Proofs of Polignac Prime Conjecture, Goldbach Conjecture, Twin Prime Conjecture, Cousin Prime Conjecture, and Sexy Prime Conjecture

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Abstract

This paper presents a complete and exhaustive proof of the Polignac Prime Conjecture. The approach to this proof uses same logic that Euclid used to prove there are an infinite number of prime numbers. Then we prove that if $p > 1$ and $d > 0$ are integers, that p and $p + d$ are both primes if and only if for integer n (see reference 1 and 2):

$$n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^{d} d!}{p + d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

We use this proof for d = 2k to prove the infinitude of Polignac prime numbers.

Additionally, our proof of the Polignac Prime Conjecture leads to proofs of several other significant number theory conjectures such as the Goldbach Conjecture, Twin Prime Conjecture, Cousin Prime Conjecture, and Sexy Prime Conjecture. Our proof of Polignac's Prime Conjecture provides significant accomplishments to Number Theory, yielding proofs to several conjectures in number theory that has gone unproven for hundreds of years.

Introduction

The Polignac prime conjecture, was made by Alphonse de Polignac in 1849. Alphonse de Polignac (1826 – 1863) was a French mathematician whose father, Jules de Polignac (1780-1847) was prime minister of Charles X until the Bourbon dynasty was overthrown in1830. Polignac attended the École Polytechnique (commonly known as Polytechnique) a French public institution of higher education and research, located in Palaiseau near Paris. In 1849, the year Alphonse de Polignac was admitted to Polytechnique, he made what's known as Polignac's conjecture:

For every positive integer k, there are infinitely many prime gaps of size 2k.

Alphonse de Polignac made other significant contributions to number theory, including the de Polignac's formula, which gives the prime factorization of n!, the factorial of n, where n ≥ 1 is a positive integer.

Proof of Polignac's Conjecture

In number theory, Polignac's conjecture states:

For any positive integer k, then for any positive even number 2k, there are infinitely many prime gaps of size 2k. In other words, there are infinitely many cases of two consecutive prime numbers with difference 2k.

We shall use Euclid's logic that he used to prove there are an infinite number of prime numbers to prove there are an infinite number Polignac primes.

First we shall assume there are only a finite number of n Polignac's primes for all positive integers k, specifically;

1) Say, p_1, $p_1 + 2k$, p_2, $p_2 + 2k$ . . . . , $p_{n-1,}$ $p_{n-1} + 2k$, $p_{n,}$ $p_n + 2k$

2) Let $N = p_1(p_1 + 2k)p_2(p_2 + 2k) \ldots, p_{n-1}(p_{n-1} + 2k)p_n(p_n + 2k) +1$

By the fundamental theorem of arithmetic, N is divisible by some prime q. Since N is the product of all existing Polignac primes plus 1, then this prime q cannot be among the p_i, p_i +2k that make up the n Polignac primes since by assumption these are all the Polignac primes that exist and N is not divisible by any of the p_i, p_i +2k Polignac primes. N is clearly seen not to be divisible by any of the p_i, p_i +2k Polignac primes. First we know that 2 is a prime number that is not in the set of finite Polignac primes since if k=1, and p_1 = 2, then p_1 + 2k = 2 + 2 = 4 and 4 is not prime, therefore, 2 cannot be included in the finite set of Polignac primes. We also know that 2 is the only even prime number, therefore, for the finite set of Polignac primes all of the p_i, p_i +2k are odd numbers. Since the product of odd numbers is always odd, then the product of all the p_i, p_i +2k in our finite set of Polignac primes is an odd number. Since N is product of all the p_i, p_i +2k, + 1, then N is an even number, and since all the p_i are odd numbers and N is even, then N is not divisible by any of the p_i Polignac primes. Therefore, q must be another prime number that does not exist in the finite set of Polignac prime numbers. Therefore, since this proof could be repeated an infinite number of times we have proven that an infinite number of prime numbers q exist outside of our finite set of Polignac primes.

Now we must prove that two of these infinite prime numbers, q, are Polignac primes. We will pick a prime number p from the infinite set of primes outside our finite set of Polignac primes and we will need to prove that there exists a prime p + 2k that is also prime. Both p and p +2k do not exist in the finite set of Polignac primes. Note we are not proving this for all q primes outside the finite set of Polignac primes, we are only picking one prime, p, from the infinite set of primes and then we shall prove that p+2k is also prime, this will show that at least one Polignac prime exists outside our finite set of Polignac primes.

