The Lorentz Transformations and the Scale Principle

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Earlier this year I wrote a paper entitled Scale Factors and the Scale Principle. In this paper I formulated a Meta Law (Scale Principle or Scale Law) which describes a number of fundamental laws. This paper provides the derivation of the Lorentz transformations and shows that these transformations obey the Scale Law.

Keywords*: Lorentz transformations, the special theory of relativity, Cartesian coordinates, reference system.*

1. Introduction

The Lorentz transformations were derived before Einstein's theory of special relativity by the Dutch physicist Hendrix Lorentz and they are in agreement with Einstein's theory.

The transformations are equations that relate the measurements of space and time carried out by two observers which move at a constant speed with respect to the other.

2. The Scale Principle or Scale Law (Summary)

The following table summarizes two forms of the Scale Law: the Meta form and the explicit form

(1)

Meta Law: *Scale Principle* or *Scale Law*	
(1a) Meta Form (Meta Quantities)	(1b) Explicit Form (ratio, exponents and scale factor)
$M_1 \Re \; S M_2$ M_1 = dimensionless Meta Quantity 1 M_2 = dimensionless Meta Quantity 2 S = dimensionless Meta scale factor $\Re$ = Meta Relationship Type	$\left(\frac{Q_1}{Q_2}\right)^n \left[< \middle\| \leq \middle\| = \middle\| \geq \middle\| > \right] S \left(\frac{Q_3}{Q_4}\right)^m$ (See details below)

The above symbols stand for

a) Quantities:
(i) Q_1, Q_2, Q_3 and Q_4 are physical quantities of identical dimension (such as Length, Time, Mass, Temperature, etc), or
ii) Q_1 and Q_2 are physical quantities of dimension 1 or dimensionless constants while Q_3 and Q_4 are physical quantities of dimension 2 or dimensionless constants. However, if Q_1 and Q_2 are dimensionless constants then Q_3 and Q_4 must have dimensions and viceversa.
(e.g.: Q_1 and Q_2 could be quantities of Mass while Q_3 and Q_4 could be quantities of Length).
The physical quantities can be variables (including differentials, derivatives, Laplacians, divergence, integrals, etc.), constants, dimensionless constants, any mathematical operation between the previous quantities, etc.

b) Relationship type: The relationship is one of five possibilities: **less than or equal to** inequation ($\leq$), or **less than** inequation (<), or **equal to** - equation (=), or a **greater than or equal to** inequation ($\geq$), or a **greater than** inequation (>).

***c) Scale factor**: S* is a dimensionless *scale factor*. This factor could be a real number, a complex number, a real function or a complex function (strictly speaking real numbers are a particular case of complex numbers). The scale factor could have more than one value for the same relationship. In other words a scale factor can be a quantum number.

d) Exponents: *n* and *m* are integer exponents: 0, 1, 2, 3, …

Some examples are:
example 1: $n = 0$ and $m = 1$;
example 2: $n = 0$ and $m = 2$;
example 3: $n = 1$ and $m = 0$;
example 4: $n = 1$ and $m = 1$;
example 5: $n = 1$ and $m = 2$;
example 6: $n = 2$ and $m = 0$;
example 7: $n = 2$ and $m = 1$;
It is worthy to remark that:
i) The exponents, n and m, cannot be both zero in the same relationship.
ii) The number n is the exponent of both Q_1 and Q_2 while the number m is the exponent of both Q_3 and Q_4 regardless on how we express the equation or inequation (1c). This means that the exponents will not change when we express the relationship in a mathematically equivalent form such as

$$\left(\frac{Q_4}{Q_3}\right)^m \left[< \middle| \leq \middle| = \middle| \geq \middle| > \right] S \left(\frac{Q_2}{Q_1}\right)^n$$

iii) So far these integers are less than 3. However we leave the options open as we don't know whether we shall find higher exponents in the future.

The Scale Law describes fundamental laws such as the Heisenberg's uncertainty principle, the black hole entropy, the fine structure constant, Einstein's relativistic energy equation, the formula for the Schwazschild radius, the Bohr Postulate, the De Broglie wavelength-momentum relationship, Newton's law of universal gravitation, the Schrödinger equation, the Friedmann equation, and maybe many others.

References [1] and [2] provide a more complete explanation on the Scale Law.

