Snyder ostensible calculation of the gravitational contraction of a localized sphere of dust are at odds with the Principle of Equivalence: notably the timelike geodesic trajectory which that Principle requires for every dust particle precludes any such dust particle from encountering an infinite redshift. Sweeping abstract arguments on the incompatibility of the Principle of Equivalence with an infinite redshift or other singularities have also been advanced. Scrutiny of the Oppenheimer-Snyder calculation shows that it never deals from scratch with localized dust; its one calculation from scratch is a Friedmann model variant with the archetypal Friedmann uniform dust density that pervades the entirety of space. This very unrealistic model, which inter alia has infinite energy, exhibits singularities and violations of the Principle of Equivalence. Since the Birkhoff theorem doesn't apply to Friedmann models because they have no empty space region, the Oppenheimer-Snyder follow-on instruction to utilize a spliced-on Schwarzschild metric beyond some arbitrarily specified radius contravenes logic.

The physical cogency of the ostensible solution of Oppenheimer and Snyder for a gravitationally contracting dust sphere [1] has been subject to strong criticism for at least the past decade and a half [2, 3, 4] on the basis that the Principle of Equivalence implies that the the geodesic trajectory of any individual dust particle must always be timelike. This precludes a dust particle's redshift ever becoming infinite, which is the redshift value at a gravitational event horizon, because attaining an infinite redshift value is tantamount to the dust particle's geodesic trajectory *changing its nature from timelike to lightlike*.

The physical unattainability of gravitational event horizons is also *an intrinsic aspect* of Christoph Schiller's principle of maximum power or force, which Schiller proves is *equivalent* to the Einstein equation [5] in an environment where the Principle of Equivalence holds.

Physical attainment of an infinite redshift value as well contravenes the theorem that immediately follows from the Principle of Equivalence which states that *the set of the four signatures* of the eigenvalues of the metric tensor *must always be the same as that of the Minkowski metric tensor, namely* $\{+1, -1, -1, -1\}$ [6].

The contents of the preceding three paragraphs comprise a persuasive indictment of the ostensible solution of Oppenheimer and Snyder for a gravitationally contracting dust sphere. It is, however, an indictment which merely points out the clash of the Principle of Equivalence with the gravitational event horizon that is part of the Oppenheimer-Snyder ostensible solution.

One would *in addition* like to identify *those specific steps* of the Oppenheimer-Snyder calculation procedure in which unphysical considerations that contravene the Principle of Equivalence or other aspects of gravitational physics *are inadvertently introduced*. Such identification could conceivably be a valuable innoculation against the *repetition* of similar missteps in *other* calculations.

The Oppenheimer-Snyder procedure for the solution of a gravitationally-contracting dust sphere, as it is presented in Steven Weinberg's monograph and textbook on gravitation [7], begins on an apparently physically flawless note with the Einstein equation for the time-dependent spherically-symmetric comoving metric tensor, as given by Weinberg's Eq. (11.9.1) on his page 342, that results from a zero-pressure fluid of dust which has the spherically-symmetric proper energy density $\rho(r, t)$, and is further described in detail by Weinberg's Eqs. (11.9.2) through (11.9.4) on the same page. The result is a nonlinear, coupled second-order partial differential system of four equations for $\rho(r, t)$ and the two unknown comoving-metric functions $U(r, t)$ and $V(r, t)$ that is given by Weinberg's Eqs. (11.9.9) through (11.9.12) on his page 343.

To proceed further analytically, the idea of *separating* both $U(r, t)$ and $V(r, t)$ into a product of two factors, one of which is a function of t only and the other of which is a function of r only, naturally comes to mind. Upon substituting this factorization Ansatz into Weinberg's Eqs. (11.9.9) through (11.9.12), it becomes apparent that *still further* analytic progress would be considerably aided by making *the additional assumption* that,

$$\rho(r, t) = \rho(t), \tag{1}$$

i.e., that the dust's proper energy density *is independent of position* [1]. While this last assumption might seem innocuous upon merely *cursory* consideration, it in fact is encumbered by *the drastic implication* that

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the dust uniformly *fills the entirety of space at every time* t! Such a system is obviously extremely unphysical; inter alia its dust component possesses *infinite energy*. The assumption of Eq. (1), while analytically helpful, obviously exacts a wholly unacceptable price in terms of physical cogency, bringing *to an abrupt end* the heretofore physically sensible Oppenheimer-Snyder modeling of the gravitational contraction *of an actually localized* dust sphere.

