Analytic Functions for Clifford Algebras

Celebrating the 200^{th} anniversary of Cauchy Integration Theorem

Hans Detlef Hüttenbach

To the Memory of Augustin Louis Cauchy

Abstract. In this article, the Cauchy theory is applied and extended to n dimensional functions in (Clifford) algebras.
I already touched on this in [3] for path integration of classical fields, which might not be evident. So, I publish the details here as a celebration of Cauchy's outstanding lecture "Sur les intégrales définies", held on the 22nd of August 1814, so 200 years ago, which has been re/published in 1825, and is publicly accessible online in [2].

1. Introduction

At the end of the 19th century, Henri Poincaré discovered as was then to be known as Poincaré's Lemma; it states that on star-shaped open regions closed differential forms are necessarily exact (see [1]). This triggered the beginning of algebraic geometry, which became one of the most important branches of mathematics, especially in France.
It is often overlooked that Poincaré's intention at that time was not the abstract development of a theory of cohomologies, but he wanted to unravel the curious nature of electromagnetic fields. With the help of this lemma, he could show that (in 3 dimensions) an electromagnetic field of zero divergence is the curl of a vector potential **A**, which in turn is fundamental to derive gauge invariance and the Lorentz representation of Maxwell's equations.

In 1895 Volterra showed that Poincaré's Lemma extends as the equivalence of closed and exact differential forms, and Elie Cartan independently rediscovered this a decade or two later. As can be seen from [1], for 1-forms that means that an n-tuple of continuously differential function $(f_1, \ldots, f_n)$ on an open, star-shaped region $U \subset \mathbb{R}^n$ is integrable (i.e. defines an exact 1-form) in that region, if and only if its derivative (the Jacobi matrix) is symmetric.

Obviously, that is not what Cauchy understood as what an integrable function should look like: Cauchy examined a pair of differentable real-valued functions (u, v) on $\mathbb{R}^2$, and then he came to a remarkable solution: He defined $u(x+iy) := u(x, y), v(x+iy) := v(x, y)$, and then he defined a complex-valued function f on $U \subset \mathbb{C}$ as $f : U \ni x + iy \mapsto u(x + iy) + iv(x + iy) \in \mathbb{C}$. He then proved that f is integrable in U if it is complex differentiable, that complex differentiability is equivalent to analyticity, and more over, this is equivalent to the Jacobi matrix of $(u(x, y), v(x, y))$ to be anti-symmetric in its off-diagonal elements, namely to follow the Cauchy-Riemann equations $\partial u/\partial x = \partial v/\partial y$, and $\partial u/\partial y = -\partial v/\partial x$!

This not only antedates Poincaré's differential geometric results 80 years later, even then, Cauchy's work appears to be ahead of that time:
Let's examine the reason for the seemingly controversial results of the conditions of integrability:
The answer is that Poincaré is integrating within Euclidean geometry, whereas Cauchy is integrating in the complex plane. There are two reasons in favour of Cauchy's technique: Firstly, one cannot divide by a vector of two or more dimensions, but one can divide by complex numbers. It is this clever substitution $(x, y) \mapsto x + iy$ that allowed Cauchy to rigorously define (complex) differentiability. Secondly, by doing so, Cauchy instantaneously carried out the path integration in a vector space $\mathbb{C}$ with its intrinsic hyperbolic metric, and not in the Euclidean metric: Whereas Poincaré used $(a_1, a_2) \cdot (b_1, b_2) := a_1b_1 + a_2b_2$ as inner product, for Cauchy it is $(a_1, ia_2) \cdot (b_1, ib_2) := a_1b_1 - a_2b_2$. That explains, why Cauchy's results lead to the unsymmetric Jacobi matrix, whereas Poincaré's Jacobi matrix is to be symmetric. So, Cauchy was also the first one to carry integration out in hyperbolic vector spaces, something that Poincaré himself never thought of, even after he and H.A. Lorentz derived the covariant Maxwell equations, which A. Einstein and H. Minkowski then proved to be a consequence of space-time being hyperbolic, rather than Euclidean!

