Notes On the Proof of Second Hardy-Littlewood Conjecture

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October 13, 2014

Abstract

In this paper a slightly stronger version of the Second Hardy-Littlewood Conjecture, namely the inequality $\pi(x) + \pi(y) > \pi(x + y)$ is examined, where $\pi(x)$ denotes the number of primes not exceeding x. It is shown that there the inequalty holds for all sufficiently large x and y.

1 Introduction

The original version of the conjecture is $\pi(x) + \pi(y) \geq \pi(x + y)$ for all x,$y \geq 2$. It had been suggested by E. Landau that $\pi(2x) < 2\pi(x)$ for all $x \geq 3$ and this eas subsequently proved by Rosser and Schoenfeld. There are some known inequalities that are similar in spirit of the Hardy-Littlewood Conjecture. For example, C. Karanikolov showed that if $a \geq e^{\frac{1}{4}}$ and $x \geq 364$ then we have,

$$\pi(ax) < a\pi(x)$$

V. Udrescu proved that if $0 < \epsilon \leq 1$ and $\epsilon x \leq y \leq x$ then $\pi(x)+\pi(y) > \pi(x+y)$ for x and y sufficiently large. L. Panaitopol made these two results sharper by proveing that if $a > 1$ and $x > e^{4(\ln a)^{-2}}$ then $\pi(ax) < a\pi(x)$ and If $a \in (0, 1]$ and $x \geq y \geq ax$, $x \geq e^{9a^{-2}}$, then $\pi(x) + \pi(y) > \pi(x + y)$. However, as may be noted that the inequality has been proved under the hypothesis that $x \geq y \geq ax$. In the same paper Panaitopol deduced an unconditional inequality which states that for all positive integer x and y such that $x, y \geq 4$ we have,

$$\frac{1}{2}\pi(x + y) \leq \pi\left(\frac{x}{2}\right) + \pi\left(\frac{y}{2}\right)$$

In this paper we prove that for all sufficienly large x and y the Hardy-Littlewood Ineqaulity holds. For this purpose we will examine the inequality $\pi(ky) + \pi(y) > \pi((k+1)y)$ and try to find out the range of valued of y for which for all $k >$ we the inequality.

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2 The Theorem

In this section we present the only theorem of our paper. But before that we agree to call the following inequality as *Poussin's Inequality* which arises quite naturally from de la Vallée-Poussin's proof of Prime Number Theorem, namely the result that for all $\epsilon > 0$ and for all sufficiently large x we have,

$$\frac{x}{\ln x - (1-\epsilon)} < \pi(x) < \frac{x}{\ln x - (1+\epsilon)}$$

Now let us state the theorem.

- **Theorem**

  For all $k > 1$ and for all y such that,

  $$\frac{y}{\ln y - (1-\epsilon)} < \pi(y) < \frac{y}{\ln y - (1+\epsilon)}$$

  we have $\pi(ky) + \pi(y) > \pi((k+1)y)$ for all ϵ satisfying $0 < \epsilon \leq \ln \sqrt{2}$.

  Proof

  We strat by noing that,

  $$\pi(ky) + \pi(y) > \frac{ky}{\ln ky - (1-\epsilon)} + \frac{y}{\ln y - (1-\epsilon)}$$

  and,

  $$\frac{(k+1)y}{\ln(k+1)y - (1+\epsilon)} > \pi\left((k+1)y\right)$$

  Hence proving,

  $$\frac{ky}{\ln ky - (1-\epsilon)} + \frac{y}{\ln y - (1-\epsilon)} \geq \frac{(k+1)y}{\ln(k+1)y - (1+\epsilon)}$$

  Or equivalently,

  $$\frac{k}{\ln ky - (1-\epsilon)} + \frac{1}{\ln y - (1-\epsilon)} \geq \frac{k+1}{\ln(k+1)y - (1+\epsilon)}$$

  will imply our inequality with some condition(s) imposed on k and y.

