Proof of Infinite Number of Fibonacci Primes

Stephen Marshall

22 May 2014

Abstract

This paper presents a complete and exhaustive proof of the Fibonacci Prime Conjecture. The approach to this proof uses same logic that Euclid used to prove there are an infinite number of prime numbers. Then we prove that if $p > 1$ and $d > 0$ are integers, that p and $p + d$ are both primes if and only if for integer n (see reference 1 and 2):

$$n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^{d} d!}{p + d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

We use this proof for $p = F_{y-1}$ and $d = F_{y-2}$ to prove the infinitude of Fibonacci prime numbers.

Introduction

The Fibonacci prime conjecture, was made by Alphonse de Fibonacci in 1849. Alphonse de Fibonacci (1826 – 1863) was a French mathematician whose father, Jules de Fibonacci (1780-1847) was prime minister of Charles X until the Bourbon dynasty was overthrown in1830. Fibonacci attended the École Polytechnique (commonly known as Polytechnique) a French public institution of higher education and research, located in Palaiseau near Paris. In 1849, the year Alphonse de Fibonacci was admitted to Polytechnique, he made what's known as Fibonacci's conjecture:

For every positive integer k, there are infinitely many prime gaps of size 2k.

Alphonse de Fibonacci made other significant contributions to number theory, including the de Fibonacci's formula, which gives the prime factorization of n!, the factorial of n, where n ≥ 1 is a positive integer.

Proof of Infinite Number of Fibonacci Primes

In number theory, a Fibonacci prime is a Fibonacci number that is also prime. It is not known whether there are infinitely many Fibonacci primes; this poof shall prove their infinitude. Fibonacci numbers or Fibonacci series or Fibonacci sequence are the numbers in the following integer sequence:

The sequence F_n of Fibonacci numbers is defined by the following reoccurring relation:

$$F_n = F_{n-1} + F_{n-2},$$

with seed values,

$$F_0 = 0,\ F_1 = 1.$$

By definition, the first two numbers in the Fibonacci sequence are 0 and 1, and each subsequent number is the sum of the previous two. The first several Fibonacci numbers are:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ….

Of these Fibonacci numbers the following are Fibonacci primes:

2, 3, 5, 13, and 89

We shall use Euclid's logic that he used to prove there are an infinite number of prime numbers to prove there are an infinite number Fibonacci primes.

First we shall assume there are only a finite number of n Fibonacci's primes for all Fibonacci numbers, specifically;

1) Say, 2, 3, 5, 13, $F_{11,}$ . . . . , $F_x = F_{x-1} + F_{x-2}$, $F_n = F_{n-1} + F_{n-2}$

Where, F_x is the next to the last Fibonacci prime, x is used instead of n-1, because the last Fibonacci prime may not be the next Fibonacci number in the Fibonacci sequence.

2) Let N = 2*3*5*13*(F_{11}) . . . . , $(F_{x-1} + F_{x-2})(F_{n-1} + F_{n-2})$ +1

By the fundamental theorem of arithmetic, N is divisible by some prime q. Since N is the product of all existing Fibonacci primes plus 1, then this prime q cannot be among the F_i that make up the n Fibonacci primes since by assumption these are all the Fibonacci primes that exist and N is not evenly divisible by any of the, F_i Fibonacci primes. N is clearly seen not to be divisible by any of the F_i Fibonacci primes. Therefore, q must be another prime number that does not exist in the finite set of Fibonacci prime numbers.

The only thing left to prove that there are an infinite number of Fibonacci primes is to prove that $q = F_y = F_{y-1} + F_{y-2}$ where F_y is Fibonacci prime that is not in the set of finite Fibonacci primes, since q is not in that set.

First we shall show that if $q = F_y = F_{y-1} + F_{y-2} = p + d$, where $F_{y-1} = p$ and $F_{y-2} = d$, then if q is prime it cannot be in the set of finite p_i Fibonacci primes above. Since q is a prime number that does not exist in the set of finite p_i Fibonacci primes, then if there exists a prime number equal to $F_{y-1} + F_{y-2}$ that is also prime, it would be a Fibonacci prime; therefore a prime $F_{y-1} + F_{y-2}$ cannot be in the set of finite n Fibonacci primes otherwise q would be in the set of n finite Fibonacci primes and we have proven that q is not in the set of fine Fibonacci primes, therefore if p + d is prime it cannot be in the finite set of Fibonacci primes since it would be Fibonacci to q.

Now we shall proceed to prove p + d = $F_{y-1} + F_{y-2}$ is prime as follows:

We will prove that if p > 1 and d > 0 are integers, that p and p + d are both primes if and only if for integer n (see reference 1 and 2):

$$n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^d \mathrm{d}!}{\mathrm{p} + \mathrm{d}}\right) + \frac{1}{\mathrm{p}} + \frac{1}{\mathrm{p} + \mathrm{d}}$$

Proof:

The equation above can be reduced and re-written as:

$$3) \quad \frac{(p-1)! + 1}{\mathrm{p}} + \frac{(-1)^d d!\,(p-1)! + 1}{\mathrm{p} + \mathrm{d}}$$

Since $(p + d - 1)! = (p + d - 1)(p + d - 2) \cdots (p + d - d)(p - 1)!$, we have $(p + d - 1)! \equiv (-1)^d d!(p - 1)! \pmod{p + d}$, and it follows that equation 4 above is an integer if and only if:

$$4) \quad \frac{(p-1)! + 1}{\mathrm{p}} + \frac{(\mathrm{p} + d - 1)! + 1}{\mathrm{p} + \mathrm{d}}$$

is an integer. From Wilson's Theorem, if p and $p + d$ are two prime numbers, then each of the terms of, equation 4 above, is an integer, which proves the necessary condition. Wilson's Theorem states:

That a natural number $n > 1$ is a prime number if and only if

$$(n-1)! \equiv -1 \pmod{n}.$$

That is, it asserts that the factorial $(n-1)! = 1 \times 2 \times 3 \times \cdots \times (n-1)$ is one less than a multiple of n exactly when n is a prime number. Another way of stating it is for a natural number $n > 1$ is a prime number if and only if:

When (n − 1)! is divided by n, the remainder minus 1 is divides evenly into (n-1)!

