Limit theorems for k-subadditive lattice group-valued capacities in the filter convergence setting

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Abstract

We investigate some properties of lattice group-valued positive k-subadditive set functions, and in particular we give some comparisons between regularity and continuity from above. Moreover we prove different kinds of limit theorems in the non-additive case with respect to filter convergence, in which it is supposed that the involved filter is diagonal.

Definitions 0.1 (a) Given a free filter $\mathcal{F}$ of $\mathbb{N}$, we say that a subset of $\mathbb{N}$ is *$\mathcal{F}$-stationary* iff it has nonempty intersection with every element of $\mathcal{F}$. We denote by $\mathcal{F}^*$ the family of all $\mathcal{F}$-stationary subsets of $\mathbb{N}$.

(b) A free filter $\mathcal{F}$ of $\mathbb{N}$ is said to be *diagonal* iff for every sequence $(A_n)_n$ in $\mathcal{F}$ and for each $I \in \mathcal{F}^*$ there exists a set $J \subset I$, $J \in \mathcal{F}^*$ such that $J \setminus A_n$ is finite for all $n \in \mathbb{N}$

Let R be a Dedekind complete lattice group, G be any infinite set, Σ be a σ-algebra of subsets of G, and k be a fixed positive integer.

Definitions 0.2 (a) A *capacity* $m : \Sigma \to R$ is a set function, increasing with respect to the inclusion and such that $m(\emptyset) = 0$.

(b) A capacity m is said to be *k-subadditive* on Σ iff

$$m(A \cup B) \leq m(A) + k\, m(B) \qquad \text{whenever } A, B \in \Sigma,\ A \cap B = \emptyset. \tag{1}$$

(c) We say that a capacity m is *continuous from above at $\emptyset$* iff

$$(O) \lim_n m(H_n) = \bigwedge_n m(H_n) = 0$$

whenever $(H_n)_n$ is a decreasing sequence in Σ with $\displaystyle\bigcap_{n=1}^{\infty} H_n = \emptyset$.

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(d) A capacity m is *k-σ-subadditive* on Σ iff

$$m\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq m(E_1) + k \sum_{n=2}^{\infty} m(E_n) \tag{2}$$

for any sequence $(E_n)_n$ from Σ.

Proposition 0.3 *Let $m : \Sigma \to R$ be a k-subadditive capacity, continuous from above at $\emptyset$. Then m is k-σ-subadditive.*

Definitions 0.4 (a) A capacity $m : \Sigma \to R$ is said to be *continuous from above* (resp. *below*) iff

$$\begin{aligned} m(E) &= (O)\lim_n m(E_n) = \bigwedge_n m(E_n) \\ \text{(resp. } m(E) &= (O)\lim_n m(E_n) = \bigvee_n m(E_n)\,) \end{aligned}$$

whenever $(E_n)_n$ is a decreasing (resp. increasing) sequence in Σ, with $E = \bigcap_{n=1}^{\infty} E_n$ (resp. $E = \bigcup_{n=1}^{\infty} E_n$).

(b) A capacity $m : \Sigma \to R$ is (s)-*bounded* on Σ iff there exists an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ and for every disjoint sequence $(C_h)_h$ in Σ there is a positive integer h_0 with $m(C_h) \leq \sigma_p$ whenever $h \geq h_0$.

(c) Let τ be a Fréchet-Nikodým topology on Σ. A capacity $m : \Sigma \to R$ is said to be *τ-continuous* on Σ iff for each decreasing sequence $(H_n)_n$ in Σ, with $\tau\text{-}\lim_n H_n = \emptyset$, we get

$$(O)\lim_n m(H_n) = (O)\bigwedge_n m(H_n) = 0.$$

(d) Let $\mathcal{G}$, $\mathcal{H} \subset \Sigma$ two lattices, such that $\mathcal{H}$ is closed under countable unions, and the complement of every element of $\mathcal{H}$ belongs to $\mathcal{G}$. We say that a capacity $m : \Sigma \to R$ is *regular* iff for every $E \in \Sigma$ there are two sequences $(F_n)_n$ in $\mathcal{H}$ and $(G_n)_n$ in $\mathcal{G}$, with

$$F_n \subset F_{n+1} \subset E \subset G_{n+1} \subset G_n \quad \text{for any } n, \tag{3}$$

and $(O)\lim_n m(G_n \setminus F_n) = \bigwedge_n m(G_n \setminus F_n) = 0$.

The next result links continuous from above at $\emptyset$ and regularity of capacities.

Theorem 0.5 *Let R be a Dedekind complete weakly σ-distributive lattice group, (G, d) be a compact metric space, Σ be the σ-algebra of all Borel sets of G, $\mathcal{G}$ and $\mathcal{H}$ be the lattices of all open and all compact subsets of G respectively. Then every k-subadditive regular capacity $m : \Sigma \to R$ is continuous from above at $\emptyset$.*

Conversely, if R is also super Dedekind complete, then every k-subadditive capacity $m : \Sigma \to R$, continuous from above at $\emptyset$, is regular.