2. Preliminaries: Clifford Algebras

I want to deal with 2 or more dimensions of complex numbers. Then, according to Cauchy, I have to represent these vectors as numbers, in order to be able to divide by these. Because only then, I'll be able to go with his strong notion of differentiability.

The technique to use have been readily exposed by Hermann Graßman and William Kingdon Clifford:

Let X be an n-dimensional complex vector space with $n \geq 1$, and let Q be a non-degenerate quadratic form on X. This means that one can find a linear basis $a_1, \ldots, a_n \in X$, w.r.t. which Q is defined through a symmetric, invertable $n \times n$-matrix A. Then there is an orthogonal transformation U

on $\mathbb{C}^n$, i.e.: $U^{-1} = U^t$, where $U^t_{ij} := U_{jt}$ is the transpose of U, such that UAU^{-1} is a diagonal matrix with real eigenvalues $\lambda_1, \dots \lambda_n \in \mathbb{R}^n$, and by scaling the basis elements a_j with a positive factor $|\lambda_j|^-1$, we arrive at: every non-degenerate quadratic form in n dimensions defines an orthonormal basis, and it falls into one of n possible categories: $(n, 0), (n-1, 1), \dots, (0, n)$, where $(p, n-p)$ signifies that the first p eigenvalues are $+1$, and the $n-p$ others are -1. This is termed the *signature* of the quadratic form.

Proposition 2.1. *For each $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ and n $m \times m$-matrices $\alpha_1, \dots, \alpha_n$, such that $\alpha_1^2 = \cdots \alpha_n^2 = \mathbb{1}_m$ and is the $m \times m$-unit matrix.*

Proof. The statement is trivial for $n = 1$. So, let $n > 1$. Then suitable matrices can be picked from the vector space of all endomorphisms on the $n \times n$-dimensional space of $n \times n$-matrices. □

Remark 2.2. Actually, it can be shown that $m = 2^{n/2}$ is the minimal m, if n is even, and then $m = 2^{(n+1)/2}$ will do for uneven n, but that is irrelevant for now.

Definition 2.3. For a non-degenerate quadratic form of signature, $(p, n-p)$, the above matrices $\alpha_1, \dots, \alpha_p, i\alpha_{p+1}, \dots, i\alpha_n$ are called the *generators* of the Clifford algebra $Cl_{p,n-p}(\mathbb{C})$. The n matrices $\alpha_1, \dots, \alpha_n$. $Cl_{p,n-p}(\mathbb{C})$ is defined as the (non-commutative) algebra of all complex linear combinations of the α_k and all products of these. Let $X(n)$ be the (complex) vector space spanned by the α_k. $Cl_{p,n-p}(\mathbb{C})$ becomes a (finite dimensional) Banach space, when equipped with its natural supremum norm

$$\|\cdot\| : x \mapsto \sup_{\|\chi\| \leq 1} x(\chi),$$

and $X(n)$ then becomes a closed subspace of $Cl_{p,n-p}(\mathbb{C})$.

The important point now is that $\sum \lambda_k \alpha_k$ is an invertible matrix, if and only if $(\lambda_k)_{1 \leq k \leq n}$ is unequal zero. Therefore x is invertible for all $x \in X(n) \setminus \{0\}$, and right as well as left division are well-defined on $Cl_{p,n-p}(\mathbb{C})$ for every $x \in X(n) \setminus \{0\}$. This allows

Definition 2.4. A continuous mapping $f : X(n) \supset U \to Cl_{p,n-p}(\mathbb{C})$ of an open subset $U \subset X(n)$ is said to be *differentiable* in $x_0 \in U$ if and only if $\lim_{x \to x_0} (f(x) - f(x_0))(x - x_0)^{-1}$ and $\lim_{x \to x_0} (x - x_0)^{-1}(f(x) - f(x_0))$ both exist and are equal. For short, I'll write this limit as $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$.

