Try to re-prove 4-Color Theorem

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Abstract

The point of inside closed curve could not directly connect to the point of outside. If a point of outside closed curve directly connect to points on closed curve then the maximum number of color of map is 4 under only 3 conditions, and the end point of closed curve is located inside newly created closed curve. If another outside point is added then we could prevent maximum color from exceeding 4 by using to rearrange colors of existing points.

1. Introduction

4-Color problem is already proved, but we try to re-prove because of feeling sorry for using computer. We classify a map expressed in graph to point,line,face to and study the color for each type minimum. We study conditions of minimum color number when the point of inside closed curve and the point of outside closed curve with face are connected. And, we try to re-prove 4-Color Theorem without computer to use the property that the point of inside closed curve could not be connected to the point of outside closed curve.when the other point of outside closed curve are connected to the map connected in the above way,

2. Try to re-prove 4-Color Theorem

Definition 1. We define P_k(P is abbreviation of Point) as arbitrary k'th point of map and C_k (C is abbreviation of Color) as the color of this point. And, we define P_{cm} as m'th point of c'th color.

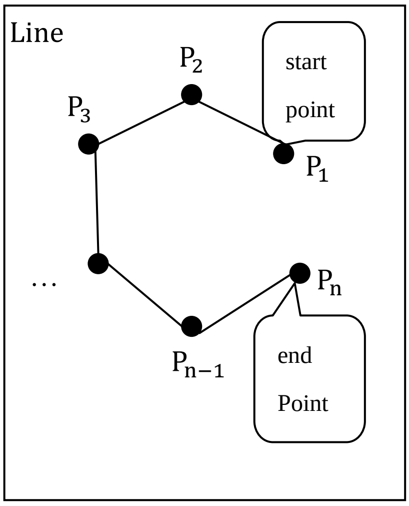
Definition 2. We define "connecting line" between P_k, P_m as the line between P_k and P_m when arbitrary 2 points P_k and P_m are connected. If any points are not present on "connecting line" between P_k, P_m, we define "direct connecting line" as this connection and transcribe $P_k \leftrightarrow P_m$. And, we define "direct connecting line" as "connecting line" of $P_k \leftrightarrow P_m$.

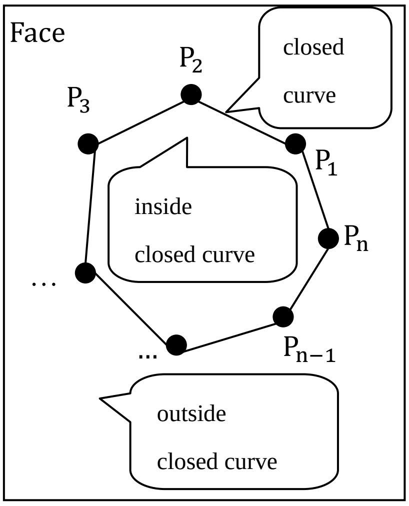
And, we transcribe $P_k \nleftrightarrow P_m$ if P_k and P_m are not connected, that is, if "connecting line" between P_k, P_m is not present,.

For reference, "connecting" includes $P_k \leftrightarrow P_i, P_i \leftrightarrow P_m$ in addition to "direct connecting".

Definition 3. We define "line" as all the points of a map are connected biting tail as shown in the following figure on the left, that is, if $P_{k-1} \leftrightarrow P_k, P_k \leftrightarrow P_{k+1}$,$P_{k-1} \nleftrightarrow P_{k+1}$ and $P_1 \nleftrightarrow P_n$ for arbitray point P_k. At this time, we define "start point" as P_1, "end point" as P_n.

We define "face" as start point and end point of a map is connected as shown in the following figure on the right. This means $P_1 \leftrightarrow P_n$ on "line". And, we define "closed curve" as all for the "direct connecting line" on "face", and we define "inside closed curve" as inner region of closed curve, and we define "outside closed curve" as outside region of closed curve.

