What Mathematics Is The Most Fundamental?

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Abstract

Standard mathematics involves such notions as infinitely small/large, continuity and standard division. This mathematics is usually treated as fundamental while finite mathematics is treated as inferior. Standard mathematics has foundational problems (as follows, for example, from Gödel's incompleteness theorems) but it is usually believed that this is less important than the fact that it describes many experimental data with high accuracy. We argue that the situation is the opposite: standard mathematics is only a degenerate case of finite one in the formal limit when the characteristic of the ring or field used in finite mathematics goes to infinity. Therefore foundational problems in standard mathematics are not fundamental.

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The problem of infinities has two major independent aspects:

- Problem 1: Does mathematics involving infinities ensure correct calculations of all phenomena on classical and quantum levels?
- Problem 2: Can mathematics involving infinities be substantiated as an abstract science?

Standard mathematics involves such notions as infinitely small/large, continuity and standard division although in traditional and constructive versions some of them are treated differently. Historically those notions have arisen from a belief based on everyday experience that any macroscopic object can be divided into arbitrarily large number of arbitrarily small parts. However, the very existence of elementary particles indicates that those notions have only a limited meaning. Indeed, we can divide any macroscopic body by ten, million, etc. but when we reach the level of atoms and elementary particles the division operation loses its meaning and we cannot obtain arbitrarily small parts. Analogously, in any computer the number of bits can be only a positive integer and such notions as 1/2 bit, 1/3 bit etc. are meaningless.

Those examples show that mathematics involving the set of all rational numbers has only a limited applicability and using standard mathematics in quantum physics and computer science is at least unnatural. As a consequence, any description of macroscopic phenomena using continuity and differentiability can be only approximate. Water in the ocean can be described by differential equations of hydrodynamics but this is only an approximation since matter is discrete.

Problem 2) has a long history described in numerous textbooks and monographs (see e.g. Ref. [1]). As shown by Russell and other mathematicians, the Cantor set theory contains several fundamental paradoxes. To avoid them several axiomatic set theories have been proposed and the most known of them is the ZFC theory developed by Zermelo and Fraenkel. However, the consistency of ZFC cannot be proven within ZFC itself and it was shown that the continuum hypothesis is independent of ZFC. Gödel's incompleteness theorems state that no system of axioms can ensure that all facts about natural numbers can be proven and the system of axioms in traditional mathematics cannot demonstrate its own consistency.

In constructive mathematics proposed by Brouwer there is no law of the excluded middle and it is required that any proof of existence be algorithmic. That is why constructive mathematics is treated such that, at least in principle, it can be implemented on a computer. Here "in principle" means that the number of steps might be not finite. With this meaning constructive mathematics, as well as traditional one, assumes that one can operate with any desired amount of resources and it is theoretically possible to consider an idealized case when a computer can operate with any desirable number of bits.

The absolute majority of mathematicians prefer the traditional version. Physics is also based only on traditional mathematics. Hilbert was a strong opponent of constructive mathematics. He said: *"No one shall expel us from the paradise that Cantor has created for us"* and *"Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists"*.

Some well known results of traditional mathematics are counterintuitive. For example, since the mapping tgx from $(-\pi/2, \pi/2)$ to $(-\infty, \infty)$ is a bijection, those intervals have the same number of elements although the former is a part of the latter. Another example is Hilbert's Grand Hotel paradox (see e.g. Wikipedia). However, in traditional mathematics those examples are not treated as contradictory.

Let us look at mathematics from the point of view of philosophy of quantum theory according to which there should be no statements accepted without proof (i.e. axioms). The theory should contain only those statements that can be verified, where by "verified" physicists mean an experiment involving only a finite number of steps.