Remark 2.5. An analytic function, which vanishes along any one of its n complex axis, must be zero throughout.

Remark 2.6. Also, given two differentiable, $Cl_{p,n-p}(\mathbb{C})$-valued functions f, g on an open $U \subset X(n)$, then the product function $fg : z \mapsto f(z)g(z) \in Cl_{p,n-p}(\mathbb{C})$ is differentiable either, and $(fg)'(z) = d(f(z)g(z))/dz = f'(z)g(z) + f(z)g'(z)$.

To get rid of the factors i for the following, let me write $\gamma_k := \alpha_k$, whenever the metrics is positive for that component, and $\gamma_k := i\alpha_k$, when it is negative.

3. Proceeding with Cauchy's theorems

We are now in the position to harvest the fruit from Cauchy's work:

Definition 3.1. Let $U \subset X(n)$ be an open subset of $X(n)$ and $z_0 \in U$. A function $f : U \ni z \mapsto Cl_{p,n-p}(\mathbb{C})$ is called *analytic* in z_0, if and only if there is a neighbourhood $V(z_0) \subset U$ of z_0, such that $f(z) = \sum_{k\geq 0} a_k(z-z_0)^k$ for all $z \in V(z_0)$, where $a_0, a_1, \dots \in \mathbb{C}$.

Clearly, then: f is analytic in z_0 if and only if there is an analytic function $g : \mathbb{C} \supset \Omega \ni \chi \mapsto g(\chi) \in \mathbb{C}$ on an ϵ-environment Ω of some $\chi_0 \in \mathbb{C}$, such that $f(z) = \sum_{k\geq 0} a_k(z-z_0)^k$, where the a_k are the coefficients of the Taylor series expansion $g(\chi) = \sum_{k\geq 0} a_k(\chi - \chi_0)^k$.

It means that the k-th (right) derivatives of f in z_0 exist as complex number (times unit matrix), and that these are the k-th derivatives of an analytic function g in χ_0. I'll say that f is *generated* by g.

For a strictly positive, real valued $r > 0$, let us define the r-ball $B_r(0) := \{z = (z_1, \dots, z_n) \in X_n : \sum_{1\leq k\leq n} \left|z_j\right|^2 \leq r^2\}$, and let $S_n(r)$ be its boundary. Because each $z_k = (x_k + iy_k)\gamma_k$, where x_k and y_k are the real and imaginary part of z_k, $B_1(0)$ is a $2n$-dimensional ball, and S_n its $2n-1$-dimensional boundary.

Then the $B_\epsilon(0)$, $\epsilon > 0$, are base of zero neighbourhoods of $X(n)$, which means that for every open neighbourhood U of 0 there is some $\epsilon > 0$ such that $B_\epsilon(0) \subset U$.

Definition 3.2. Although with $z = \sum_k z_k\gamma_k$ with $z_k \in \mathbb{C}$ the derivative $f'(z)$ is well-defined, the partial derivatives $\partial f(z)/\partial z_k$ are *not*: Generally, when f is differentiable in z, right and left partial derivatives will be unequal! In order to deal with partial derivatives, we need to confine to partial derivatives to the right, which will be denoted by $\partial_r/\partial x_k$, $(1 \leq k \leq n)$.

A function $If : X_n \supset U \to Cl_{p,n-p}(\mathbb{C})$ will be called integral of $f : U \to Cl_{n,n-p}(\mathbb{C})$, if $(If)'(z) = f(z)$ for all $z \in U$. More generally, for $k \in \mathbb{N}$, the k-th integral of f is a function $I^{(k)}f : U \to Cl_{p,n_p}(\mathbb{C})$, auch that $d^k f(z)/dz^k = f(z)$ for all $z \in U$.

Remark 3.3. Once again, be warned: The notion of partial differentiation $\partial_r/\partial x_k$ is inadequate: It tacitly introduces an Euclidean metrics, which is not given. The clean notion would be the directional derivative $\partial/\partial(\gamma_k x_k)$, which however is just d/dz!

