Thin Spherical Matter Shell as Gravity Theory Filter

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Introduction

In GR (general relativity) a static thin solid and uniform spherical matter shell is the source of an external SM (Schwarzschild metric) and interior flat MM (Minkowski metric), plus a shell wall transitional metric not needing consideration here. It will be shown in part 1 the above implies a physically absurd disappearing dependence on gravitational potential for just the radial spatial metric component, in crossing the shell wall. In part 2 mathematical inconsistency is found. In the gravitationally small regime, treating each element of shell mass as an independent point source of SM and linearly summing over all such contributions ought to but manifestly does not yield an interior spatial metric consistent with the usual matching scheme of part 1. A conformally flat exterior metric as necessary cure is discussed in part 3.

1: GR patching prescription = anomaly A

In the weak gravity limit, any viable theory of gravity must reduce to Newtonian gravity. For which interior to the wall of a rigid spherical shell of matter of mass *M* and radius *r* = *R* (assumed for simplicity here to have an infinitesimal wall thickness *dR*) a Newtonian equipotential region with $\boldsymbol{g} = -\nabla\phi = 0$ applies:

$$\phi = -GM/r \to -GM/R\big|_{r\le R} = \text{constant for all}\, r \le R \quad (1\text{-}1)$$

In GR Birkhoff's theorem similarly demands an interior equipotential region [1]. To find the consequences of that, start with expression for standard SM line element in polar form:

$$ds^2 = c^2 d\tau^2 = (1 - r_s/r)c^2dt^2 - (1 - r_s/r)^{-1}dr^2 - r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right) \quad (1\text{-}2),\ \text{or}$$

$$ds^2 = c^2 d\tau^2 = g_{tt}c^2dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2 \quad (1\text{-}3),\quad \text{where}$$

$$r_s = 2GM/c^2 = -2\phi r/c^2 \quad (1\text{-}4)$$

is the Schwarzschild radius for a spherically symmetric source of mass *M*. Holding respectively *r*,*t*, constant and equating the relevant terms in (1-3) yields the well-known relations

$$d\tau/dt = \sqrt{g_{tt}} \quad (1\text{-}5)$$ (coordinate determined clock-rate slows further down a potential well),

$$dr/ds = 1\big/\sqrt{-g_{rr}} = \sqrt{g_{tt}} \quad (1\text{-}6)$$ (radially oriented ruler coordinate length shrinks further down the well).

As inferred by a stationary observer at spatial infinity (where nominally $\phi = 0$), coordinate clock-rate and radial length scale are everywhere in the SM region equally affected by potential ϕ according to the factor

$$\sqrt{g_{tt}} = \sqrt{1 - r_s/r} = \sqrt{1 + 2\phi/c^2} = \sqrt{1 - 2GM/(rc^2)} \quad (1\text{-}7)$$

Yet transverse components $g_{\theta\theta}$, $g_{\varphi\varphi}$ are entirely unaffected. A ruler transversely oriented has an invariant coordinate length. While such coordinate determined spatial anisotropy is not directly observable *locally*, it presents a consistency issue. Not remedied via recourse to the so-called Isotropic form of SM, as will be shown in part3. Standard SM forces that patching anisotropic exterior to flat and isotropic MM interior region can only be 'sensibly' satisfied one way. Given $g_{\theta\theta}$, $g_{\varphi\varphi}$ are *everywhere* unaffected by ϕ, then $\left|1\big/\sqrt{-g_{rr}}\right| \equiv \left|\sqrt{g_{tt}}\right| \to 1 (r < R)$. Abruptly losing all dependence on Newtonian potential ϕ in crossing from just exterior at $r = R^+$, to just interior at $r = R^-$ to the shell wall. A physically unjustifiable, selective magic disappearing act! A mathematical force-fit having unphysical consequences. For it's universally a given that according to (1-1) into (1-5), (1-7), redshifted clock-rate transitions essentially unchanged from exterior to interior. No mysterious break in potential dependence for clocks. (By contrast, vanishing of Newtonian $\boldsymbol{g} = -\nabla\phi \rightarrow 0\,(r < R)$ is completely consistent with **g**'s *invariant functional dependence on* ϕ *for all r*.)

2: Direct determination of interior spatial metric = worse anomaly B

Any physically reasonable non-linear situation must reduce arbitrarily close to linearity, in some lower parameter limit, or over a sufficiently narrow range. In GR it's a given that situation corresponds to the so-called weak gravity regime typically characterised by $r_s/R \ll 1$. Apply to Fig.1 where for concreteness the thin shell is say a steel globe atlas (minus support stand), radius *R* = 10cm, mass *m* = *M* = 200gm. This yields $r_s/R \simeq 3.0\times 10^{-26}$, weak gravity indeed! Arguing that a direct summation is invalid owing to not being a proper solution of the EFE's is in this context just silly. Any fractional error *in the final result* will reasonably be bounded to around the above value! As per intro then, treat each element of shell mass *dM* as an independent gravitating point source of SM - according to (1-3), (1-6), (1-7).