3. Derivation of the Lorentz Transformations

From the Scale Law we shall derive the following Lorentz's transformation's equations

3.1) Space transformations

$x' = f(x,\ t)$	*Direct transformation*
$x\ = f(x',\ t')$	*Reverse transformation*

3.2) Time Transformations

$t' = f(x,\ t)$	*Direct transformation*
$t\ = f(x',\ t')$	*Reverse transformation*

3.1 Space Transformations

The data we have is what we call "**all the necessary information**" for this particular case:

i) the *Galilean transformations*, and

ii) For low speeds, the *Lorentz transformations* (the transformation we are looking for) have to yield the *Galilean transformation,* and
iii) Einstein's postulate*: "Any ray of light moves in the "stationary" system of co-ordinates with the determined velocity c, whether the ray is emitted by a stationary or by a moving body"* [3].

This information is necessary and sufficient to find the Lorentz transformation through the Scale Law. Let us see how.

Let us imagine two Cartesian coordinate systems: the *x, y, z, t* reference system (or system A); and the *x', y', z', t'* reference system (or system B). . We assume that reference system B moves with respect to system A at a constant speed denoted by *v* towards the direction of the the positive x-axis. We also imagine that we have a body (called X) moving along the x axis at constant speed. An observer from system A measures the speed of this body as w_x, while an observer from system B measures $w_x^{'}$ for the same body. The body can also be a ray of light or photon.

We don't need the scale table in this specific case because the transformations' equations are mathematically very simple (but not necessarily the physical concepts behind them). Thus, according to the Scale Law, and assuming that system B moves in the direction of the positive x-axis of system A, we can write

$$\frac{x'}{x - vt} = S \qquad \text{(Assumption)} \qquad (2)$$

Where *S* is a dimensionless scale factor whose value we intend to derive.
The corresponding equation for the *x* co-ordinate of the inverse transformation is

$$\frac{x}{x' + vt'} = S \qquad \text{(Assumption)} \qquad (3)$$

It is worthy to remark the minus sign of the denominator of equation (2) was replaced by a plus sign in equation (3) to account for the fact that, for an observer from system B, system A moves towards the direction of the negative x'-axis of system B .

We can write the above relationship as follows

$$x' = S\left(x - vt\right) \qquad (4)$$

$$x = S\left(x' + vt'\right) \qquad (5)$$

To find the expression for the scale factor we start from equation (4) where we substitute *x* with the second side of equation (5), this gives

$$x = S\left[S\left(x - vt\right) + vt'\right] \qquad (6)$$

Solving for *t'*

$$\frac{x}{S} = S\left(x - vt\right) + vt' \qquad (7)$$

$$t' = \frac{1}{S}\frac{x}{v} - S\frac{x}{v} + St = S\left(\frac{1}{S^2}\frac{x}{v} - \frac{x}{v} + t\right) \qquad (8)$$

$$t' = S\left[t + \left(\frac{1}{S^2} - 1\right)\frac{x}{v}\right] \qquad (9)$$

Now we differentiate with respect to time *t*

$$\frac{dt'}{dt} = S\left[1 + \left(\frac{1}{S^2} - 1\right)\frac{1}{v}\frac{dx}{dt}\right] \qquad (10)$$

But *dx/dt* is the velocity w_x of a body X with respect to the system A

$$w_x \equiv \frac{dx}{dt} \qquad (11)$$

Thus equation (12) can be expressed as

$$\frac{dt'}{dt} = S\left[1 + \left(\frac{1}{S^2} - 1\right)\frac{w_x}{v}\right] \qquad (12)$$

Now we differentiate equation (4) with respect to *t'*

$$\frac{dx'}{dt'} = S\frac{dx}{dt'} - Sv\frac{dt}{dt'} \qquad (13)$$

Now we multiply the first term of the second side by *dt/dt*

$$\frac{dx'}{dt'} = S\frac{dx}{dt'}\frac{dt}{dt} - Sv\frac{dt}{dt'} \qquad (14)$$

But the velocity of the body X with respect to system B is $w_x^{'}$, which is given by

$$w_x^{'} \equiv \frac{dx'}{dt'} \qquad (15)$$

Thus substituting *dx'/dt'* and *dx/dt* in equation (14) with the second side of equations (11) and (15), respectively, yields

$$w_x^{'} = S\, w_x \frac{dt}{dt'} - S\, v \frac{dt}{dt'} = S\left(w_x - v\right)\frac{dt}{dt'} \tag{16}$$

$$\frac{dt'}{dt} = S\left(\frac{w_x - v}{w_x^{'}}\right) \tag{17}$$

Now eliminating dt'/dt from equations (12) and (17)

$$S\left[1 + \left(\frac{1}{S^2} - 1\right)\frac{w_x}{v}\right] = S\left(\frac{w_x - v}{w_x^{'}}\right) \tag{18}$$

$$1 + \left(\frac{1}{S^2} - 1\right)\frac{w_x}{v} = \frac{w_x - v}{w_x^{'}} \tag{19}$$