We now give the following limit theorems for non-additive lattice group-valued capacities with respect to filter convergence.

Theorem 0.6 *Let $\mathcal{F}$ be a diagonal filter of $\mathbb{N}$, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be an equibounded sequence of k-subadditive capacities, such that $m_0(E) := (O\mathcal{F}) \lim_j m_j(E)$ exists in R for every $E \in \Sigma$, m_0 is continuous from above at $\emptyset$ and m_j is (s)-bounded on Σ for every $j \geq 0$.*

If $R \subset C_\infty(\Omega)$ is as in the Maeda-Ogasawara-Vulikh representation theorem, then for every $I \in \mathcal{F}^*$ and for each disjoint sequence $(C_h)_h$ in Σ there exist a set $J \subset I$, $J \in \mathcal{F}^*$, and a meager set $N \subset \Omega$ with

$$(O) \lim_h \Big(\bigvee_{j \in J} m_j(C_h) \Big) = 0 \tag{4}$$

and

$$\lim_h (\sup_{j \in J} m_j(C_h)(\omega)) = 0 \qquad \text{for each } \omega \in \Omega \setminus N. \tag{5}$$

Theorem 0.7 *Let R, Ω, $\mathcal{F}$ be as in Theorem 0.6, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be an equibounded sequence of k-subadditive capacities. Assume that $m_0(E) := (O\mathcal{F}) \lim_j m_j(E)$ exists in R for every $E \in \Sigma$.*

Then for every $I \in \mathcal{F}^*$ and for each decreasing sequence $(H_n)_n$ in Σ with

$$(O) \lim_n m_j(H_n) = \bigwedge_n m_j(H_n) = 0 \qquad \text{for every } j \geq 0 \tag{6}$$

there are a set $J \subset I$, $J \in \mathcal{F}^*$, and a meager set $N^* \subset \Omega$ with

$$\lim_n (\sup_{j \in J} m_j(H_n)(\omega)) = \inf_n (\sup_{j \in J} m_j(H_n)(\omega)) = 0 \tag{7}$$

and

$$(O) \lim_n \Big(\bigvee_{j \in J} m_j(H_n) \Big) = \bigwedge_n \Big(\bigvee_{j \in J} m_j(H_n) \Big) = 0. \tag{8}$$

Theorem 0.8 *Let $\mathcal{F}$, R, Ω, k, G, Σ be as in Theorem 0.7, τ be a Fréchet-Nikodým topology on Σ, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be an equibounded sequence of k-subadditive capacities, τ-continuous (resp. continuous from above at $\emptyset$) on Σ. Let $m_0(E) := (O\mathcal{F}) \lim_j m_j(E)$ exist in R for every $E \in \Sigma$, and suppose that m_0 is τ-continuous (resp. continuous from above at $\emptyset$) on Σ.*

Then for every $I \in \mathcal{F}^*$ and for each decreasing sequence $(H_n)_n$ in Σ, with τ-$\lim_n H_n = \emptyset$ (resp. $\bigcap_{n=1}^{\infty} H_n = \emptyset$), there exist a set $J \subset I$, $J \in \mathcal{F}^*$, and a meager set $N \subset \Omega$, satisfying (7) and (8).

Theorem 0.9 *Let $\mathcal{F}$, R, Ω, G, Σ be as in Theorem 0.7, $\mathcal{G}$, $\mathcal{H} \subset \Sigma$ be two lattices, such that the complement of every subset of $\mathcal{H}$ belongs to $\mathcal{G}$, and $\mathcal{H}$ is closed under countable unions. Let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of k-subadditive regular capacities, such that $m_0(E) = (O\mathcal{F}) \lim_j m_j(E)$ for*

any $E \in \Sigma$ and m_0 is regular. Then we get:
(R3) *for every $E \in \Sigma$ and $I \in \mathcal{F}^*$ there are $J \in \mathcal{F}^*$, $J \subset I$, and two sequences $(F_n)_n$ in $\mathcal{H}$, $(G_n)_n$ in $\mathcal{G}$, satisfying (3) and with*

$$(O) \lim_n \Big(\bigvee_{j \in J} m_j(G_n \setminus F_n) \Big) = \bigwedge_n \Big(\bigvee_{j \in J} m_j(G_n \setminus F_n) \Big) = 0,$$

and furthermore there exists a meager set $N \subset \Omega$ with

$$(O) \lim_n \big(\sup_{j \in J} m_j(G_n \setminus F_n)(\omega) \big) = \inf_n \big(\sup_{j \in J} m_j(G_n \setminus F_n)(\omega) \big) = 0$$

for each $\omega \in \Omega \setminus N$.