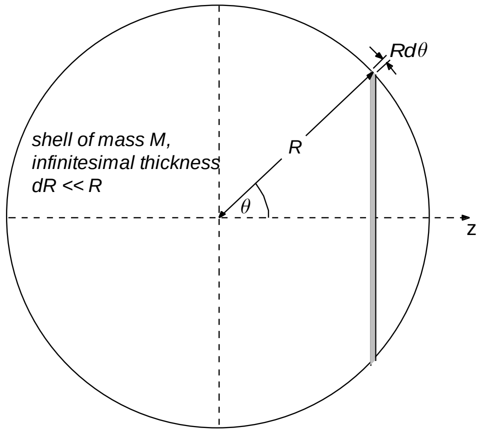

Fig. 1

Mathematically it is particularly easy to choose as target field point the shell centre *r* = 0 at which to evaluate a given spatial metric component, which as per Fig.1 will be taken as along axis **z**. Symmetry guaranteed Isotropy at *r* = 0 then demands the net result will equally apply along any other axis.

Using that in the gravity-free limit, $\sqrt{g_{tt}}\big|_{m\to 0} = 1$, (1-6) can be recast as a fictitious mechanical 'strain' ε_{rr}

$$|\varepsilon_{rr}| = \left(\sqrt{g_{tt}}\big|_{m=0} - \sqrt{g_{tt}}\big|_{m=M}\right)\Big/\left(\sqrt{g_{tt}}\big|_{m=0}\right) = 1 - \sqrt{g_{tt}} \qquad \text{(2-1)}$$

No actual mechanical strain is implied - it's a useful representation of *coordinate determined* radial metric contraction. Of course both transverse 'strains' $\varepsilon_{\theta\theta}, \varepsilon_{\varphi\varphi} = 0$, virtually by definition. Expressing radial metric component as an 'at a point' purely uniaxial strain will be useful. First consider the case along axis **z** of Fig.1, just exterior to shell wall at $r = R^+$. Shell mass *M* can in that instance be treated as concentrated at the centre *r* = 0, so $|\varepsilon_{zz}| \equiv |\varepsilon_{rr}| = 1 - \sqrt{g_{tt}}$ is a maximum that from (1-7), (2-1) is given by

$$|\varepsilon_{zz}|_{r=R^+} = 1 - \sqrt{g_{tt}}\big|_{r=R^+} = 1 - \sqrt{1 - 2GM/(Rc^2)} \qquad \text{(2-2)}$$

That will be taken as a base value against which the computed interior metric 'strain' at *r* = 0, $|\varepsilon_{zz}|_{r=0}$, will be most easily expressed as a fraction $|\varepsilon^*_{zz}|_{r=0} = |\varepsilon_{zz}|_{r=0} \big/ |\varepsilon_{zz}|_{r=R^+}$. All shell mass elements have the same spatial displacement *R* from centre, just as for the effective point mass in calculation of (2-2). Now *if* metric isotropy applied, angular position θ of any contributing mass element is immaterial and a simple scalar summing over all mass elements would yield $|\varepsilon^*_{zz}|_{r=0} = 1$, implying an isotropic compressive strain at *r* = 0 equal in amplitude to the radial strain of (2-2). In conflict with $|\varepsilon^*_{zz}|_{r=0} = 0$ enforced by the patching procedure in part 1. Here though SM anisotropy requires a slightly more involved integration procedure, which still won't deliver $\left|\varepsilon_{zz}^{*}\right|_{r=0} = 0$! Consider the shown shell annular element, of radius $R\sin\theta$ thus area A_a and mass M_a given by

$$A_a = 2\pi(R\sin\theta)Rd\theta = 2\pi R^2 \sin\theta d\theta \quad \text{(2-3)},$$

$$M_a = \sigma A_a = (M/(4\pi R^2))2\pi R^2 \sin\theta d\theta = \tfrac{1}{2} M \sin\theta d\theta \quad \text{(2-4)},$$

where $\sigma = M/\left(4\pi R^2\right)$ is shell area mass density. All annulus sub-elements dM_a as point SM sources at distance $r = R$ from centre as target point, project at angle θ to axis **z** differential contributions $d\varepsilon_{rr}$ at $r = 0$. An initially s*calar* summing over the annular element, dividing by the maximal base value per eqn. (2-2), yields a nominal fractional contribution to $\left|\varepsilon_{zz}^{*}\right|_{r=0}$ that is just the mass ratio