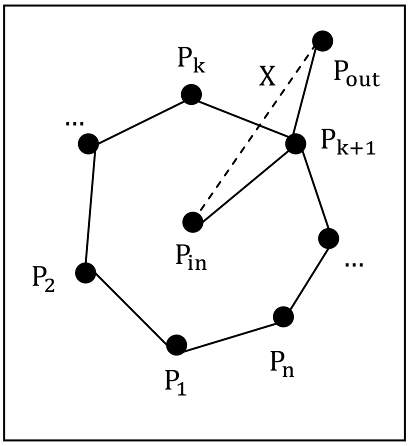




Definition 4. We define N(M) as the number of minimum colors in map M, if M(M is abbreviation of map) is a map.

Theorem 1. Connetion inside to outside of closed curve

The point of inside closed curve could not be directly connected to the point of outside closed curve, and these points should pass through a point on the closed curve to connct to each other.



[Figure 1. Conncting between inside and outside of face]

Proof 1. Let map M be face without P_{in}, P_{out} in Figure 1.

Let us assume that P_{in} the point of inside closed curve and P_{out} the point of outside closed curve could be directly connected. That is, let us assume that direct connecting line between P_{in}, P_{out} could be pass throuth direct connecting line between arbitrary P_k, P_{k+1}.

According to the upper assumption, direct connecting line between P_{in}, P_{out} split direct connecting line between P_k, P_{k+1}. This violates "direct connecting line could not be splitted" implicitly defined condition. And, if we insert P_m onto the intersection between the direct connecting line of P_k, P_{k+1} and the direct connecting line of P_{in}, P_{out} in order to overcome the direct connecting line is divided, a point which was not present in the existing map M is added. This violates "map M should not be changed" implicitly defined condition. Therefore, the upper assumption is wrong, so, P_{in} of inside closed curve and P_{out} of outside closed curve could not be connected.
And, if P_{in}, P_{out} is just connected, P_{in}, P_{out} should be connected to an arbitrary P_k or P_{k+1} on the closed curve because we could not direct connect to P_{in}, P_{out}.

The results described above, prerequisites of 4-color problem is because planar map. If map is not plane but space, we think that direct connecting between P_{in}, P_{out} is possible without dividing the connecting line between P_k, P_{k+1}. It is that the 2 points P_k, P_{k+1} of any closed curve is directly connected means that 2 country on the map are bordered. If we describe by way of example travel this content,when a traveler who has starting from country P_{in} to go to country P_{out}, he has to move along the border (in this cae, he has no choice but to step into one of the country P_k or country P_{k+1}) or pass through the country P_k or P_{k+1}. ■

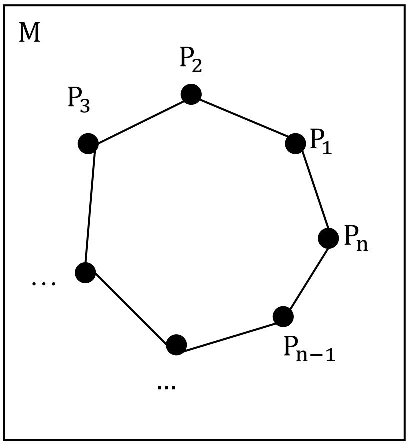
Theorem 2. $N(M)$ **of point,line,face**

If M_{point} is map "point", $N(M_{point}) = 1$

If M_{line} is map "line", regardless of the number of points $N(M_{line}) = 2$

If M_{face} is map "face", if the number of points is even, $N(M_{face}) = 2$, if odd, $N(M_{face}) = 3$.

Especially, if the number of points is odd, the color of end point is different only, all other points are enough two colors.



[Figure 2. face]

Proof 2. Let M_{point} be map "point", only one point is not connected to any point, so, $N(M_{point}) = 1$ is self-evident.

Let M_{line} be map "line" and P_1 be start point and P_n be end point. This is the case of $P_1 \nleftrightarrow P_n$ in the above Figure 2.

For arbitray P_k, $C_k \neq C_{k-1}, C_k \neq C_{k+1}$. So $\{C_1 = C_3 = C_5 = C_7 = \cdots\} \neq \{C_2 = C_4 = C_6 = \cdots\}$, that is, the color of map is minimum when $C_{k-1} = C_{k+1}$.

Therefore, the minimum colors of M_{line} are sufficient C_1, C_2, and $N(M_{line}) = 2$ regardless of the number of points.