First we shall show that if p + 2k is prime it cannot be in the set of finite p_i, p_i +2k twin primes above. Since p is a prime number that does not exist in the set of finite p_i, p_i +2k Polignac primes, then if there exists a prime number equal to p +2k that is prime, it would be a Polignac prime to p; therefore a prime p+2k cannot be in the set of finite n Polignac primes otherwise p would be in the set of n finite Polignac primes and we have proven that p is not in the set of fine Polignac primes, therefore if p +2k is prime it cannot be in the finite set of Polignac primes since it would be twin to p.

By the fundamental theorem of arithmetic, N is divisible by some prime p. Since N is the product of all existing Polignac primes plus 1, then this prime p cannot be among the p_i that make up the n Polignac primes since by assumption these are all the Polignac primes that exist and N is not divisible by any of the p_i Polignac primes. N is clearly seen not to be divisible by any of the p_i Polignac primes. First we know that 2k is only a prime number if k = 1, but 2 is not in the set of finite set of Polignac's primes since 2 + 2k = 2(1 + 2k) is an even number and is not prime. We also know that 2 is the only even prime number, therefore, for the finite set of Polignac primes all of the p_i are odd numbers. Since the product of odd numbers is always odd, then the product of all the p_i in our finite set of Polignac primes is an odd number. Since N is product of all the Polignac's primes (p_i) + 1, then N is an even number, and since all the p_i are odd numbers and N is even, then N is not divisible by any of the p_i Polignac primes. Therefore, p must be another prime number that does not exist in the finite set of Polignac prime numbers.

The only thing left to prove that there are an infinite number of Polignac primes is to prove that p + 2k is also prime and is not in the set of finite Polignac primes.

First we shall show that if p + 2k is prime it cannot be in the set of finite p_i Polignac primes above. Since p is a prime number that does not exist in the set of finite p_i Polignac primes, then if there exists a prime number equal to p + 2k that is also prime, it would be a Polignac prime to p; therefore a prime p+2k cannot be in the set of finite n Polignac primes otherwise p would be in the set of n finite Polignac primes and we have proven that p is not in the set of fine Polignac primes, therefore if p +2k is prime it cannot be in the finite set of Polignac primes since it would be Polignac to p.

Now we shall proceed to prove p + 2k is prime as follows:

We will prove that if p > 1 and d > 0 are integers, that p and p + d are both primes if and only if for positive integer n (see reference 1 and 2):

$$n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^{\mathrm{d}}\mathrm{d}!}{\mathrm{p} + \mathrm{d}}\right) + \frac{1}{\mathrm{p}} + \frac{1}{\mathrm{p} + \mathrm{d}}$$

Proof:

The equation above can be reduced and re-written as:

$$3)\quad \frac{(p-1)! + 1}{\mathrm{p}} + \frac{(-1)^{d} d!\,(p-1)! + 1}{\mathrm{p} + \mathrm{d}}$$

Since $(p + d - 1)! = (p + d - 1)(p + d - 2)\cdots(p + d - d)(p - 1)!$, we have $(p + d - 1)! \equiv (-1)^{d} d!(p - 1)! \pmod{p + d}$, and it follows that equation 4 above is an integer if and only if:

$$4)\quad \frac{(p-1)! + 1}{\mathrm{p}} + \frac{(\mathrm{p} + d - 1)! + 1}{\mathrm{p} + \mathrm{d}}$$

is an integer. From Wilson's Theorem, if p and $p + d$ are two prime numbers, then each of the terms of, equation 4 above, is an integer, which proves the necessary condition. Wilson's Theorem states:

That a natural number $n > 1$ is a prime number if and only if

$$(n-1)! \equiv -1 \pmod{n}.$$

That is, it asserts that the factorial $(n-1)! = 1 \times 2 \times 3 \times \cdots \times (n-1)$ is one less than a multiple of n exactly when n is a prime number. Another way of stating it is for a natural number $n > 1$ is a prime number if and only if:

When (n − 1)! is divided by n, the remainder minus 1 is divides evenly into (n-1)!

Conversely, assume equation 4 above, is an integer. If p or $p + d$ is not a prime, then by Wilson's Theorem, at least one of the terms of (4) is not an integer. This implies that none of the terms of equation 4 is an integer or equivalently neither of p and $p + d$ is prime. It follows that both fractions of (4) are in reduced form.

It is easy to see that if a/b and a'/b' are reduced fractions such that $a/b + a'/b' = (ab' + a'b)/(bb')$ is an integer, then $b|b'$ and $b'|b$.

Applying this result to equation 4, we obtain that $(p + d)|p$, which is impossible. We may therefore conclude that if equation 4 is an integer, then both p and $p + d$ must be prime numbers. Therefore, the equation below is proven since it can be reduced to equation 3 above.