Proposition 3.4. *Let $f : U \to Cl_{p,n-p}(\mathbb{C})$ be differentiable. Then, writing $z = z_1\gamma_1 + \cdots + z_n\gamma_n$ for $z \in U$,*

1. $\partial_r f(z)/\partial z_k = \partial_r f(z)/\partial z_l \gamma_l^{-1}\gamma_k$, $(1 \leq k \neq l \leq n)$.

2. *If f is twice differentiable, then* $\partial_r^2 f(z)/\partial z_k \partial z_l = -\partial_r^2 f(z)/\partial z_l \partial z_k$, $(1 \leq k \neq l \leq n)$.

Proof. We have $\partial f(z)/\partial z_k = f'(z)\partial z/\partial z_k = f'(z)\gamma_k$ for all k, from which the statement 1 follows. Taking a second partial derivative, delivers 2. □

Remark 3.5. Let $f(z) = \sum_{1\leq k\leq n} \gamma_k f_k(z)$ with $f_k(z) \in \mathbb{C}$ for all $k = 1, \dots, n$ and $z = \gamma_1 z_1 + \dots + \gamma_n z_n in X(n)$. The (ordinary) Cauchy-Riemann equations here appear for each component k, if we write $z_k = x_k + iy_k$ and split $f_k = g_k + ih_k$ into the sum of its real and imaginary part. We can then define real-valued functions u_k, v_k on open subsets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ by $u_k(x_1, \dots, x_n) + iv_k(y_1, \dots, y_n) := f_k(z) = g_k(z) + ih_k(z)$. The Cauchy-Riemann equations then hold for u_k and v_k as functions of x and y, resp.. Consequently, the functions u_k and v_k will be called *harmonic* functions.

Remark 3.6. An analytic function $f : X_n \supset B_r(z_0) \to Cl_{p,n-p}(\mathbb{C})$ for $z_0 \in X_n$ is evidently integrable in that region. Therefore its real and imaginary parts also are. These, however are not differentiable in the above defined sense: These functions contradict the Cauchy-Riemann conditions. For f analytic in $B_r(z_0)$ as above, let $f_k := \partial f(\gamma_1 z_1, \dots \gamma_n z_n)/\partial z_k$, $(1 \leq k \leq n)$ be the gradient of f. Because of the preceeding proposition, $\partial_l f_k = -\partial_k f_l, (1 \leq k \neq l \leq n)$ follows. Conversely, given $\partial_l f_k = -\partial_k f_l, (1 \leq k \neq l \leq n)$ for continously differentiable $f_k : B_r(z_0) \to Cl_{p,n-p}(\mathbb{C})$, integrating f_k along $\gamma_l dz_l$ followed by integration along $\gamma_k dz_k$ commute, which means that $(f_k)_{1\leq k\leq n}$ integrable within $B_r(z_0)$ to some function $f : B_r(z_0) \to Cl_r(\mathbb{C})$, which again is differentiable, since it obeys the Cauchy-Riemann conditions.
The takeaway is: The integrability of functions is independent of the signature of the Clifford algebra.

We want to extend the Cauchy integral theorem from 1 to $n \in \mathbb{N}$ dimensions. So, we need to have a notion of surface integration over the unit ball. One might think that should be trivial, given that X_n is isomorphic to $\mathbb{C}^n$, so one could rely on differential $2n-1$-forms. But that is not true, because the $Cl_{p,n-q}(\mathbb{C})$-valued functions on open subsets of X_n do not locally map ϵ-balls with n complex dimensions in X_n into regions, that by themselves would be homeomorphic to δ-balls of n complex dimensions, again. So, the embedding $\Omega : X_n \to Cl_{p,n-q}(\mathbb{C})$ impedes the definition of surface integrals, and with it a direct approach to extend Cauchy integration.