Let M_{face} be map "face" and P_1 be start point and P_n be end point.

If the number of points is even, that is, $n = 2k$. If we apply a method similar to the proof of the above M_{line}, $\{C_1 = C_3 = \cdots = C_{2k-1=n-1}\} \neq \{C_2 = C_4 = \cdots = C_{2k-2} = C_{2k=n}\}$, that is $C_1 \neq C_n$, so the minimum colors of M_{face} are C_1, C_2 and $N(M_{face}) = 2$.

If the number of points is even, that is, $n = 2k + 1$. If we apply similar method, $\{C_1 = C_3 = \cdots = C_{2k-1} = C_{2k+1=n}\} \neq \{C_2 = C_4 = \cdots = C_{2k-2} = C_{2k=n-1}\}$, so $C_1 = C_n$.

In order to avoid this result, we should set $C_n \neq \{C_2 = C_{n-1}\}, C_n \neq C_1$.

Therefore, the minimum colors of M_{face} are C_1, C_2, C_n and $N(M_{face}) = 3$.

And, $\{C_1 = C_3 = \cdots\} \neq C_n$, $\{C_2 = C_4 = \cdots\} \neq C_n$, so, the color of end point P_n is different only, all other points are sufficient 2 colors C_1, C_2. ■

Theorem 3. **Condition of $N(M) = 4$**

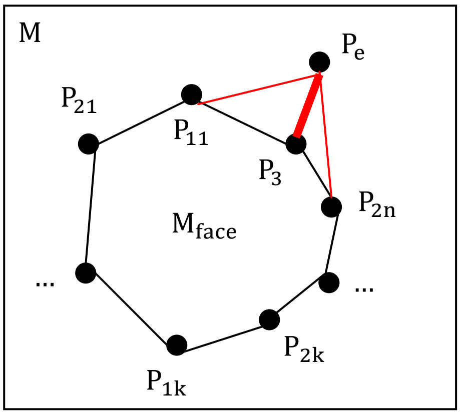
If a point P_e of outside closed curve directly connect to points on closed curve, when it satisfies all the following three conditions, $N(M) = 4$, otherwise $N(M) < 4$.

1) The number of “face” should be odd.

2) P_e should be directly connected to the end point.

3) P_e should be directly connected to two or more points of different colors one of the remaining points of the closed curve except the end point.

In addition, when set to satisfy the above three conditions, the end point of closed curve is located inside the closed curve newly created containing P_e.



[Figure 3. connecting “face”, one point]

Proof 3. Let M_{face} be map “face” like Figure 3, M be the map M_{face} and P_e is connected.

1) In the case of “line” or “point” or the number of points in “face” is an even number, we omit a detailed proof because $N(M) \leq 3$ is self-evident according to Theorem 2. Therefore, the number of points in “face” should be odd for $N(M) > 3$.

2) We colud select arbitrary start point and end point because all points in “face” unlike “line” are connected biting tail. So, Let P_{11} be start point, P_3 be end point.

In the case of $P_e \nleftrightarrow P_3$, even if P_e is directly connected to all points on the closed curve except for P_3, if $C_e = C_3$, $\{C_{11} = \cdots = C_{1k} = \cdots\} \neq \{C_{21} = \cdots = C_{2k} = \cdots = C_{2n}\} \neq \{C_3 = C_e\}$, so $N(M) = 3$.

Therefore, it should be $P_e \leftrightarrow P_3$ for $N(M) > 3$.

3) If $P_e \nleftrightarrow P_{1k}, P_e \nleftrightarrow P_{2k}$ for arbitrary k, that is, If P_e is not connected directly to the any point of closed curve on except for P_3, it could be $C_e = C_{1k}$ or $C_e = C_{2k}$ or $C_e = C_3$, so $N(M) = 3$.

If $P_e \leftrightarrow P_{1k}, P_e \nleftrightarrow P_{2k}$, that is, even if P_e is connected directly to the one or more point of $P_{11}, P_{12}, \ldots$, it could be $C_e = C_{2k}$,

If $P_e \nleftrightarrow P_{1k}, P_e \leftrightarrow P_{2k}$, $N(M) = 3$ is because it may be $C_e = C_{1k}$.

