On Goldbach Conjecture

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Abstract

Goldbach conjecture asserts that every even integer greater than 4 is sum of two odd primes. Stated in a letter to Leonard Euler by Christian Goldbach in 1842, this is still an enduring unsolved problem. In this paper we develop a new simple strategy to settle this most easy to state problem which has baffled mathematical community for so long. We show that the existence of two odd primes for every even number greater than 4 to express it as their sum follows from the well known Chinese remainder theorem. We further develop a method to actually determine a pair of primes for any given even number to express it as their sum using remainders modulo all primes up to square root of that given even number.

1. **Introduction:** We begin our discussion with some elementary observations. If a number *m* is not prime and $m = pq$ then clearly either $p \leq \sqrt{m}$ or $q \leq \sqrt{m}$. Further, if $p \leq \sqrt{m}$ say, and suppose *p* is not prime then by fundamental theorem of arithmetic *p* can be factored uniquely as the product of prime powers with obviously all the primes in that unique factorization strictly less than *p*. Thus, we have the following well-known

Theorem 1.1: A number *m* is prime if and only if it is not divisible by any prime number $\leq \sqrt{m}$.

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Also let us now state a **very useful** Chinese remainder theorem [1]:

Theorem 9.2 (Chinese Remainder Theorem): Let $m_1, m_2, \cdots, m_l$ denote l positive integers which are relatively primes in pairs, and let

$a_1, a_2, \cdots, a_l$ denote any l integers. Then the following congruence system

$$x \equiv a_1 \bmod(m_1)$$
$$x \equiv a_2 \bmod(m_2)$$
$$\vdots$$
$$x \equiv a_l \bmod(m_l)$$

has common solutions. If x_0 is one such solution, then an integer x is another solution if and only if $x = x_0 + km$ for some integer k and here $m = \prod_{j=1}^{l} m_j$.

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Let $2n$ be an arbitrary positive and even integer. In order to settle Goldbach conjecture in the affirmative we have to show the existence of two prime numbers p, $q < 2n$ such that $2n = p + q$. Let $p_1(=2), p_2(=3), p_3, \cdots, p_k, p_{(k+1)}$ be the successive primes such that $p_k^2 < 2n < p_{(k+1)}^2$. Let $2n \equiv \beta_i \bmod(p_i), i = 1,2,\cdots,k$. Note that since $2n$ is even so clearly $\beta_1 = 0$. The **main idea** in this approach is to split each remainder β_i into **two nonzero parts in all possible ways:** $\beta_i = \eta_i^j + \delta_i^j$, and $1 \le \eta_i^j, \delta_i^j \le (p_i - 1)$. We then determine all possible number pairs (p, q) using the above mentioned Chinese remainder theorem such that $p \equiv \eta_i^j \bmod(p_i)$ and $q \equiv \delta_i^j \bmod(p_i)$. We call such p, q numbers the **suitable candidates**. They are **complements** of each other in the sense that $\beta_i = \eta_i^j + \delta_i^j$. If we show that among the suitable candidates there exists at least one number p (or q) such that $p < 2n$ (and so prime due to theorem 9.1) then Goldbach conjecture follows.

For the sake of **illustration** we begin with an example:

Example 1.1: Let $2n = 100$. So $\sqrt{2n} = 10$. The primes less than 10 are respectively 2, 3, 5, and 7. Cleary, $100 \equiv 0 \bmod(2)$, $100 \equiv 1 \bmod(3)$,

$100 \equiv 0 \bmod(5)$, and $100 \equiv 2 \bmod(7)$. Now we note down all the possibilities that exists for η_i^j :

(1) $\eta_1^j = \{1\}$

(2) $\eta_2^j = \{2\}$

(3) $\eta_3^j = \{1,2,3,4\}$

(4) $\eta_4^j = \{1,3,4,5,6\}$

Note that for any choice of η_i^j given above the corresponding choice for δ_i^j that get fixed by the requirement, namely, $\beta_i = \eta_i^j + \delta_i^j$, is suitable in the sense that all these δ_i^j are nonzero as is needed.

To settle Goldbach conjecture for the even number equal to100 we need numbers (**at least one**) less than 100 which can be expressed simultaneously in the forms

$2k_1 + 1, 3k_2 + 2, 5k_3 + \{1 or 2 or 3 or 4\}, 7k_4 + \{1 or 3 or 4 or 5 or 6\}$ for some positive integers k_1, k_2, k_3, k_4. For example, note that 11 has the desired representations i.e. 2×5+1, 3×3+2, 5×2+1, 7×1+4. etc. so, 11 is the suitable prime with suitable prime complement 89 so that 100 = 11+89.