Definition 3.7. Let $\mathcal{L}(\mathbb{C}^n)$ be the (non-commutative) algebra of linear mappings $T : \mathcal{C}^n \to \mathcal{C}^n$. Because each $z = \gamma_1 z1 + \cdots \gamma_n z_n \in X_n$ defines a linear operator $\Xi z : \mathbb{C}^n \ni (a_1, \dots, a_n) \mapsto (z_1 a_1, \dots, z_n a_n) \in \mathbb{C}^n$, Ξ is a vector space isomorphism $\Xi : X_n \to \mathcal{A}_n \subset \mathcal{L}(\mathbb{C}^n)$ onto a (commutative) subalgebra $\mathcal{A}_n$ of $\mathcal{L}(\mathbb{C}^n)$. In short, $\mathcal{A}_n$ is the algebra $\mathbb{C}^n$, where the multiplication is given by $(\lambda_1, \dots, \lambda_n) \cdot (\eta_1, \dots, \eta_n) := (\lambda_1\eta_1, \dots \lambda_n\eta_n)$. Giving $\mathcal{A}_n$ the canonical norm of $\mathbb{C}^n$, a function $f : V \to \mathcal{A}_n$ on an open subset $V \subset \mathcal{A}_n$ will be called analytic in in $z_0 \in V$, if there is a sequence $(a_k)_{k\in\mathbb{N}}$ of complex numbers $a_k \in \mathbb{C}$, such that $\sum_{k\in\mathbb{N}} a_k(z - z_0)^k$ exists and converges to $f(z)$ in an ϵ-ball around z_0. Clearly then, $f \mapsto \Xi f \Xi^{-1}$ is an isomorphism I of the vector space of analytic functions $f : X_n \supset U \to Cl_{p,n-p}(\mathbb{C})$ onto the space of analytic functions $g : \mathbb{C}^n \supset V \to \mathcal{A}_n$. Since k-dimensional manifolds in X_n map 1-1 into k-dimensional manifolds in $\mathcal{A}_n$ and since the integral $\int_{\mathcal{M}}(f)d\omega \in \mathbb{C}$ is well-defined in $\mathcal{A}_n$, whenever $\mathcal{M}$ is a smooth and bounded manifold of $\mathcal{A}_n$ and f is sufficiently smooth in a neighbourhood of $\mathcal{M}$, we can use this to define the integral of $f : X_n \supset U \to Cl_{p,n-p}(\mathbb{C}^n)$ over a (smooth and bounded) manifold $\mathcal{M} \subset X_n$ as:

$$\int_{\mathcal{M}} f d\omega := \int_{\Xi\mathcal{M}} (\Xi f \Xi^{-1}) d(\Xi\omega).$$

Remark 3.8. Note that $\mathcal{A}_n$ supplements $Cl_{p,n-p}(\mathbb{C})$, but does not replace it: within $\mathcal{A}_n$ we gain the notion of surface integrals, but we loose the notion of differentiability: In $\mathcal{A}_n$, $1/z$ is undefined not only for $z = 0$, but also whenever one of its components vanishes. (In line with this, care must be taken not to integrate $1/z$ along a manifold on which $1/z$ is not defined, unless the set of these points is of measure zero.)

Lemma 3.9 (Cauchy Theorem). *Let $f : U \to Cl_{p,n-p}(\mathbb{C})$ be analytic on an open subset $U \subset X_n$, $r > 0$, $B_r(z_0) \subset U$ be the closed r-ball around $z_0 \in U$, and let $S_r(z_0)$ be its $2n-1$-dimensional boundary. Then the surface integral of f over $S_r(z_0)$ vanishes, i.e.: $\int_{S_r(z_0)} f d^{2n-1}a = 0$.*