Let us pose a problem of whether 10+20 equals 30. Then we should describe an experiment which should solve this problem. Any computer can operate only with a finite number of bits and can perform calculations only modulo some number p. Say $p = 40$, then the experiment will confirm that 10+20=30 while if $p = 25$ then we will get that 10+20=5. *So the statements that 10+20=30 and even that* $2 \cdot 2 = 4$ *are ambiguous because they do not contain information on how they should be verified.* We believe the following observation is very important: although standard mathematics is a part of our everyday life, people typically do not realize that *standard mathematics is implicitly based on the assumption that one can have any desirable amount of resources.* So standard mathematics (including traditional and constructive versions) is based on the implicit assumption that we can consider a formal limit $p \to \infty$ and the correctness of the limit can be substantiated.

While Gödel's works on the incompleteness theorems are written in highly technical terms of mathematical logics, the fact that standard mathematics has foundational problems is clear from the philosophy of quantum theory. For instance, the first incompleteness theorem says that not all facts about natural numbers can be proven. However, from the philosophy of quantum theory this seems to be clear because if the number of numbers is not finite then we cannot verify that $a + b = b + a$ for any a and b.

The famous Kronecker's expression is: *"God made the natural numbers, all else is the work of man"*. However only addition and multiplication are always possible in the set of natural numbers. In order to make addition invertible we introduce negative integers. However, they do not have a direct physical meaning (e.g. the phrases "I have -2 apples" or "this computer has -100 bits of memory" are meaningless). Their only goal is to get the ring of integers Z. The next step is the transition to the field of rational numbers Q and analogous remarks can be made about division.

However, if we consider only a set F_p of p numbers 0, 1, 2, ... $p - 1$ where p is prime and the operations are defined as usual but modulo p then we get a field without adding new elements. Note that in F_p we can formally use the minus sign because, by definition, $b = -a$ if $a + b = 0$ in F_p. For example, $-1 = p - 1$ and $-(p - 1)/2 = (p + 1)/2$.

Consider first F_p as a ring of elements $\{0, \pm i\}$ $(i = 1, ...(p - 1)/2)$. Let f be a function from F_p to Z such that $f(a)$ in Z has the same notation in Z as a in F_p. Then for elements $a \in F_p$ such that $|f(a)| \ll p$, addition, subtraction and multiplication are the same as in Z. In other words, for such elements we do not feel the existence of p. Indeed, let $\tilde{F_p}$ be a subset of elements $a \in F_p$ such that $|f(a)| < [(p - 1)/2]^{1/2}$. Then for $a_1, a_2 \in \tilde{F_p}$, $f(a_1 + a_2) = f(a_1) + f(a_2)$ and $f(a_1 a_2) = f(a_1) f(a_2)$ which shows that if F_p is treated as a ring then f is a local isomorphism between F_p and Z.

As explained in textbooks, both F_p and Z are cyclic groups with respect to addition. However, an important difference between F_p and Z is that only the former has a property which we call *strong cyclicity*: for any fixed $a \in F_p$ any element of F_p distinct from a can be obtained from a by successively adding 1. In particular, by successively adding 1 to a "positive" element $a \in F_p$ (i.e. such that $f(a) > 0$)

we will get all "positive" elements, all "negative" elements (such that $f(b) < 0$) and zero. As noted below, in particle physics the presence or absence of strong cyclicity plays an important role.

The above remarks show that if elements of Z are depicted as integer points on the x axis of the plane xy then it is natural to depict the elements of F_p as points of the circumference in Fig. 1 such that the distance between the neighboring elements of F_p is unity.