  Notice that the above inequality is satisfied if and only if,

  $$k\left(\frac{1}{\ln ky - (1-\epsilon)} - \frac{1}{\ln(k+1)y - (1+\epsilon)}\right) \geq \left(\frac{1}{\ln(k+1)y - (1+\epsilon)} - \frac{1}{\ln y - (1-\epsilon)}\right)$$

  Which holds if and only if,

  $$k\left(\frac{\ln\left(1+\frac{1}{k}\right) - 2\epsilon}{\ln ky - (1-\epsilon)}\right) \geq \left(\frac{2\epsilon - \ln(k+1)}{\ln y - (1-\epsilon)}\right)$$

Or equivalently,

$$k\left(\frac{\ln y-(1-\epsilon)}{\ln ky-(1-\epsilon)}\right)\geq\left(\frac{2\epsilon-\ln(k+1)}{\ln\left(1+\dfrac{1}{k}\right)-2\epsilon}\right)$$

Now we note that the two inequalities $2\epsilon-\ln(k+1)>0$ and $\ln\left(1+\frac{1}{k}\right)-2\epsilon>0$ can't hold simultaneously for all $k>1$.

In anticipation of a contradiction, let us suppose that they do. Then $2\epsilon-\ln(k+1)>0$ and $\ln\left(1+\frac{1}{k}\right)-2\epsilon>0$. Now the first inequality implies $k<e^{2\epsilon}-1$ while the second implies $\frac{1}{k}>e^{2\epsilon}-1$. Combining we get, $\frac{1}{k}>e^{2\epsilon}-1>k$ which is impossible for all $k>1$.

Thus we can either have $2\epsilon-\ln(k+1)<0$ and $\ln\left(1+\frac{1}{k}\right)-2\epsilon<0$ or $2\epsilon-\ln(k+1)>0$ and $\ln\left(1+\frac{1}{k}\right)-2\epsilon<0$. In the first case we get, $k>e^{2\epsilon}-1$ while in the second we get $k<e^{2\epsilon}-1$. Notice that for all $0<\epsilon\leq\ln\sqrt{2}$ we would have only the first inequality.

We conclude that for all $x>y$ and for all y such that $\frac{y}{\ln y-(1-\epsilon)}<\pi(y)<\frac{y}{\ln y-(1+\epsilon)}$ for all $0<\epsilon\leq\ln\sqrt{2}$ the inequality, $\pi(x)+\pi(y)>\pi(x+y)$, is true.

3 Conclusion and Remarks

It can be easily deduced from what we have proved that for the above conditions as stated in the theorem just discussed satisfied by every x and y Hardy-Littlwood Inequality holds. However, the problem is that the inequality may fail infinitely often. For there may exist an y which doesn't satisfy the *Poussin's Inequality* for any $0<\epsilon\ln\sqrt{2}$ but the x satisfies *Poussin's Inequality* for $0<\epsilon\leq\ln\sqrt{2}$ and for that y and for such x the inequality may fail infinitely often. But there is a theoretical procedure to settle this matter completely. For from the limit $\lim_{n\to\infty}\frac{\pi(n)}{n}$ we conclude that there exists a M such that for all $n\geq M$ we have $\left|\frac{\pi(n)}{n}\right|<\epsilon$ for each $\epsilon>0$. Now from the assumption $x>y$, V. Udrescu's result that if $0<\epsilon\leq 1$ and $\epsilon x\leq y\leq x$ then $\pi(x)+\pi(y)>\pi(x+y)$ for x and y sufficiently large and L. Panaitopol's result that if $a>1$ and $x>e^{4(\ln a)^{-2}}$ then $\pi(ax)<a\pi(x)$ we notice that,

$$\pi(x+y)<\left(1+\frac{y}{x}\right)\pi(x)=\pi(x)+\frac{y}{x}\pi(x)<\pi(x)+\pi(y)$$

for all $x > e^{4(\ln a)^{-2}}$.Now if it can be shown that for each $a > 1$ and for all $y \geq 2$ we have and $\pi(x) + \pi(y) \geq \pi(x + y)$ for all sufficiently large x then in principle the conjecture can be settled to rest once and for all.

4 References

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