Proof. Suppose, there exist $r > 0$ and $z_0 \in U$, such that $B_r(z_0) \subset U$ and $\int_{S_r(z_0)} f d^{2n-1}a \neq 0$, where $B_r(z_0)$ is the closed r-ball around z_0. Then, $B_r(z_0)$ is compact, and we can find an $\epsilon > 0$, and a finite covering of $B_r(z_0)$ by ϵ-balls, for which the sum of surface integrals of f over the ϵ-spheres is unequal zero. So, there exists an ϵ-ball $B_\epsilon(\zeta_1)$ around some $\zeta_1 \in B_r(z_0)$, for which the surface integral of f over its sphere is unequal zero. By induction we get a sequence $(\zeta_k)_{k\in\mathbb{N}}$ converging to some $\zeta_0 \in B_r(z_0)$. So, for each $\epsilon > 0$ there exists an $r > 0$ with $r < \epsilon$, such that $\int_{S_r(\zeta_0)} f d^{2n-1}a \neq 0$. The proof will be complete, when it will be shown that f then cannot be analytic in ζ_0.
If f is analytic in ζ_0, there is an r-ball around ζ_0 for some $r > 0$, such that $f(z) = \sum_{k\geq 0} a_k(\zeta_0 - z)^k$ is the uniformly and absolutely converging limit of its Taylor series on $B_r(\zeta_0)$, and the surface integral over $S_r(\zeta_0)$ commutes with the summation. As elements of $\mathcal{A}_n$, we may change from (complex) Euclidean to the spherical coordinates, so, $f(z) = f(r, \phi_1, \ldots, \phi_{2n-1})$, where $r > 0$ is the radius of the ball around ζ_0, $0 \leq \phi_1 \leq 2\pi$ is the azymuthal angle of rotation in the x_1x_2-plane, and the other $2n-2$ angles range from 0 to π. Then

$$\int_{S_r(\zeta_0)} f(z) d^{2n-1}a = \int_{\phi_1=0}^{2\pi} \int_{\phi_2=0}^{\pi} \cdots \int_{\phi_{2n-1}=0}^{\pi} f(r, \phi_1, \ldots, \phi_{2n-1}) \\ (2n/r) det(J(z, r, \phi_1, \ldots)) d\phi_1 \cdots d\phi_{2n-1}, \quad (3.1)$$

where $det(J)$ is the determinant of the Jacobi matrix, given by $det(J) = r^{2n} sin^{2n-1}(\phi_1) sin^{2n-2}(\phi_2) \cdots sin(\phi_{2n-1})$.

The function $g(r) := \int_{S_r(\zeta_0)} f d^{2n-1}a$ therefore is the uniform limit of its Taylor series $g(r) = \sum_{k\geq 0} c_k r^k$ on a closed interval $[0, r_0]$ for some $r > 0$. It uniquely extends as an analytic function $g(\lambda) := \sum_{k\geq 0} c_k \lambda^k$ onto $\{\lambda \in \mathbb{C} : |\lambda| \leq r_0\}$. Now, because the derivatives of f of all orders are bounded on $B_r(\zeta_0)$, $g(0) = dg(0)/dr = d^2g(0)/dr^2 = \cdots = 0$. So, g vanishes on $B_r(\zeta_0)$, which completes the proof. □

As an immediate consequence, for an analytic function f on an (2n-dimensional) r-ball, the surface integral of f over a smooth and closed $2n-1$-dimensional manifold contained in the r-ball is zero.

Remark 3.10. With the above extension of Cauchy's theorem to $\mathcal{A}_n$, it is tempting to try to evaluate $\int_{S_n(r)} f(z)(1/z)^{2n-1} d^{2n-1}a$; however, that becomes unwieldy, because the integrand $1/z$ brings an additional phase factor $e^{-i(\phi_1+\cdots+\phi_n)}$; $1/r$ instead of $1/z$ would be more convenient, but $z \mapsto f(z)/r^{2n-1}$ is not analytic outside the origin $\{0\}$. That indicates that the Euclidean topology might not be the best choice. So, let's switch to a simpler topology/geometry:

Definition 3.11. For $r_0 > 0$ let $T_{r_0}(0) := \{z \in \mathcal{A}_n : z_k = re^{i\phi_k}, 0 \leq \phi_k < 2\pi, r \leq r_0, 1 \leq k \leq n\}$ define a base of neighbourhoods of 0 in $\mathcal{A}_n$, which make $\mathcal{A}_n$ and, via Ξ^{-1} above, X_n into a barreled, locally convex space. It is metrizable and complete, its metrics being given by $\mathcal{A}_n \ni z \mapsto \sup_{1\leq k\leq n}|z_k| \in [0, \infty)$, it is topological equivalent to $\mathbb{C}^n$, but it is not an Euclidean space.