Note that there are in all 1×1×4×5 = 20 possibilities for p (product of the cardinalities of sets η_i^j, $i = \{1,2,3,4\}$) and one can see that the numbers obtained from these choices by applying **Chinese remainder theorem** are {11, 17, 29, 41, 47, 53, 59, 71, 83, 89, 101, 113, 131, 137, 143, 167, 173, 179, 197, 209}, when written in increasing order. We see that among these choices those which are less than $2n(=100)$ are to be seen. Thus, there are in all 10 choices less than $2n(=100)$ for p and since $p + q = q + p$ therefore there are only 5 ways to express 100 as sum of two primes which are distinct, namely, 11+89, 17+83, 29+71, 41+59, 47+53. Here we have split the remainders β_i into two nonzero parts.

But instead if we allow the splitting such that some one $\eta_i^j = 0$ and $\delta_i^j \neq 0$ then it leads to additional expressions for 100 as sum of two primes, namely, 3+97. In this last representation we are actually allowing participation of a prime among primes 2, 3, 5, 7 as one of the prime in the sum of primes representation when its complement is also prime.

2. **Goldbach Conjecture:** We now proceed to to show how this famous Goldbach conjecture actually follows as a consequence of Chinese remainder theorem stated above in quite transparent way. For this let us start with the following equivalent:

Theorem 2.1 (Equivalent of Goldbach Conjecture): For every positive integer greater than or equal to 5 there exists two primes equidistant from it. In other words, let $n \geq 5$ be the positive integer then there exists distance d, $0 < d < n$, such that $p = n - d$ and $q = n + d$, where p, q are prime numbers.

The equivalence of the above statement with Goldbach conjecture is straightforward. If the above statement (equivalent) is valid then it implies that $n - p = q - n = d$ which in turn implies that $2n = p + q$, which is Goldbach Conjecture.

Let $p_1(=2), p_2(=3), p_3, \cdots, p_k, p_{(k+1)}$ be the successive primes such that $p_k^2 < 2n < p_{(k+1)}^2$. Let $n \equiv \alpha_i \bmod(p_i), i = 1,2,\cdots,k$ and further each α_i satisfies the inequality $0 \leq \alpha_i \leq (p_i - 1)$. Now, in order to find out the desired primes p, q mentioned above which are equidistant from $n \geq 5$ and at distance d, $0 < d < n$, we need to have following congruence relations, namely, $n - d \equiv \mu_i \bmod(p_i)$, and $n + d \equiv \nu_i \bmod(p_i)$, such that $\mu_i > 0, \nu_i > 0$ for all $i = 1,2,\cdots,k$. We record the **formulae of distances** for each value of α_i such that at those distances there will be positive entries (positive values for μ_i, ν_i).

1) **Mod(2) case:** Let $\alpha_1 = 0$ then for numbers at distance $d = 2k_1 + 1$, on both left and right side of the number *n* on the number line, we will have positive remainders modulo 2, where, $k_1 = 0,1,2,\cdots$ Similarly, Let $\alpha_1 = 1$ then at distance $d = 2k_1$, on both left and right side of the

number *n* on the number line, we will have positive entries, where, $k_1 = 0,1,2,\cdots$

2) **Mod(3) case:** Let $\alpha_2 = 0$ then for numbers at distance $d = 3k_2 + 1, 3k_2 + 2$, on both left and right side of the number *n* on the number line, we will have positive remainders modulo 3, where, $k_2 = 0,1,2,\cdots$ Similarly, Let $\alpha_2 = 1,2$ then for numbers at distance $d = 3(k_2 + 1)$, on both left and right side of the number *n* on the number line, we will have positive remainders modulo 3, where, $k_2 = 0,1,2,\cdots$
3) **Mod(5) case:** Let $\alpha_3 = 0$ then for numbers at distance $d = 5k_3 + 1, 5k_3 + 2, 5k_3 + 3, 5k_3 + 4$, on both left and right side of the number *n* on the number line, we will have positive remainders modulo 5, where, $k_3 = 0,1,2,\cdots$ Similarly: Let $\alpha_3 = 1$ then for numbers at distance $d = 5k_3 + 2, 5k_3 + 3, 5(k_3 + 1)$, on both left and right side of the number *n* on the number line, we will have positive remainders modulo 5, where, $k_3 = 0,1,2,\cdots$, Let $\alpha_3 = 2$ then for numbers at distance $d = 5k_3 + 1, 5k_3 + 4, 5(k_3 + 1)$, on both left and right side of the number *n* on the number line, we will have positive remainders modulo 5, where, $k_3 = 0,1,2,\cdots$, Let $\alpha_3 = 3$ then for numbers at distance $d = 5k_3 + 1, 5k_3 + 4, 5(k_3 + 1)$, on both left and right side of the number *n* on the number line, we will have positive remainders modulo 5, where, $k_3 = 0,1,2,\cdots$, Let $\alpha_3 = 4$ then for numbers at distance $d = 5k_3 + 2, 5k_3 + 3, 5(k_3 + 1)$, on both left and right side of the number *n* on the number line, we will have positive remainders modulo 5, where, $k_3 = 0,1,2,\cdots$.