$$M_a/M = \tfrac{1}{2}\sin\theta d\theta \quad \text{(2-5)} .$$

Summing in turn over all annuli would clearly, as mentioned earlier, give a unity value. Assuming an *isotropic* spatial metric that is. Each elemental $d\varepsilon_{rr}$ is in SM though not owing to one orthogonal component of an isotropic metric but is a locally *uniaxial* strain acting along each relevant ***R*** axis. By axial symmetry only the resolved **z** axis components $d\varepsilon_{zz}$ survive summing over the annulus. Application of the pertinent mechanics resolved-stress/strain relations [2] yields a resolved $d\varepsilon_{zz}$ for each $d\varepsilon_{rr}$ according to

$$\left|d\varepsilon_{zz}\right| = \left|d\varepsilon_{rr}\right|\cos^2\theta \quad \text{(2-6)}$$

Combining (2-5), (2-6), angular integration over all annuli gives the total **z** axis *relative* strain at *r* = 0 as

$$\left|\varepsilon_{zz}^{*}\right|_{r=0} = \tfrac{1}{2}\int_0^{\pi} \sin\theta\cos^2\theta d\theta = \tfrac{1}{4}\int_0^{\pi} \sin 2\theta \cos\theta d\theta = \tfrac{1}{8}\int_0^{\pi} (\sin\theta + \sin 3\theta) d\theta \quad \text{(2-7)}$$

This readily evaluates 'by hand' to

$$\left|\varepsilon_{zz}^{*}\right|_{r=0} = \tfrac{1}{8}\left[-\cos\theta - \tfrac{1}{3}\cos 3\theta\right]_0^{\pi} = \tfrac{1}{8}\left(\left[1+\tfrac{1}{3}\right] - \left[-1-\tfrac{1}{3}\right]\right) = \tfrac{1}{3} \quad \text{(2-8)}$$

Direct evaluation thus predicts a coordinate determined isotropic compressive 'strain' at the shell centre whose relative magnitude is ⅓ that of the maximal radial 'strain' at the shell wall outer surface. Just by inspection of Fig.1 that value has to be about right. *Certainly not zero as demanded by the patching prescription of section 1*! A more involved treatment for arbitrary internal location would likely predict non-flatness. A pointless exercise, given the simplest case studied here has already undermined validity of SM.

3: Intrinsically isotropic aka conformally flat exterior metric as cure

3a: What won't work

It may be argued the above two issues are only apparent as evidenced by choosing rather than standard SM, the *notionally* physically equivalent ISM (isotropic SM) [3]:

$$ds^2 = c^2 d\tau^2 = \frac{(1 - r_s/(4r_1))^2}{(1 + r_s/(4r_1))^2} c^2 dt^2 - (1 + r_s/(4r_1))^4 \left(dr_1^2 + r_1^2\left(d\theta^2 + sin^2\theta d\varphi^2\right)\right) \quad \text{(3-1)}$$

$$r_1 = \tfrac{1}{2} r - \tfrac{1}{4} r_s + \sqrt{\tfrac{1}{4} r(r - r_s)} \quad \text{(3-2)}$$

Which if applied in part 1 permits a shell transition without any break in ϕ dependency for any of the transformed metric components. And in part 2 yields a ‘genuinely’ scalar addition as per (2-5), eliminating the $\cos^2\theta$ factor in (2-6), leading to a unity value in (2-8). Thus an interior metric consistent with the ‘new’ patching result of part 1. While *formally* that would follow, is it really the case SM and ISM represent the same physical metric? For SM, proper volume between two closely spaced concentric spherical surfaces - of given area difference - and centred about a central point mass *M*, is greater by the factor $1/\sqrt{g_{tt}}$ than in the Euclidean gravity-free limit *M*=0. Owing to the proper radial spacing being greater than for the Euclidean case. Similarly between concentric great circles (or arbitrarily small sectors from such) – proper radial gap again greater by the factor $1/\sqrt{g_{tt}}$. Such non-Euclidean geometry effects must vanish within the flat MM shell interior. So an (in-principle!) locally physically determinable transition exists for SM.

In ISM coordinate measure though the differential area-volume relation is clearly seen from (3-1) to be *by construction* locally Euclidean i.e. conformally flat. Evidently equivalent to having zero Weyl curvature. Hardly consistent with the usually claimed physical equivalence of SM and ISM. Amazing that for so long ISM – a kind of half-way house to true isotropy, has been taken as just SM with a new coat of paint. At any rate the anomalies found above using standard SM are genuine and should be dealt with as such.