With this, the above Cauchy theorem simplifies to:

Proposition 3.12. *For $r > 0$ let $\Gamma_r(0)$ be the (n-dimensional) surface of $T_r(0)$. Then $\int_{\Gamma_r(0)} f(z) d^n a = 0$ for every function f, which is analytic in an open neighbourhood of $T_r(0)$.*

Proof. $\int_{\Gamma_r(0)} f(z) d^n a = \int_0^{2\pi} \cdots \int_0^{2\pi} f(e^{i(\phi_1+\cdots+\phi_n)}) r^n e^{i\phi_1+\cdots+\phi_n} d\phi_1 \cdots d\phi_n$ is analytic in $r \geq 0$, if f is analytic, and for $r = 0$ that function vanishes with all its derivatives.
We even get that $\int_{\Gamma_r(0)} f(z) z^{-j} d^n a$ vanishes for $j = 1, \ldots, n-1$. □

Likewise, in this topology, the rest becomes straightforward:

Lemma 3.13. *1.* $\int_{\Gamma_r(0)} 1/z^n d^n a = (2\pi i)^n$.
2. $\int_{\Gamma_r(0)} 1/z^k d^n a = 0$ *for* $k > n$.

Proof. $\int_{\Gamma_r(0)} 1/z^n d^n a = (2\pi i)^n =$
$= \int_0^{2\pi} \cdots \int_0^{2\pi} r^{-n} e^{-i(\phi_1+\cdots+\phi_n)} (ir)^n e^{i(\phi_1+\cdots+\phi_n)} d\phi_1 \cdots d\phi_n$, from which the first statement follows. Because of Cauchy theorem and the fact that the integrand is analytic outside the origin, the surface $\Gamma_r(0)$ can be deformed diffeomorphically without affecting the integral, as long as the origin stays in the interior of the encompassed region, i.e.: $\int_{\Gamma_r(0)} (z-y)^n d^n a = (2\pi i)^n$ for y such that $y_1 < r, \ldots, y_n < r$. Now, for $k > n$: $1/z^k = -(k-1)^{-1} d(1/z^{k-1})/dz$,

so that by partial integration and the fact that the derivative of a constant function is zero, the other statements follow. □

Then, the following holds:

Proposition 3.14 (Cauchy integral theorem). *Let* $f : U \to Cl_{p,n-p}(\mathbb{C})$ *be analytic in an open neighbourhood* $U \subset X_n$ *of some* $z_0 \in X_n$*. Then for* f *as a function from* $U \subset \mathcal{A}_n$ *to* $\mathcal{A}_n$ *and* $\Gamma_r(z_0) \subset U$*:*
$\int_{\Gamma_r(z_0)} f(z)z^{-n} = (2\pi i)^n f(z_0)$ *and*
$d^k f(z_0)/dz^k = \frac{(n+1)\cdots(n+k)}{(2\pi i)^n} \int_{\Gamma_r(z_0)} 1/z^{n+k} d^n a$ *for* $k = 1, 2, \dots$*.*

Proof. Because of continuity of f in z_0,

$$(2\pi i)^n f(z_0) = \lim_{r\to 0} \int_{\Gamma_r(z_0)} f(z)/(z-z_0)^n d^n a,$$

and because of the Cauchy theorem above, the integral is independent of $r > 0$ (as long as r is sufficiently small).This proves the first statement. The other statements follow analogously through integration by parts. □

Conversely, if $f : X_n \supset U \to Cl_{p,n-p}(\mathbb{C})$ is differentiable in some $z_0 \in U$, then (as a function from $U \subset \mathcal{A}_n$ to $\mathcal{A}_n$), an $r_0 > 0$ exists, such that $\int_{\Gamma_r(z_0)} f(z)d^n a = f(z_0)$ for $0 < r < r_0$, which means that (complex) differentiability (as defined above) implies analyticity.

