Divergence in the Stefan-Boltzmann law at High Energy Density Conditions

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It was recently detected an unidentified emission line in the stacked X-ray spectrum of galaxy clusters. Since this line is not catalogued as being the emission of a known chemical element, several hypotheses have been proposed, for example that it is of a known chemical element but with an emissivity of 10 or 20 times the expected theoretical value. Here we show that there is a divergence in the Stefan-Boltzmann equation at high energy density conditions. This divergence is related to the correlation between gravitational mass and inertial mass, and it can explain the increment in the observed emissivity.

Key words: Stefan-Boltzmann law, Thermal radiation, Emissivity, gravitational mass and inertial mass.

1. Introduction

The recent detection of an unidentified emission line in the stacked X-ray spectrum of galaxy clusters [1] originated several explanations for the phenomenon. It was proposed, for example that the unidentified emission line, spite to be non-catalogued, it is of a known chemical element but with intensity (emissivity) of 10 to 20 times the expected value.

Here we show that there is a divergence in the Stefan-Boltzmann equation at high energy density conditions. This divergence is related to the correlation between gravitational mass and inertial mass, and it can explain the increment in the observed emissivity.

2. Theory

The quantization of gravity shows that the *gravitational mass* m_g and *inertial mass* m_i are not equivalents, but correlated by means of a factor χ, which, under certain circumstances can be negative. The correlation equation is [2]

$$m_g = \chi \; m_{i0} \tag{1}$$

where m_{i0} is the *rest* inertial mass of the particle.

The expression of χ can be put in the following forms [2]:

$$\chi = \frac{m_g}{m_{i0}} = \left\{1 - 2\left[\sqrt{1+\left(\frac{W}{\rho\, c^2}n_r\right)^2} - 1\right]\right\} \tag{2}$$

$$\chi = \frac{m_g}{m_{i0}} = \left\{1 - 2\left[\sqrt{1+\left(\frac{D\, n_r^2}{\rho c^3}\right)^2} - 1\right]\right\} \tag{3}$$

where W is the density of electromagnetic energy on the particle (J/kg); D is the radiation power density; ρ is the matter density of the particle (kg/m^3); n_r is the index of refraction, and c is the speed of light.

Equations (2) and (3) show that *only* for $W = 0$ or $D = 0$ *the gravitational mass is equivalent to the inertial mass* $(\chi = 1)$. Also, these equations show that the gravitational mass of a particle can be significtively reduced or made strongly *negative* when the particle is subjected to high-densities of electromagnetic energy.

Another important equations obtained in the quantization theory of gravity is the new expression for the *kinetic energy* of a particle with gravitational mass m_g and velocity V, which is given by [2]

$$E_{kinetic} = \tfrac{1}{2} m_g V^2 = \chi \tfrac{1}{2} m_{i0} V^2 \tag{4}$$

Only for $\chi = 1$ the equation above reduces to the well-known expression $E_{kinetic} = \frac{1}{2} m_{i0} V^2$.

The *thermal energy* for a single particle calculated starting from this equation is $k_B T = \frac{1}{2} m_{i0} \overline{V^2}$ [3], where the line over the velocity term indicates that the average value is calculated over the entire ensemble; $k_B = 1.38\times10^{-23}\,J/K$ is the Boltzmann constant.

Now, this expression can be rewritten as follows $|\chi| k_B T = |\chi|\left(\frac{1}{2} m_{i0}\overline{V^2}\right) = \frac{1}{2}|m_g|\overline{V^2}$. We have put $|\chi|$ because $k_B T$ is always positive, and χ can be positive and negative. Thus, we can write that

$$E_{thermal} = \tfrac{1}{2}|m_g|\overline{V^2} = |\chi|\left(\tfrac{1}{2} m_{i0}\overline{V^2}\right) = |\chi| k_B T \qquad (5)$$

Only for $\chi = 1$ the expression of $E_{thermal}$ reduces to $k_B T$.

In the derivation of the Rayleigh-Jeans law, the assumption that $E_{thermal} = k_B T$, and that each radiation mode can have *any* energy E led to a wrong expression for the electromagnetic radiation emitted by a black body in thermal equilibrium at a definite temperature, i.e., Since the continuous Boltzmann probability distribution shows that

$$P(E) \propto \exp\left(\frac{-E}{E_{thermal}}\right) = P(E) \propto \exp\left(\frac{-E}{k_B T}\right) \qquad (6)$$

One can conclude that the average energy per mode is

$$\langle E\rangle = \frac{\int_0^\infty E P(E)\,dE}{\int_0^\infty P(E)\,dE} = k_B T \qquad (7)$$

This result was later corrected for Planck, which postulated that the mode energies are not continuously distributed, but rather they are *quantized* and given by $E = nhf$, $n = 1,2,3,\ldots$, where n is the number of photons in that mode. Thus

$$P(E) = P(nhf) \propto \exp\left(\frac{-nhf}{k_B T}\right) \qquad (8)$$

and the average energy per mode can be calculated assuming over only the discrete energies permitted instead integrating over all energies, i.e.,

$$\langle E\rangle = \frac{\sum_{n=0}^{\infty} nhf\ P(nhf)}{\sum_{n=0}^{\infty} P(nhf)} = \frac{\sum_{n=0}^{\infty} nhf\ \exp\left(\frac{-nhf}{k_B T}\right)}{\sum_{n=0}^{\infty} \exp\left(\frac{-nhf}{k_B T}\right)}$$

whose result is

$$\langle E\rangle = \frac{hf}{e^{\left(\frac{hf}{k_B T}\right)} - 1}$$

or

$$\frac{\langle E\rangle}{k_B T} = \frac{hf/k_B T}{e^{\left(\frac{hf}{k_B T}\right)} - 1} \qquad (9)$$