One can continue in this way and determine the formulae for distance d, $0 < d < n$, for all primes $p_1(=2), p_2(=3), p_3, \cdots, p_k, p_{(k+1)}$ such that we will have positive remainders modulo corresponding prime under consideration for numbers at the distance d satisfying these formulae, on both left and right side of the number *n* on the number line, and which (we aim to) should ultimately lead to the following congruence relations, namely, $n - d \equiv \mu_i \bmod(p_i)$, and $n + d \equiv \nu_i \bmod(p_i)$, such that $\mu_i > 0, \nu_i > 0$ for all $i = 1,2,\cdots,k$, which settles Goldbach conjecture. Thus, in order settle Goldbach conjecture we need to find distance d, $0 < d < n$, which takes the form $d = p_i k_i + r_i$, for all primes $p_1(=2), p_2(=3), p_3, \cdots, p_k, p_{(k+1)}$ and $r_i \in \{0,1,\cdots,p_i - 1\}$ such that we will have positive remainders modulo corresponding prime under consideration for numbers at the distance d satisfying these formulae, on both sides, left and right side of the number n on the number line.

Proof of theorem 2.1: Now, given $n \geq 5$ we can determine uniquely the remainders α_i satisfying the inequality $0 \leq \alpha_i \leq (p_i - 1)$ such that $n \equiv \alpha_i \bmod(p_i), i = 1,2,\cdots,k$. Using these values of α_i we can find formulae (expressions) for distance d, as is concretely done above for Mod(2), Mod(3), Mod(5) cases, in the form $d = p_i k_i + r_i$, for all primes $p_1(=2), p_2(=3), p_3, \cdots, p_k, p_{(k+1)}$ and where we will get $r_i \in \{0,1,\cdots,p_i - 1\}$ such that we will have positive remainders modulo corresponding prime under consideration for numbers at the distance d satisfying these formulae, on both sides, left and right side of the number n on the number line. But what does these formulae for distances imply? These formulae for distances $d = p_i k_i + r_i$ equivalently imply that we in possession of following congruence system and are after finding a positive number d satisfying this congruence system:

$$\begin{gathered} d \equiv \alpha_1 \bmod(p_1) \\ d \equiv \alpha_2 \bmod(p_2) \\ \vdots \\ d \equiv \alpha_k \bmod(p_k) \end{gathered}$$

and clearly, this system of congruence **has a solution by Chinese Remainder Theorem**. Thus, the proof of Goldbach conjecture.

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Example 2.1: Let $2n = 34$, therefore we have $n = 17$. Now, it is clear to see that clearly, $17 \equiv 1 \bmod(2)$, $17 \equiv 2 \bmod(3)$, and $17 \equiv 2 \bmod(5)$. Therefore considering Mod(2) case: $\alpha_1 = 1$ and so $d = 2k_1$. Similarly, considering Mod(3) case: $\alpha_2 = 2$ and so $d = 3k_2$. Similarly, considering Mod(5) case: $\alpha_3 = 2$ and so $d = 5k_3 + 1$. Therefore, we have

$$d \equiv 0 \bmod(2)$$
$$d \equiv 0 \bmod(3)$$
$$d \equiv 1 \bmod(5)$$

so as a solution of this congruence system we get $d = 6$. Therefore, it implies that $n - p = q - n = d$ equivalent to $17 - 13 = 23 - 17 = 6$ which in turn from relation $2n = p + q$ implies that $34 = 17 + 23$.

References

1. Ivan Niven, Herbert S. Zuckerman, Hugh L. Montgomery, An Introduction to The Theory of Numbers, Fifth Edition, John Wiley & Sons, Inc, 2004.