4. Conclusion

Let me come back to H. Poincaré's motivation: He was wondering what the Lorentz condition

$$\partial_0 j_0(x_0, \dots, x_3) + \dots + \partial_3 j_3(x_0, \dots, x_3) = 0$$

meant to electrodynamics. The j_μ represent charge density j_0 and charge flux (j_1, j_2, j_3) on regions of $\mathbb{R}^4$, so are to be taken as real-valued (and smooth) functions.
Therefore they can be extended to the complex by defining

$$j_\mu(z_0, \dots, z_3) := j_\mu(x+iy) := j_\mu(x) - i j_\mu(y), (0 \le \mu \le 3).$$

Poincaré knew that the Lorentz condition stated nothing but the charge conservation law. But why is this not Lorentz invariant, when everything else is? Therefore, that law has to be re-established by an operation called "gauge" upon every Lorentz transformation of the inertial system.

Let us now accept that space-time is not Euclidean, but to be described in Minkowski metrics, instead, so is of signature $(1, 3)$. So, let's rewrite:

$$f_\mu(z_0\gamma_0 + \dots + z_3\gamma_3)\gamma_\mu := j_\mu(z_0, \dots, z_3), (0 \le \mu \le 3).$$

Then the Lorentz condition becomes

$$\sum_{0\le\mu\le 3} \partial(f_\mu(z_0\gamma_0 + \dots + z_3\gamma_3)\gamma_\mu)/\partial(z_\mu\gamma_\mu) = 0.$$

By the above, we know that $(f_\mu \gamma_\mu)_{0\le\mu\le 3}$ is integrable to a function $F : X_n \ni z = z_0\gamma_0 + z_n\gamma_3 \mapsto F(z) \in Cl_{1,3}(\mathbb{C})$ if and only if $\partial_\mu j_\nu = -\partial_\nu j_\mu$ for $0 \le \mu \ne \nu \le 3$. That means that $dF(z)/dz = \sum_{0\le\mu\le 3} f_\mu(z)\gamma_\mu$. So, $\partial F(z)/\partial(z_\mu\gamma_\mu) = f_\mu(z)\gamma_\mu$, hence the Lorentz condition enforces:

$$\Box F(z) := (\partial_0^2 - \cdots - \partial_3^2)F(z) = 0.$$

Note that integrability here means integrability in the Clifford algebra, but not in the standard Euclidean space: it's $\gamma_\mu\partial_\mu f_\nu\gamma_\nu = \gamma_\nu\partial_\nu f_\mu\gamma_\mu$, and it's *not* $\partial_\mu f_\nu = \partial_\nu f_\mu$ for $0 \le \mu \ne \nu \le 3$!

In other words: the mapping

$$\Psi : \sum_{0\le\mu\le 3} \gamma_\mu f_\mu(\gamma_0 z_0 + \cdots + \gamma_3 z_3) \mapsto (f_0(z_0, \ldots, z_3), \ldots, f_3(z_0, \ldots, z_3))$$

maps spinor-functions that are integrable within the Clifford algebra into Euclidean vector fields with a generally nontrivial rotation, which are therefore not integrable within the Euclidean space - the electromagnetic fields just fall into this class: see [3] for details.

References

[1] H. Cartan, *Differential Forms* Herman Kershaw, 1971.

[2] A. L. Cauchy, *Memoires sur les intégrales définies*, https://archive.org/details/mmoiresurlesin00cauc, 1825.

[3] H. D. Hüttenbach, *The Action Function of Adiabatic Systems*, http://vixra.org/pdf/1404.0441v2.pdf, 2014.

Hans Detlef Hüttenbach
e-mail: detlef.huettenbach@computacenter.com