Note that only for $hf \ll k_B T$, this expression reduces to $\langle E\rangle = k_B T$ (the classical assumption that breaks down at high frequencies). Equation (9) is therefore the *quantum* correction factor, which transforms the Rayleigh-Jeans equation $\left(2kTf^2/c^2\right)$ into the Planck's equation, i.e.,

$$I(f,T) = \frac{2kTf^2}{c^2}\left[\frac{hf/k_B T}{e^{\left(\frac{hf}{k_B T}\right)} - 1}\right] = \frac{2hf^3}{c^2}\frac{1}{e^{\frac{hf}{k_B T}} - 1} \qquad (10)$$

However, in the derivation of the Planck's law the wrong assumption that $E_{thermal} = k_B T$ was maintained. Now, Eq. (5) tells us that we must replace $k_B T$ for $|\chi| k_B T$. Then the Planck's equation must be rewritten as

$$I(f,T) = \frac{2hf^3}{c^2}\frac{1}{e^{\frac{hf}{|\chi| k_B T}} - 1} \qquad (11)$$

$I(f,T)$ is the amount of energy per unit surface area per unit time per unit solid angle emitted at a frequency f by a black body at temperature *T*.

Starting from Eq. (11) we can write the expression of the power density D (watts/m^2) for emitted radiation

$$D = \frac{P}{A} = \int_0^\infty I(f,T)\,df \int d\Omega \qquad (12)$$

To derive the Stefan–Boltzmann law, we must integrate Ω over the half-sphere and integrate f from 0 to ∞. Furthermore, because black bodies are *Lambertian* (i.e. they obey Lambert's cosine law), the intensity observed along the sphere will be the actual intensity times the cosine of the zenith angle φ, and in spherical coordinates, $d\Omega = \sin\varphi \; d\varphi \; d\theta$. Thus,

$$D = \frac{P}{A} = \int_0^{\infty} I(f,T)df \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \cos\varphi \sin\varphi d\varphi =$$

$$= \pi \int_0^{\infty} I(f,T)df = \frac{2\pi\, h}{c^2} \int_0^{\infty} \frac{f^3}{e^{\frac{hf}{|\chi| k_B T}} - 1} df \qquad (13)$$

Then, by making

$$u = \frac{hf}{|\chi| k_B T}$$

$$du = \frac{h}{|\chi| k_B T} df$$

Then Eq. (13) gives

$$D = \chi^4 \frac{2\pi\, h}{c^2} \left(\frac{k_B T}{h} \right)^4 \underbrace{\int_0^{\infty} \frac{u^3}{e^u - 1} du}_{\pi^4/15}$$

The integral above can be done in several ways. The result is, $\pi^4/15$ [4]. Thus, we get

$$D = \chi^4 \left(\frac{2\pi^5 k_B^4}{15 c^2 h^3} \right) T^4 = \chi^4 \sigma_B T^4 \qquad (14)$$

where $\sigma_B = 5.67 \times 10^{-8} \, watts / m^2 \, {}^{\circ}K^4$ is the *Stefan-Boltzmann's constant*.

Note that, for $\chi = 1$ (gravitational mass equal to inertial mass), Eq. (14) reduces to the well-known *Stefan-Boltzmann's* equation. However, *at high energy density conditions* the factor χ^4 can become much greater than 1 (See Eqs. (2) and (3)). This divergence, which is related to the correlation between gravitational mass and inertial mass, can explain the increment of 10 to 20 times in the recently observed emissivity. In this case, we would have $\chi^4 = 10 \;\; to \;\; 20 \rightarrow \chi \cong -2$.

If we put $\chi \cong -2$ and $W = B^2/\mu_0$ into Eq. (2) the result is

$$B = \sqrt{\frac{\sqrt{21}\, \mu_0 \rho c^2}{2 n_r}} = 5.1 \times 10^5 \sqrt{\rho / n_r} \qquad (15)$$

For example, in the case of a intergalactic plasma with $\rho << 1 kg.m^{-3}$ and $n_r \cong 1$, Eq. (15) gives

$$B << 5.3 \times 10^5 \, Tesla \qquad (16)$$

Magnetic fields with these intensities are relatively common in the Universe, and even much more intense as for example, the magnetic field of *neutron stars* (10^6 to 10^8 *Tesla*) and of the *magnetars* (10^8 to 10^{11} *Tesla*) [5, 6, 7].

In the case of *Thermal radiation*, considering Eq. (14), we can put Eq. (3) in the following form

$$\chi = \left\{ 1 - 2 \left[\sqrt{1 + \left(\frac{\chi^4 \sigma_B T^4 \, n_r^2}{\rho c^3} \right)^2} - 1 \right] \right\} \qquad (17)$$

For $\chi \cong -2$, we get

$$T = 9.08 \times 10^7 \sqrt[4]{\frac{\rho}{n_r^2}} \qquad (18)$$

For $\rho << 1 kg.m^{-3}$ and $n_r \cong 1$ Eq. (18) gives

$$T << 9.08 \times 10^7 K \qquad (19)$$

Temperatures $T \approx 10^6 K$ are relatively common in the Universe (close to a star, for example).

Thus, we can conclude that there are several ways to produce $\chi \cong -2$ in an intergalactic plasma (or interstellar plasma) in the Universe.

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