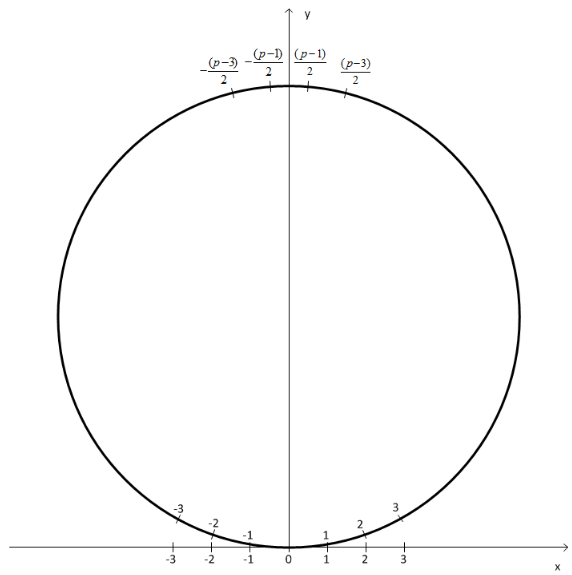

Figure 1: Relationship between F_p and Z

When p increases, the bigger and bigger part of $\tilde{F}_p$ becomes the same as Z. Hence we can conclude that Z can be treated as a degenerate case of F_p in the formal limit $p \to \infty$ because in this limit operations modulo p disappear and strong cyclicity is broken. *Therefore, at the level of rings standard mathematics is a degenerate case of finite one when formally $p \to \infty$.*

In standard mathematics there exists a similar construction called stereographic projection. We again consider the xy plane and treat the points of the x axis as elements of the set of real numbers $\mathcal{R}$. Let $\mathcal{C}$ be a circumference with the center at $(0, R)$ and the radius R. This set has a property similar to strong cyclicity because if we take a point $b \in \mathcal{C}$ then any other point can be obtained from b by a rotation by an angle $\varphi \in [0, 2\pi)$ and after rotating by 2π we will come back to b.

Let $a = (0, 2R)$ be the North pole of $\mathcal{C}$ and $f(x, y)$ be a function from $\mathcal{C} - a$ to $\mathcal{R}$ such that the line containing the points a and (x, y) crosses the x axis at $X = f(x, y)$. Let $\alpha \in (-\pi/2, \pi/2)$ be the angle between this line and the line connecting a with the origin. Then $X = 2Rtg\alpha$, $x = Rsin(2\alpha)$ and $y = 2Rsin^2\alpha$. Then the stereographic projection is a bijection between $\mathcal{C} - a$ and $\mathcal{R}$.

A natural distance between the elements $X_1, X_2 \in \mathcal{R}$ is $|X_1 - X_2|$ while a natural distance in $\mathcal{C}$ is such that if two points are characterized by polar angles φ_1 and φ_2 then the distance between them is $R \cdot min(|\varphi_1 - \varphi_2|, (2\pi - |\varphi_1 - \varphi_2|))$. The map f does not conserve distances but in a vicinity of $\mathcal{C}$ near the origin such that $|\alpha| < \alpha_0$, $\alpha_0 \ll 1$ the distance between any two points is approximately the same as the distance between their images in $\mathcal{R}$. In the formal limit $(R\alpha_0 \to \infty, R\alpha_0^3 \to 0)$ this vicinity becomes $\mathcal{R}$ and strong cyclicity is broken. The situation is similar to the transition from F_p to Z, and $\mathcal{R}$ can be treated as a degenerate case of $\mathcal{C}$ in this limit.

The above constructions have a well-known historical analogy. For many years people believed that our Earth was flat and infinite, and only after a long period of time they realized that it was finite and had a curvature. It is difficult to notice the curvature when we deal only with distances much less than the radius of the curvature. Analogously one might think that the set of numbers describing physics has a "curvature" defined by a very large number p but we do not notice it when we deal only with numbers much less than p.

One might argue that introducing a new fundamental constant p is not justified. However, the history of physics tells us that new theories arise when a parameter, which in the old theory was treated as infinitely small or infinitely large, becomes finite. For example, from the point of view of classical nonrelativistic physics, the velocity of light c is infinitely large and the Planck constant $\hbar$ is infinitely small but in relativistic quantum theory they are finite. Therefore, it is natural to think that in the future quantum physics the quantity p will be not infinitely large but finite. A problem arises whether p is a constant or is different in various periods of time. In view of the problem of time in quantum theory, an extremely interesting scenario is that the world time is defined by p.

At the level of fields, finite and standard mathematics already considerably differ each other. For example, $1/2$ in F_p equals $(p+1)/2$, i.e. a very large number if p is large. However, this does not mean that mathematics modulo p cannot describe physics because spaces in quantum theory are projective. In Refs. [2, 3] we have proposed an approach called GFQT where quantum theory is based on a Galois field with characteristic p. It has been shown that in the formal limit $p \to \infty$ GFQT recovers predictions of standard continuous theory. Then the fact that standard mathematics describes many experiments with a high accuracy is a consequence of the fact that in real life the number p is very large.