The analytic upshot of Weinberg's Eqs. (11.9.9) through (11.9.12) and the aforementioned factorizations of $U(r,t)$ and $V(r,t)$ *plus the completely unphysical assumption of* Eq. (1) is *a Friedmann model* that embraces *the Robertson-Walker metric form* given in Weinberg's Eq. (11.9.16) on his page 344, namely,

$$ds^2 = (cdt)^2 - (R(t))^2 \left[(1 - kr^2)^{-1}dr^2 + r^2((d\theta)^2 + (\sin\theta d\phi)^2)\right], \tag{2a}$$

together with the Friedmann equation for the dimensionless Robertson-Walker metric function $R(t)$ that is given by Weinberg's Eq. (11.9.21) on the same page,

$$(\dot{R}(t))^2 = ((8\pi G\rho(t=0))/(3c^2R(t))) - c^2k, \tag{2b}$$

and this Friedmann equation's initial condition $R(t = 0) = 1$ that is given by Weinberg's Eq. (11.9.17). From the assumptions summarized at the beginning of this paragraph it as well follows that (see Weinberg's Eq. (11.9.18)),

$$\rho(t) = (\rho(t=0)/(R(t))^3). \tag{2c}$$

Imposing in addition the condition $\dot{R}(t = 0) = 0$ (Weinberg's Eq. (11.9.22) on page 344) determines from the Eq. (2b) Friedmann equation the constant k of dimension of inverse length squared that occurs in the Eq. (2a) Robertson-Walker metric form. The resulting value of k is,

$$k = ((8\pi G\rho(t=0))/(3c^4)). \tag{3}$$

With the insertion of Eq. (3), the Friedmann Eq. (2b) for the Robertson-Walker metric function $R(t)$ can be reexpressed as,

$$\dot{R}(t) = -c\sqrt{k[(1/R(t)) - 1]}, \tag{4a}$$

which additionally implies that,

$$\ddot{R}(t) = -((c^2k)/(2(R(t))^2)). \tag{4b}$$

Given the initial condition $R(t = 0) = 1$ of Weinberg's Eq. (11.9.17), it is clear from Eqs. (4a) and (4b) that $R(t)$ decreases monotonically *and increasingly rapidly* with increasing t, and thus would be expected to fall to *zero* after the lapse of a *finite* time. Indeed Weinberg's Eq. (11.9.25) on page 345, which *implicitly solves* Eq. (4a) for the initial condition $R(t = 0) = 1$, can be written in the form,

$$t = \left[\arccos(2R(t) - 1) + 2\sqrt{R(t)(1 - R(t))}\right]/(2ck^{\frac{1}{2}}), \tag{4c}$$

from which one reads off that,

$$R(T) = 0, \tag{4d}$$

where,

$$T \stackrel{\text{def}}{=} (\pi/(2ck^{\frac{1}{2}})). \tag{4e}$$

From Eqs. (4d), (2c), (4a) and (4b), we see that $\rho(T)$, $\dot{R}(T)$, and $\ddot{R}(T)$ *all diverge*, which is *manifestly unphysical behavior* that confirms the extremely unphysical nature of the spatially uniform dust density assumption $\rho(r,t) = \rho(t)$ set out in Eq. (1) above.

In addition, it is apparent from the Robertson-Walker metric form given by Eq. (2a) that at $t = T$, $ds^2 = (cdt)^2$, whose eigenvalue signature set is $\{+1, 0, 0, 0\}$ in gross violation of the Principle of Equivalence, which requires that any metric's eigenvalue signature set be everywhere that of the Minkowski metric, namely $\{+1, -1, -1, -1\}$ [6]. It is thus apparent that the completely unphysical uniform dust density assumption $\rho(r,t) = \rho(t)$ of Eq. (1) *is also incompatible with the Principle of Equivalence*.

But *failing to notice that fact*, Oppenheimer and Snyder proceeded to make an inadequately thought through *further supposition*, namely that the infinitely-extended uniform dust density of Eq. (1) *can be effectively metamorphosed into a localized spherical dust density* by "applying" the Birkhoff theorem to that infinitely-extended uniform dust density's gravitational field.

To implement this supposition Oppenheimer and Snyder spliced, at an arbitrary radius $a > 0$, onto the Friedmann-Robertson-Walker metric determined by Eqs. (2a), (3) and (4c) a "standard" form Schwarzschild metric written in terms of the "barred" space-time variables $\bar{\phi} = \phi$, $\bar{\theta} = \theta$, $\bar{r} = rR(t)$ and an integral expression for $\bar{t}(r, R(t), k; a)$ which is given by Weinberg's Eqs. (11.9.29) and (11.9.30) on his page 345. In addition, the spliced-on "standard" form Schwarzschild metric's effective mass M is determined by continuity at the splicing radius a to be (see Weinberg's Eqs. (11.9.37) and (11.9.38) on his page 346),

$$M(a) = ((c^2ka^3)/(2G)) = (4/3)\pi a^3(\rho(t = 0)/c^2). \tag{5}$$

Unlike Oppenheimer and Snyder, however, we need to *stop* at this point in order *to try to reflect on the meaning* of the a-dependent *infinite family* of "standard" form Schwarzschild metrics which Oppenheimer and Snyder thus constructed from the *single* Friedmann-Robertson-Walker metric solution that is completely determined by Eqs. (2a), (3) and (4c). We need to understand in particular *exactly how* this Oppenheimer-Snyder a-dependent family of "standard" form Schwarzschild metrics *relates to the Birkhoff theorem*.