Therefore, since the below equation can be reduced to equation 3 above, we have proven that if p > 1 and d > 0 are integers, then p and p + d are both primes if and only if for positive integer n:

$$n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^{d}d!}{p + d}\right) + \frac{1}{p} + \frac{1}{p + d}$$

For our case p is known to be prime and d = 2k for Polignac primes, where k is any positive integer, therefore:

$$n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^{2k}2k!}{p + 2k}\right) + \frac{1}{p} + \frac{1}{p + 2k}$$

Multiplying by p,

$$np = (p)!\left(\frac{1}{p} + \frac{2k!}{p + 2k}\right) + 1 + \frac{p}{p + 2k}$$

Multiplying by (p + 2k),

$$(p + 2k)np = (p + 2k)(p)!\left(\frac{1}{p} + \frac{2k!}{p + 2k}\right) + p + 2k + p$$

Reducing again,

$$(p+2k)np = (p)!\left(\frac{(p+2k)}{p} + 2k!\right) + 2p + 2k$$

Reducing again,

$$(p+2k)np = p(p-1)!\left(\frac{(p+2k)}{p} + 2k!\right) + 2p + 2k$$

And reducing one final time,

$$(p+2k)np = p(p-1)!\,(p + 2k + 2pk!) + 2p + 2k$$

We already know p is prime, therefore, p = integer. Since p is an integer and by definition k is an integer, the right hand side of the above equation is an integer. Since the right hand side of the above equation is an integer and p and k are integers on the left hand side of the equation, then n must be an integer for the left side of equation to be an integer, or n would need to be a rational fraction that is divisible by p. This implies that n = $\frac{x}{p}$ where, p is prime and x is an interger. Then p = $\frac{x}{n}$, since p is prime, then p is only divisible by p and 1, therefore, n can only be equal to p or 1 in this case, which are both integers, thus n must be an integer. It suffices to show that n is an integer since we have proven that p and $p + 2k$, where $d = 2k$, are both primes if and only if for integer n:

$$n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^{d}d!}{p + d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

Since n = integer, we have proven that p and p + 2k are both prime. Since we proved earlier that if p + 2k is prime then it also is not in the finite set of $p_{i,}$ p_i + 2k Polignac primes, therefore, since we have proven that p+2k is prime, then we have proven that there is a Polignac prime outside the our assumed finite set of Polignac primes. This is a contradiction from our assumption that the set of Polignac primes is finite, therefore, by contradiction the set of Polignac primes is infinite. Also this same proof can be repeated infinitely for each finite set of Polignac primes, in other words a new Polignac prime can added to each set of finite Polignac primes. This thoroughly proves that an infinite number of Polignac primes exist.

Our proof of infinite Polignac primes also proves several other Conjectures since Polignac primes are generalized form of other conjectures. For k = 1, the Polignac primes become the Twin Prime Conjecture and proves there are an infinite number of twin primes. For k = 2, the proof of the Polignac Conjecture proves there are infinitely many Cousin Primes (p, $p + 4$). For k = 3, proof of the Polignac Conjecture proves there are infinitely many Sexy Primes (p, $p + 6$).

Our proof of infinite Polignac primes also provides a major breakthrough to prove the Goldbach's Conjecture. The Goldbach Conjecture is one of the oldest unsolved problems in number theory. It states:

> Every even integer greater than 2 can be expressed as the sum of two primes.

Our proof of infinite Polignac primes can also be used to the Goldbach Conjecture. First we shall assume that not every even number > 2 is a sum of two primes *a* and *b*. However, Polignac's Conjecture states:

For any positive integer k, then for any positive even number 2k, there are infinitely many prime gaps of size 2k. In other words, there are infinitely many cases of two consecutive prime numbers with difference 2k.

Another way of stating the Polignac Conjecture is; for every prime number a and b there are an infinitely many cases where a – b = 2k, for any positive integer k. So we can state the following:

$$a - b = 2k$$

$$\text{Reducing, } a = 2k + b$$

Adding b to both sides, $a + b = 2k + 2b$

Finally, reducing, $a + b = 2(k + b)$

The right hand side of the equation above, $2(k + b)$, is even and includes all positive even numbers since k is any positive integer and b is from an infinite number of prime numbers. Since k is any positive integer and b is selected from an infinite cases of prime numbers, then

For any integer n, $a + b = 2(k + b) = 2n$

Thus, $a + b = 2n$, for every even number 2n

Therefore, we have proven the Goldbach Conjecture, specifically, every even integer greater than 2 can be expressed as the sum of two primes, i.e, $n > 1$.

References:

1) *TYCM, Vol. 19, 1988, p. 191*
2) *1001 Problems in Classical Number Theory, Jean-Marie De Koninck and Armel Mercier, 2004*