In addition, GFQT gives a new look at many fundamental problems in physics (see the discussion in Refs. [4, 5]). We consider two examples which are strong indications that nature is described by finite mathematics.

A well-known fact of particle physics is that a particle and its antiparticle have equal masses. This follows from the requirement that their irreducible representations (IRs) should be combined into a local field satisfying a covariant equation (e.g. the Dirac equation). A question arises that if locality is only approximate then the masses of a particle and its antiparticle remain equal or can differ each other?

This question is legitimate because, since local fields are described by non-unitary representations, their probabilistic interpretation is problematic.

In standard theory, IRs corresponding to particles are constructed from a state where energy=mass. When representation operators act on this state the energy can only increase and we get the spectrum of energies in the range $[mass, \infty)$. In mathematics such IRs are called IRs with the minimum weight. Analogously, in the case of antiparticles we start from a state where energy=-mass. Then the energy can only decrease and we get an IR with the maximum weight where the spectrum of energies is in the range $(-\infty, -mass]$. Hence in standard theory a particle and its antiparticle are described by different IRs and the equality of their masses is a consequence of additional requirements.

In GFQT we also start from a state where energy=mass and gradually increase the energy by acting by representation operators on this state. However, in such a way we are moving not along a straight line but along the field F_p in Fig. 1. Then sooner or later we will arrive at the point where energy=-mass which shows that a finite field analog of an IR with the minimum weight is simultaneously a finite field analog of an IR with the maximum weight. Therefore in GFQT a particle and its antiparticle belong to the same IR and have the same masses because the field F_p is finite and has the property of strong cyclicity. This effect cannot be reproduced in standard theory where there are no operations modulo p.

Another striking example is that gravity can be treated not as an interaction but simply as a manifestation of the fact that nature is described by a Galois field of characteristic p. In this approach the gravitational constant G is not a parameter taken from the outside (e.g. from the condition that theory should describe experiment) but a quantity which should be calculated. The actual calculation is problematic but reasonable qualitative arguments show [5] that G is proportional to $1/lnp$. Therefore, gravity is a consequence of the finiteness of nature and disappears in the limit $p \to \infty$. A qualitative estimation based on additional assumptions gives that p is a huge number of the order of $exp(10^{80})$.

Also as noted above, in quantum theory division has a limited applicability. This might be an indication that (as Metod Saniga pointed out), in the spirit of Refs. [6, 7, 8], the ultimate quantum theory will be based even on a finite ring and not a field.

The above discussion indicates that the answer to Problem 1 is probably negative. However, regardless of whether or not this is the case, Problem 2 still remains. The absolute majority of physicists and mathematicians think that standard mathematics is fundamental while finite one is inferior. Typical reasons are that standard mathematics contains more numbers than finite one and that the whole history of mankind has proven that standard mathematics describes reality with an unprecedented accuracy. For those reasons, the fact that standard mathematics has foundational problems is usually treated as less important.

However, any realistic calculations can involve only a finite number of bits and any experiment has a finite accuracy. As explained above, in the formal limit $p \to \infty$ operations modulo p disappear and strong cyclicity is broken. Therefore standard mathematics can be treated as a degenerate case of finite one in the formal limit $p \to \infty$. This fact is obvious and probably it has been overlooked by mathematicians.

An illusion of continuity arises because p is very large. Standard mathematics might be treated only as a technique which in many cases describes reality with a high accuracy while the fact that this mathematics has foundational problems indeed does not have a fundamental role. The philosophy of Brouwer, Cantor, Fraenkel, Gödel, Hilbert, Kronecker, Russell, Zermelo and other great mathematicians working on foundation of standard mathematics was based on macroscopic experience in which the notions of infinitely small, infinitely large, continuity and standard division are natural. However, as noted above, those notions contradict the existence of elementary particles and are not natural in quantum theory.

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