Now, we assume that body X is a photon. Then according to Einstein's postulate (iii) an observer in system A and an observer in system B shall measure the same velocity, *c*, for the photon. Mathematically this is expressed as

$$w_x = w_x^{'} = c \tag{20}$$

If we replace these two variables in equation (19) with the speed of light, *c*, we get

$$1 + \left(\frac{1}{S^2} - 1\right)\frac{c}{v} = \frac{c - v}{c} \tag{21}$$

Solving for *S* yields we easily obtain the equation for the scale factor

$$S(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{(Scale factor for the Lorentz transformations)} \tag{22}$$

In the literature, the scale factor is normally denoted by *γ (gamma)*. However. In the literature, there are no references as *gamma* being the scale factor of the transformation in the sense we gave it here.

Thus we found the scale factor for this transformation is a function of the speed, *v*, between the two reference systems. Substituting *S* in equations (2) and (3) with the value given by equation (22) we obtain the *x'* coordinate and the *x* coordinate of the Lorentz transformations, respectively

$$\frac{x'}{x - vt} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{(}x'\text{ coordinate's equation of the Lorentz transformations)} \tag{23}$$

$$\frac{x}{x'+vt'} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$ (x coordinate's equation of the Lorentz transformations) (24)

We can rewrite these equations, (24) and (25), in a more familiar way as follows

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$ (x' coordinate's equation of the Lorentz transformations) (25)

$$x = \frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$ (x coordinate's equation of the Lorentz transformations) (26)

Thus we have proved that both the *x'* coordinate equation and the *x* coordinate equation of the Lorentz transformations obey the Scale Law and we have also found the expression for the scale factor *S*. The Scale Law works perfectly well in this case because we have all the necessary information: i), ii) and iii) about the phenomenon under study.

3.2 Time Transformations

From equation (5)

$$\frac{x}{S} = x' + vt' \qquad (27)$$

$$\frac{x}{S} - vt' = x' \qquad (28)$$

Substituting x' with the second side of equation (4) we get

$$\frac{x}{S} - vt' = S\left(x - vt\right) \qquad (29)$$

Multiplying by *S* both sides we get

$$x - S\,vt' = S^2\left(x - vt\right) \qquad (30)$$

$$-\,S\,vt' = S^2\left(x - vt\right) - x = S^2x - S^2vt - x = \left(S^2 - 1\right)x - S^2vt \qquad (31)$$

$$S\,vt' = S^2\,vt - \left(S^2 - 1\right)x \qquad (32)$$

Dividing by Sv both sides

$$t' = \frac{S^2 vt}{S v} - \left(S^2 - 1\right)\frac{x}{Sv} = St - \left(\frac{S^2 - 1}{Sv}\right)x \tag{33}$$

$$t' = S\left[t - \left(\frac{S^2 - 1}{S^2}\right)\frac{x}{v}\right] \tag{34}$$

Now we work with the factor $\left(\frac{S^2 - 1}{S^2}\right)$

$$\left(\frac{S^2 - 1}{S^2}\right) = 1 - \frac{1}{S^2} \tag{35}$$

If we replace S in the above equation by the value of S given by equation (22) we obtain

$$\left(\frac{S^2 - 1}{S^2}\right) = 1 - \frac{1}{\left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\right)^2} \tag{36}$$

$$\left(\frac{S^2 - 1}{S^2}\right) = 1 - \left(1 - \frac{v^2}{c^2}\right) \tag{37}$$

$$\left(\frac{S^2 - 1}{S^2}\right) = \frac{v^2}{c^2} \tag{38}$$

Now equation (34) can be expressed as

$$t' = S\left[t - \frac{v^2}{c^2}\frac{x}{v}\right] \tag{39}$$

$$t' = S\left(t - \frac{v\,x}{c^2}\right) \tag{40}$$

In accordance with the Scale Law we can write this equation as follows

$$\frac{t'}{t - \frac{v\,x}{c^2}} = S \qquad \text{(t' coordinate's equation of the Lorentz transformations)} \qquad (41)$$