If $P_e \leftrightarrow P_{1k}$, $P_e \leftrightarrow P_{2k}$, it should be $\{C_{11} = \cdots = C_{1k} = \cdots\} \neq \{C_{21} = \cdots = C_{2k} = \cdots = C_{2n}\} \neq C_3 \neq C_e$,so $N(M) = 4$.

Therefore, by summarizing the above contents, only if the number of M_{face} is odd, $P_e \leftrightarrow P_3$, $P_e \leftrightarrow P_{1k}$, $P_e \leftrightarrow P_{2k}$ then $N(M) = 4$, otherwise $N(M) < 4$.

And, when it satisfys the above, P_e is to be directly connected to the 3 points on the closed curve at least. The 1 point of 3 points in the closed curve will be positiond between the 2 points remaining, start and end points could be arbitrarily selected by the closed curve, therefore, we could set the end point located between the 2 points.

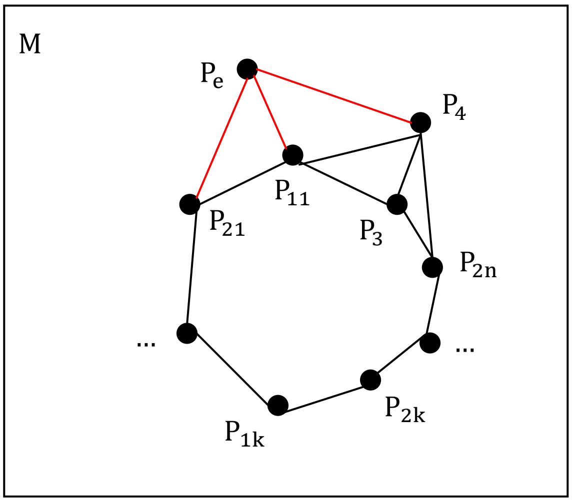
For example, when $P_e \leftrightarrow P_{21}, P_e \leftrightarrow P_{11}, P_e \leftrightarrow P_3$ in Figure 3, end point P_3 is not located between P_{21}, P_{11}, but if we set that P_{21} is start point and P_{11} is end point, end point P_{11} is located between P_3, P_{21}

That is, in any case P_e is directly connected to the 3 points on the closed curve, we could set P_3 to end point between 2 points as shown in Figure 3.

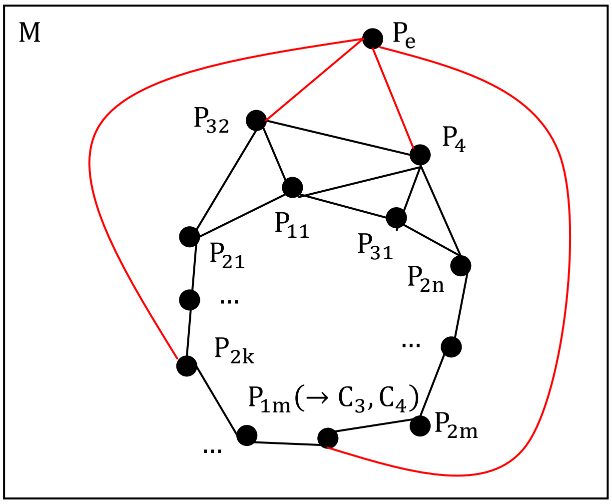
And, when P_3 is end point in Figure 3, P_e and P_{11}, P_{2n} of closed curve form another closed curve "$P_e, P_{11}, P_3, P_{2n}, P_e$", so P_3 will be present in the internal a new closed curve "$P_e, P_{11}, P_{21}, \ldots, P_{2n}, P_e$" except P_3 ■

Theorem 4. Rearrange color of existing points and conclusion

If another outside point is added to the map of theorem 3 then we could prevent maximum color from exceeding 4 by using to rearrange colors of existing points. Therefore,for all map M, $N(M) = 4$.