At the end of the next-to-last paragraph on his page 337 in his Section 11.7 Weinberg writes that the *content* of the Birkhoff theorem is, "... that a spherically symmetric gravitational field in empty space must be static, with a metric given by the Schwarzschild solution." But because of the Eq. (1) condition that $\rho(r, t) = \rho(t)$, which implies that dust of uniform density pervades *all* of space, *there exists no empty-space region whatsoever in conjunction with the gravitational field that is described by the Friedmann-Robertson-Walker metric solution set out in* Eqs. (2a), (3) and (4c), *so the Birkhoff theorem doesn't apply to this field*.

Therefore the splicing-radius dependent *infinite family* of "standard" form Schwarzschild metrics which Oppenheimer and Snyder constructed on the basis of the "nowhere-empty space" Friedmann-Robertson-Walker metric solution set out in Eqs. (2a), (3) and (4c) *patently has no gravitational physics significance or utility whatsoever*.

As a matter of fact, that statement is virtually self-evident *just from the pathologies* that are manifested by the Oppenheimer-Snyder spliced "exterior time" $\bar{t}(r, R(t), k; a)$ which is given by the integral expression in Weinberg's Eqs. (11.9.29) and (11.9.30) on his page 345. Noting that $0 \leq kr^2 < ka^2 < 1$, it can be verified that $\bar{t}(r, R(t), k; a)$ *diverges logarithmically* for *all* $R(t)$ which satisfy $R(t) < ka^2$. Indeed for $R(t)$ which satisfy $R(t) < [1 - (1 - ka^2)^{\frac{3}{2}}]$, there always exists a *range* of sufficiently *small* r-values such that $\bar{t}(r, R(t), k; a)$ *still diverges*.

Still another way to grasp that the a-dependent infinite family of "standard" form Schwarzschild metrics which Oppenheimer and Snyder constructed is devoid of gravitational physics significance or utility is to realize that *just myopically focusing* on a localized subregion of an infinitely-extended system *can't change that infinitely-extended system by a single iota*. Any *localized* system *definitely differs* from that infinitely-extended system however, and, as such, can certainly be expected to manifest *different dynamical behavior from that of the infinitely-extended system in any nontrivial subregion whatsoever*.

We conclude our discussion with a bird's-eye overview of the organization, progression and end result of the Oppenheimer-Snyder calculation procedure, *which has three distinct phases*. Its *first phase* produces the Friedmann-Robertson-Walker metric solution set out in Eqs. (2a), (3) and (4c), whose sole independent input is $\rho(t = 0) > 0$, the initial dust proper energy density *which uniformly pervades all of space*. That the source proper energy density *continues to uniformly pervade all of space at subsequent times* is crystal-clear from Eq. (2c), namely,

$$\rho(t) = (\rho(t = 0)/(R(t))^3),$$

about which we of course know that $R(t)$ decreases monotonically from unity at $t = 0$ to zero at $t = T$. The fact that *an empty space region never exists* implies that *the Birkhoff theorem never applies*.

That notwithstanding, the *second phase* of the Oppenheimer-Snyder calculation procedure generates the a-dependent infinite family of "standard" form Schwarzschild metrics, where the Schwarzschild metric family member which has parameter a is spliced onto the Friedmann-Robertson-Walker metric at $r = a$. In detail, the family member which has parameter a is given by Weinberg's Eqs. (11.9.27) through (11.9.30) on his page 345, plus Eq. (5) above for these "standard" form Schwarzschild metrics' effective masses $M(a)$.

The *third phase* of the Oppenheimer-Snyder calculation procedure nominates a specific value of the parameter a that is ostensibly of interest (e.g., Weinberg writes of "a star" whose radius just before its gravitational contraction process begins is a). This third phase *utilizes* the "standard" form Schwarzschild metric which has the splicing radius a *as though* the region $r > a$ of the Friedmann-Robertson-Walker solution *is actually empty space and the Birkhoff theorem therefore actually applies to this region*. But as we have taken pains to carefully document above, for the Friedmann-Robertson-Walker solution *nothing could be further from the truth*!

Therefore *nothing whatsoever can be salvaged* from this three-phase Oppenheimer-Snyder calculation procedure which *claims* to model the gravitational contraction of a dust sphere of initial *finite* radius a, but which *in fact is completely based on an extremely unphysical infinitely-extended Friedmann system*.

The *only* conceivable path to progress is to *return* to the comoving Einstein equation as it is set out in Weinberg's Eqs. (11.9.9) through (11.9.12) on his page 343 and insert into it *a physically sensible localized initial proper energy density* $\rho(r, t = 0)$, *resolutely forsaking entirely the infinitely-extended Friedmann models which adhere to* $\rho(r, t) = \rho(t)$. Doing that might preclude analytic solution, but even so a decent understanding of such a system's behavior might still eventuate.

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