Conversely, assume equation 4 above, is an integer. If p or $p + d$ is not a prime, then by Wilson's Theorem, at least one of the terms of (4) is not an integer. This implies that none of the terms of equation 4 is an integer or equivalently neither of p and $p + d$ is prime. It follows that both fractions of (4) are in reduced form.

It is easy to see that if a/b and a'/b' are reduced fractions such that $a/b + a'/b' = (ab' + a'b)/(bb')$ is an integer, then $b|b'$ and $b'|b$.

Applying this result to equation 4, we obtain that $(p + d)|p$, which is impossible. We may therefore conclude that if equation 4 is an integer, then both p and $p + d$ must be prime numbers. Therefore, the equation below is proven since it can be reduced to equation 3 above.

Therefore, since the below equation can be reduced to equation 3 above, we have proven that if $p > 1$ and $d > 0$ are integers, then p and $p + d$ are both primes if and only if for positive integer n:

$$n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^{d} d!}{p + d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

For our case p = F_{y-1} and d = F_{y-2} are Fibonacci numbers, and, $F_y = F_{y-1} + F_{y-2}$, our objective is to prove that if p = F_{y-1} and d = F_{y-2} , then p + d = $F_{y-1} + F_{y-2}$ = prime number = F_y = Fibonacci prime. Since, as discussed earlier F_y is outside our finite set of Fibonacci Prime, therefore, if we prove F_y is prime then we will prove the infinitude of Fibonacci Primes.

$$n = (F_{y-1} - 1)!\left(\frac{1}{F_{y-1}} + \frac{(-1)^{F_{y-2}}(F_{y-2})!}{F_{y-1} + F_{y-2}}\right) + \frac{1}{F_{y-1}} + \frac{1}{F_{y-1} + F_{y-2}}$$

Multiplying by F_{y-1} ,

$$n(F_{y-1}) = (F_{y-1})!\left(\frac{1}{F_{y-1}} + \frac{(-1)^{F_{y-2}}(F_{y-2})!}{F_{y-1} + F_{y-2}}\right) + 1 + \frac{F_{y-1}}{F_{y-1} + F_{y-2}}$$

Multiplying by $(F_{y-1} + F_{y-2})$ and since F_{y-1} , is greater than 2 and is prime, then F_{y-1} is prime and therefore F_{y-2} must be even for F_y to be prime:

$$n(F_{y-1} + F_{y-2})(F_{y-1}) = (F_{y-1} + F_{y-2})(F_{y-1})!\left(\frac{1}{F_{y-1}} + \frac{(F_{y-2})!}{F_{y-1} + F_{y-2}}\right) + F_{y-1} + F_{y-2} + F_{y-1}$$

Reducing,

$$n(F_{y-1} + F_{y-2})(F_{y-1}) = (F_{y-1})!\left(\frac{F_{y-1} + F_{y-2}}{F_{y-1}} + (F_{y-2})!\right) + 2F_{y-1} + F_{y-2}$$

Reducing again,

$$n(F_{y-1} + F_{y-2})(F_{y-1}) = ((F_{y-1)} - 1))!\,(F_{y-1} + F_{y-2} + F_{y-1}(F_{y-2})!) + 2F_{y-1} + F_{y-2}$$

We already know F_{y-1} and F_{y-2} are Fibonacci numbers therefore; F_{y-1} and F_{y-2} are both positive integers. We also know that by the definition of Fibonacci numbers that the sum of F_{y-1} and F_{y-2} is a Fibonacci number, specifically, $F_y = F_{y-1} + F_{y-2}$, therefore F_y is also a positive integer. Since F_{y-1} and F_{y-2} are integers the right hand side of the above equation is an integer. Since the right hand side of the above equation is an integer and F_{y-1} and F_{y-2} are integers on the left hand side of the equation, then n must be an integer for the left side of equation to be an integer. It suffices to show that n is an integer since we have proven that F_{y-1} and $F_{y-1} + F_{y-2} = F_y$ are both primes if and only if for integer n:

$$n = (F_{y-1} - 1)!\left(\frac{1}{F_{y-1}} + \frac{(-1)^{F_{y-2}}(F_{y-2})!}{F_{y-1} + F_{y-2}}\right) + \frac{1}{F_{y-1}} + \frac{1}{F_{y-1} + F_{y-2}}$$

Since n = integer, we have proven that $F_{y-1} + F_{y-2}$ must be prime, and since $F_y = F_{y-1} + F_{y-2}$, then F_y is also prime. Since we proved earlier that if $F_{y-1} + F_{y-2}$ is prime then it also is not in the finite set of p_i Fibonacci primes, therefore, since we have proven that $F_{y-1} + F_{y-2} = F_y$ is prime, then we have proven that there is a Fibonacci prime outside our assumed finite set of Fibonacci primes. This is a contradiction from our assumption that the set of Fibonacci primes is finite, therefore, by contradiction the set of Fibonacci primes is infinite. Also this same proof can be repeated infinitely for each finite set of Fibonacci primes, in other words a new Fibonacci prime can added to each set of finite Fibonacci primes. This thoroughly proves that an infinite number of Fibonacci primes exist.

References:

1) *TYCM, Vol. 19, 1988, p. 191*

2) 1001 Problems in Classical Number Theory, Jean-Marie De Koninck and Armel Mercier, 2004