[Figure 4.1. Figure 3 + 1 point]



[Figure 4.2. Figure 4.1 + 1 point]

Proof 4. Figure 4.1 is a case when another P_e is connected directly to the map of Figure 3. Even though P_e is connected directly to all points on the map, P_e is point of outside closed curve "$P_4, P_{11}, P_{21}, \dots, P_{2n}, P_4$" and P_3 is point of inside, so, it will be $P_e \nleftrightarrow P_3$ according to theorem 1. Therefore, $C_e = C_3$ and $N(M) = 4$.

Figure 4.2 is a case when another P_e is connected directly to the map of Figure 4.1.
If $N(M)$ is the maximum, $P_e \leftrightarrow P_{32}, P_e \leftrightarrow P_4, P_e \leftrightarrow P_{2k}, P_e \leftrightarrow P_{1m}$.
P_e is the point of outside closed curve "$P_{32}, P_{21}, \dots, P_{2n}, P_4, P_{32}$" and P_{11}, P_{31} is inside.
According to theorem 1, $P_e \nleftrightarrow P_{31}, P_e \nleftrightarrow P_{11}$ but $P_e \leftrightarrow P_{32}$, it should not be $C_e = C_{31}$, and it should not be $C_e = C_{11}$ because $P_e \leftrightarrow P_{1m}$, we think it will be $N(M) = 5$.
However, if we change the color of points on line "$P_{21}, \dots, P_{2k}, \dots, P_{1m}, P_{2n}$" to C_{32} or C_4, we could make $N(M) = 4$.
Let us define C_r be the color of $P_{r1}, P_{r2}, P_{r3}, \dots$ to simplify the notation. Even though we change the color of $P_{21}, \dots, P_{2k}, \dots, P_{1m}, P_{2n}$, it should be $C_e = C_1$ or $C_e = C_2$ because $P_e \leftrightarrow P_{32}, P_e \leftrightarrow P_4$.
First of all, let us study to change the color of P_{2k}. The color of point direct connected to P_{2k} is C_1 according to theorem 2, C_{2k} should be changed C_3 or C_4. If $C_{2k} = C_3$ then $P_{21} \leftrightarrow P_{32}$, so we should not change, if $C_{2k} = C_4$ then $P_{2n} \leftrightarrow P_4$, so we should not change. And, if we do not change C_{2k} then $P_e \leftrightarrow P_{2k}$, so it should be $C_e = C_1$.

Now, let us study to change the color of P_{1m}. If we change C_{1m} to C_3, $P_{1m} \leftrightarrow P_{2m}$ is good, $P_e \leftrightarrow P_{1m}$ is good because $C_e = C_1$ and $C_{1m} = C_3$, $P_e \leftrightarrow P_{32}, P_e \leftrightarrow P_4, P_e \leftrightarrow P_{2k}$ is good because $C_e = C_1$. Therefore, $C_e = C_1$ is good.

If we change C_{1m} to C_4, $P_{1m} \leftrightarrow P_{2m}$ is good, $P_e \leftrightarrow P_{1m}$ is good because $C_e = C_1, C_{1m} = C_4$. $P_e \leftrightarrow P_{32}, P_e \leftrightarrow P_4, P_e \leftrightarrow P_{2k}$ is good because $C_e = C_1$. Therefore, $C_e = C_1$ is good.

The results described above, if we change $C_{1m} = C_3$ or $C_{1m} = C_4$ and $C_e = C_{11}$, then $P_e \leftrightarrow P_{32}, P_e \leftrightarrow P_4, P_e \leftrightarrow P_{2k}, P_e \leftrightarrow P_{1m}$ or $P_e \leftrightarrow P_{32}, P_e \leftrightarrow P_4, P_e \leftrightarrow P_{2k}, P_e \leftrightarrow P_{1m}$. That is, if we change the color of $P_{12}, P_{13}, \ldots$ the same color as the P_{11} inside closed curve to C_3 or C_4, the point P_{11} of same color as C_1 is unique in the map. Because P_{11} is point of inside closed curve, any point of outside closed curve could not be directly connected .

When a point is added to the arbitrary map M, after we set the color of a point of the inside closed curve to the color of the point added, if this does not work, then, when using the method as described above to change the color of the remaining points of the closed curve, it is sufficient $N(M) = 4$ for all map M because we could rearrange to prevent from $N(M) > 4$. ■

References

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(This is Korean book. I translate, sorry . Original book is

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