Equation (40) can be written in a more familiar form as follows

$$t' = \frac{t + \frac{v\,x}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{(t' coordinate's equation of the Lorentz transformations)} \qquad (42)$$

Similarly we could prove that

$$t = S\left(t' - \frac{v\,x'}{c^2}\right) \qquad (43)$$

In accordance with the Scale Law we can write this equation as follows

$$\frac{t}{t' + \frac{v\,x'}{c^2}} = S \qquad \text{(t coordinate's equation of the Lorentz transformations)} \qquad (44)$$

Equation (40) can be written in a more familiar form as follows

$$t = \frac{t' + \frac{v\,x'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{(t coordinate's equation of the Lorentz transformations)} \qquad (45)$$

Thus the derivations of the time transformations are complete.

4. Conclusions

This paper shows that the Lorentz transformations obey the present formulation: the Scale Law. More importantly, the scale factor for the transformation was found from the previous knowledge we had: a) the *Galilean transformations* were known before the formulation of the theory of relativity; b) the invariance of the speed of light was also known before the formulation of Einstein's theory (Michelson-Morley experiments).

In summary, the following table shows the Lorentz transformations we derived in the previous sections

Transformation Type	Lorentz transformations according to the Scale Law	Lorentz transformations
Space transformations	$\frac{x'}{x - vt} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ $y' = y$ $z' = z$	$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$ $y' = y$ $z' = z$
Time transformation	$\frac{t'}{t - \frac{vx}{c^2}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$	$t' = \frac{t + \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$
Inverse space Transformations	$\frac{x}{x' + vt'} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ $y = y'$ $z = z'$	$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$ $y = y'$ $z = z'$
Inverse time Transformation	$\frac{t}{t' + \frac{vx'}{c^2}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$	$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$

Table 1: This table shows the Lorentz transformations

By observing this table it seems that the Scale Law is just a different way of writing a natural law such as the *Lorentz transformations*. However this is not the case for the following reasons:

a) ***The Scale Law was originally a Quantum Mechanical Formulation.***

The Scale Law was originally a *quantum mechanical formulation* that, later on, was extended to include both special and general relativity (including the following laws: Einstein's relativistic energy, the Lorentz transformations and the Friedmann equation.)

Because special relativity is not explained by any quantum mechanical model, there is not known way of expressing the Lorentz transformations in terms of Planck units. However this does not means that the Lorentz transformations do not obey the Scale Law. On the contrary, it means that all we need to do is to rearrange the normal equations as we did above to show them in their fundamental form. Thus, there is much less work to do in cases like this one.

b) ***In General the Fundamental Form of the Equation is Different to the Corresponding Normal Form.***

For example when we apply the Scale Law to find the black hole entropy the corresponding equations turned out to be:

	Fundamental form of the law or constant according to the Scale Law (dimensionless equation)	How humans formulated this law or constant
Black Hole Entropy Formula	$\frac{R^2}{L_P^2} = \frac{1}{\pi}\frac{S_{BH}}{k_B}$	$S_{BH} = \frac{k_B c^3}{4\hbar G} A_H$

Where

R = black hole radius
L_P = Planck length
S_{BH} = Berkenstein-Hawking's black hole entropy
k_B = Boltzmann's constant

Thus according to this Meta Law, the law for the black hole entropy is explained in terms of the two ratios: R^2/L_P^2 and S_{BH}/k_B and a scale factor, *1/π* (for the derivation of the formula see [4]).

The conclusion is that, in general, the fundamental form for the equation (in this case the equation shown on the second column of the above table) looks very different to the corresponding normal form (in this case the Berkenstein-Hawking formula, third column of the above table). It is only when we carry out simple algebraic steps that we discover that the two forms are identical.

***c) The Interpretation is Different*.**

This new interpretation means that all laws of physics and the mysterious fine structure constant were spawned by Meta Laws. In other words the laws of physics exist because of the existence of Meta Laws.

This is a similar situation we encountered in the past with respect to the Lorentz transformations. Both Lorentz and Einstein found the same transformations but they interpreted these transformations differently. Lorentz derived these equations assuming a contraction of the objects that were moving through the ether, while Einstein derived the same equations assuming that the speed of light was invariant. Because of this conceptual difference Einstein's theory was a more general formulation.

One interpretation is to think we have hundreds of natural laws with no connection whatsoever among them. On the other hand, the other interpretation is to consider that behind these laws there is a common origin - a Meta Law - that unifies the laws of physics and provides a deeper understanding of nature. We think that the latter interpretation is the correct one.

REFERENCES

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