Notes On the Proof of Second Hardy-Littlewood Conjecture

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Abstract

In this paper a slightly stronger version of the Second Hardy-Littlewood Conjecture, namely the inequality $\pi(x) + \pi(y) > \pi(x + y)$ is examined, where $\pi(x)$ denotes the number of primes not exceeding x. It is shown that there the inequalty holds for all sufficiently large x and y.

1 Introduction

The original version of the conjecture is $\pi(x) + \pi(y) \geq \pi(x + y)$ for all x,$y \geq 2$. It had been suggested by E. Landau that $\pi(2x) < 2\pi(x)$ for all $x \geq 3$ and this eas subsequently proved by Rosser and Schoenfeld. There are some known inequalities that are similar in spirit of the Hardy-Littlewood Conjecture. For example, C. Karanikolov showed that if $a \geq e^{\frac{1}{4}}$ and $x \geq 364$ then we have,

$$\pi(ax) < a\pi(x)$$

V. Udrescu proved that if $0 < \epsilon \leq 1$ and $\epsilon x \leq y \leq x$ then $\pi(x)+\pi(y) > \pi(x+y)$ for x and y sufficiently large. L. Panaitopol made these two results sharper by proveing that if $a > 1$ and $x > e^{4(\ln a)^{-2}}$ then $\pi(ax) < a\pi(x)$ and If $a \in (0, 1]$ and $x \geq y \geq ax$, $x \geq e^{9a^{-2}}$, then $\pi(x) + \pi(y) > \pi(x + y)$. However, as may be noted that the inequality has been proved under the hypothesis that $x \geq y \geq ax$. In the same paper Panaitopol deduced an unconditional inequality which states that for all positive integer x and y such that $x, y \geq 4$ we have,

$$\frac{1}{2}\pi(x + y) \leq \pi\left(\frac{x}{2}\right) + \pi\left(\frac{y}{2}\right)$$

In this paper we prove that for all sufficienly large x and y the Hardy-Littlewood Ineqaulity holds. For this purpose we will examine the inequality $\pi(ky) + \pi(y) > \pi((k+1)y)$ and try to find out the range of valued of y for which for all $k >$ we the inequality.

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2 The Theorem

In this section we present the only theorem of our paper. But before that we agree to call the following inequality as *Poussin's Inequality* which arises quite naturally from de la Vallée-Poussin's proof of Prime Number Theorem, namely the result that for all $\epsilon > 0$ and for all sufficiently large x we have,

$$\frac{x}{\ln x - (1-\epsilon)} < \pi(x) < \frac{x}{\ln x - (1+\epsilon)}$$

Now let us state the theorem.

- **Theorem**

  For all $k > 1$ and for all y such that,

  $$\frac{y}{\ln y - (1-\epsilon)} < \pi(y) < \frac{y}{\ln y - (1+\epsilon)}$$

  we have $\pi(ky) + \pi(y) > \pi((k+1)y)$ for all ϵ satisfying $0 < \epsilon \leq \ln\sqrt{2}$.

  Proof

  We strat by noing that,

  $$\pi(ky) + \pi(y) > \frac{ky}{\ln ky - (1-\epsilon)} + \frac{y}{\ln y - (1-\epsilon)}$$

  and,

  $$\frac{(k+1)y}{\ln(k+1)y - (1+\epsilon)} > \pi\left((k+1)y\right)$$

  Hence proving,

  $$\frac{ky}{\ln ky - (1-\epsilon)} + \frac{y}{\ln y - (1-\epsilon)} \geq \frac{(k+1)y}{\ln(k+1)y - (1+\epsilon)}$$

  Or equivalently,

  $$\frac{k}{\ln ky - (1-\epsilon)} + \frac{1}{\ln y - (1-\epsilon)} \geq \frac{k+1}{\ln(k+1)y - (1+\epsilon)}$$

  will imply our inequality with some condition(s) imposed on k and y.