3b: What will work

A necessary *true* cure is an *intrinsically* isotropic aka genuinely conformally flat exterior metric. One maybe viable gravity theory consistent with that is YG (Yilmaz gravity) [4] (whereas say Nordstrom’s 1912 theory, also conformally flat, is elsewise non-viable). The YG equivalent line element to that for SM is [5]

$$ds^2 = c^2 d\tau^2 = e^{-2\phi}c^2dt^2 - e^{2\phi}\left(dr^2 + r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right)\right) \quad (3\text{-}3),$$

with ϕ the Newtonian potential of (1-1). Purely by inspection of (3-3) not only is anomaly A automatically avoided in respect of shell spatial metric match, but further *all* 4 metric components keep pace in *all* regions, if temporal component is expressed as clock-rate. Unlike say for the assumed gravity-free interior spatial values that GR’s SM-to-MM patching procedure insists on. Anomaly B is also avoided - each shell point-mass element is in YG the source of an isotropic thus effectively scalar spatial metric field. Summing therefore in identical manner to the anomaly-free temporal component. No nasty shell-shock surprises using YG.

At least one site currently and perhaps maliciously references in a comparison table only the earlier pre 1974 scalar gravity versions of YG dealing just with static gravity [6]. Sort of like comparing 1911 GR with later rival theories in order to ‘disprove’ 1915 GR. According to proponents of YG it passes all the usual observational ‘acid tests’. Yet interestingly GR and YG are opposites in a sense. In the vacuum region, SM hence GR has zero Ricci curvature and non-zero Weyl curvature, while the reverse is true of YG.

It’s not the intention here to plug YG as necessarily *the* classical gravity theory, but to point it out as a rare example readily passing the thin shell filter test. Yilmaz possibly erred in one or more aspects of his theory [7]. On the other hand, the underlying assumptions of such critiques may themselves need critiquing. Regardless, conformal flatness as an integral component of that theory **is** surely correct.

4: Conclusion and discussion

SM pathology has been shown in two distinct but related ways. Requiring both a physically absurd selective break in functional dependence on Newtonian potential ϕ , and yielding self-contradictory results for interior spatial metric. Such anomalies *should* be GR’s death sentence - given SM is touted as the unique solution to the EFE’s for a static spherically symmetric mass distribution. Normally observational evidence is king, but when a theory claims the equivalent of 2+2=5, logic *should* take over as final arbiter. To try and counter-argue that e.g. coordinate spatial anisotropy is not a relativistic invariant, is unobservable locally 'at least for small field curvature' etc. thus supposedly 'physically irrelevant', just ducks a fatal key issue. Internal consistency, or lack thereof. Which unfortunately may not count for much arrayed against the string of so far observational successes and massive incumbency enjoyed by GR.

SM and thus GR, predicts its own downfall in entirely different and much simpler, classical ways than e.g. seemingly perennial 'singularity' or 'black hole information paradox' issues currently fashionable in quantum gravity circles. Why now glaringly obvious elementary failings weren't uncovered at around the outset of GR is perhaps a mystery for historians to ponder. One conceivable reason was emphasis on an ever accreting edifice of sophisticated mathematical theorems, to the exclusion of Einstein's earlier gedanken experiment approach emphasising physical intuition.

Whichever theory of gravity finally succeeds, the admittedly low-level arguments given here strongly imply it **must**, to avoid logical inconsistencies, incorporate conformal flatness at least in the classical regime.

References

[1] http://en.wikipedia.org/w/index.php?title=Birkhoff%27s_theorem_%28relativity%29&oldid=599313098 (under 'Implications')

[2] http://www.mae.ncsu.edu/zhu/courses/mae314/lecture/Lecture2_Stress-Strain.pdf (slide 28, - 'strain' will follow the same resolving relation as for uniaxial stress considered there, as here there is no finite valued, material dependent Poisson's ratio to complicate matters.)

[3] http://en.wikipedia.org/w/index.php?title=Schwarzschild_metric&oldid=606158991 (sec 4)

[4] E.g. http://www.powershow.com/view/1bbc8-ZjhlZ/P1246341516SeoJH_flash_ppt_presentation (Note downloadable version is corrupted, and for legibility hit full-screen tab.)

[5] http://iopscience.iop.org/0004-637X/515/1/365/36337.text.html (adapted from eqns (1), (2) there)

[6] http://en.wikipedia.org/w/index.php?title=Alternatives_to_general_relativity&oldid=616572453 (under Scalar Field theories, sec 7.1)

[7] https://arxiv.org/abs/0705.0080