  Notice that the above inequality is satisfied if and only if,

  $$k\left(\frac{1}{\ln ky - (1-\epsilon)} - \frac{1}{\ln(k+1)y - (1+\epsilon)}\right) \geq \left(\frac{1}{\ln(k+1)y - (1+\epsilon)} - \frac{1}{\ln y - (1-\epsilon)}\right)$$

  Which holds if and only if,

  $$k\left(\frac{\ln\left(1+\dfrac{1}{k}\right) - 2\epsilon}{\ln ky - (1-\epsilon)}\right) \geq \left(\frac{2\epsilon - \ln(k+1)}{\ln y - (1-\epsilon)}\right)$$

Or equivalently,

$$k\left(\frac{\ln y-(1-\epsilon)}{\ln ky-(1-\epsilon)}\right)\geq\left(\frac{2\epsilon-\ln(k+1)}{\ln\left(1+\frac{1}{k}\right)-2\epsilon}\right)$$

Now we note that the two inequalities $2\epsilon-\ln(k+1)>0$ and $\ln\left(1+\frac{1}{k}\right)-2\epsilon>0$ can't hold simultaneously for all $k>1$.

In anticipation of a contradiction, let us suppose that they do. Then $2\epsilon-\ln(k+1)>0$ and $\ln\left(1+\frac{1}{k}\right)-2\epsilon>0$. Now the first inequality implies $k<e^{2\epsilon}-1$ while the second implies $\frac{1}{k}>e^{2\epsilon}-1$. Combining we get, $\frac{1}{k}>e^{2\epsilon}-1>k$ which is impossible for all $k>1$.

Thus we can either have $2\epsilon-\ln(k+1)<0$ and $\ln\left(1+\frac{1}{k}\right)-2\epsilon<0$ or $2\epsilon-\ln(k+1)>0$ and $\ln\left(1+\frac{1}{k}\right)-2\epsilon<0$. In the first case we get, $k>e^{2\epsilon}-1$ while in the second we get $k<e^{2\epsilon}-1$. Notice that for all $0<\epsilon\leq\ln\sqrt{2}$ we would have only the first inequality.

We conclude that for all $x>y$ and for all y such that $\frac{y}{\ln y-(1-\epsilon)}<\pi(y)<\frac{y}{\ln y-(1+\epsilon)}$ for all $0<\epsilon\leq\ln\sqrt{2}$ the inequality, $\pi(x)+\pi(y)>\pi(x+y)$, is true.

3 Conclusion and Remarks

It can be easily deduced from what we have proved that for the above conditions as stated in the theorem just discussed satisfied by every x and y Hardy-Littlwood Inequality holds. However, the problem is that the inequality may fail infinitely often. For there may exist an y which doesn't satisfy the *Poussin's Inequality* for any $0<\epsilon\ln\sqrt{2}$ but the x satisfies *Poussin's Inequality* for $0<\epsilon\leq\ln\sqrt{2}$ and for that y and for such x the inequality may fail infinitely often. But there is a theoretical procedure to settle this matter completely. For from the limit $\lim_{n\to\infty}\frac{\pi(n)}{n}$ we conclude that there exists a M such that for all $n\geq M$ we have $\left|\frac{\pi(n)}{n}\right|<\epsilon$ for each $\epsilon>0$. Now from the assumption $x>y$, V. Udrescu's result that if $0<\epsilon\leq1$ and $\epsilon x\leq y\leq x$ then $\pi(x)+\pi(y)>\pi(x+y)$ for x and y sufficiently large and L. Panaitopol's result that if $a>1$ and $x>e^{4(\ln a)^{-2}}$ then $\pi(ax)<a\pi(x)$ we notice that,

$$\pi(x+y)<\left(1+\frac{y}{x}\right)\pi(x)=\pi(x)+\frac{y}{x}\pi(x)<\pi(x)+\pi(y)$$

Hence it can be concluded that for each y we have a real M_x such that for all $x\geq M_x$ we have $\pi(x+y)<\pi(x)+\pi(y)$. One way for setting this conjecture once and for all is to calculate explicit bounds for which the theorem described in this paper holds and then use the procedure elaborated above.

4 References

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