ON THE PRINCIPLES OF LOGIC IN THE AGE OF UNCERTAINTY

by Ricardo Alvira

1_SUMMARY

Logic mainly refers to the way we reason [hence, we 'understand' things], which in turn relates to the way we can 'know' the world around us. But during 20th century, several advances in relation to our understanding of the qualities of reality as well as to our limits to 'fully knowing' it, have forced us to adapt and re-estate some of the principles on which logic was built; mainly those usually designated as 'classical logic'.

While some issues are –more or less- widely accepted to have been solved by new approaches to logic [non-classical logics], other still show some controversy, which we try to shed some light from a new perspective based on Communication Theory [Shannon, 1948].

To help us more clearly explain the issue, we build on an easy example such as a bottle, which will allow us to easily understand each of the reviewed issues.

2_THE PRINCIPLES OF LOGIC IN THE FRAMEWORK OF COMPLEXITY AND UNCERTAINTY

THE FOUR PRINCIPLES OF CLASSICAL LOGIC

Classical logic is usually understood as a binary logic built mainly on the ideas of Aristotle's, which reviewed qualities that can only be truth or false referred to objects.

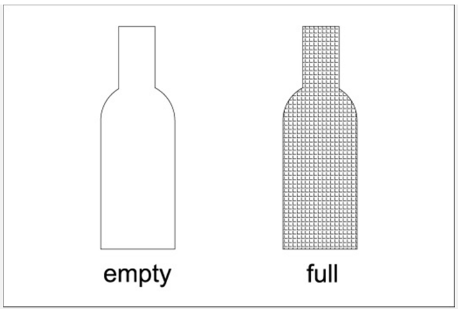

Figure 01. From classical logic, a bottle is full or empty, which implies that the statement 'the bottle is full' may only be true or false. There are only two truth values.

From classical logic, there are only two possibilities [two possible states of things] leading to four principles usually summarized as:

01 $$Identity \rightarrow \forall I; I \equiv I \qquad (1)$$

02 $$Bivalence\ [two\ truth\ values] \rightarrow |x| = 1 \vee |x| = 0 \qquad (2)$$

03 $$Non\ Contradiction \rightarrow \forall x; |x| \neq |\neg x| \qquad (3)$$

04 $$Excluded\ Middle \rightarrow \nexists y; y = x \wedge y = \neg x \qquad (4)$$

FUZZINESS, VAGUENESS AND TIME

Classical logic has been widely accepted as a general framework for producing knowledge [and more specifically, scientific knowledge], but some changes will appear along the 20th century when three ideas [fuzziness, time variation and vagueness] are progressively incorporated into logical statements and analysis, which we briefly review next.

Fuzziness leads to **Fuzzy Logic** which seeks modeling the fact that most bottles cannot only be totally full or empty, but they can be full/empty to certain extent. There are infinite possibilities/states of things, which equate saying that most statements may have infinite values of truth[1].

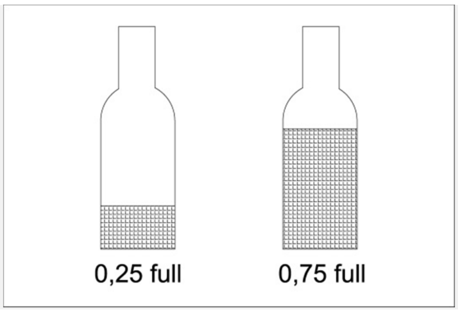

Figure 02. The bottle can be full or empty to an extent, which implies that the statement 'the bottle is full' may be truth to such extent. It has a 'degree of truth'

It allows understanding the 'sorites' paradox. If we pour a drop of water in an empty bottle, the bottle is still empty. If we pour another drop, it is a bit 'less empty'. If we keep on pouring drops of water, at some point the bottle will be full[2].

Time variation leads to **Temporal Logic** which appears to model the fact that tomorrow the bottle can be full to a different extent that it is full today; some truth values can change over time.

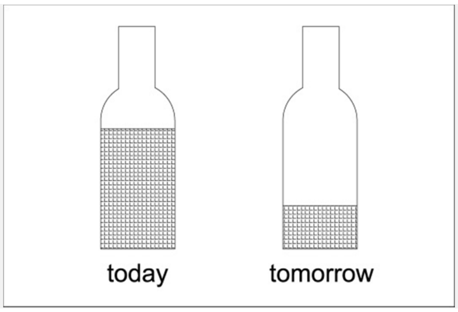

Figure 03. Today the bottle is full to a different extent than it may be tomorrow. Statements' truth degree gets linked to a certain moment in time.

And **Vagueness**[3] appears due to the fact that natural languages are composed by a limited amount of linguistic terms that designate qualities that may be present in an infinite amount of possible objects, and as a consequence these terms refer to only certain objects' information; they refer to common informational patterns to classes of objects.

But summarizing information [selecting only some part of objects' information] means that we are not fully describing the object; some information is excluded from the description and the excluded information may sometimes be the key to fully understand statements' meaning.

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[1] Even if some of them are so similar that we cannot differentiate them

[2] Though it is not the aim of this article to review in detail the 'sorites' paradox, in our view it combines two issues. The first can be solved by 'Fuzzy Logic', while the second relates to computing the truth value of a concept [a heap] from an independent variable [the grain number]. The extent to which a number of grains form a heap does not depend of the number of grains but of their shape. If we put infinite grains in a row they do not form a heap. It is therefore a paradox created partly by the choice of an inadequate variable to measure the concept. In a similar way, the truth value of the assertion 'the bottle is full' does not depend on the number of drops poured, but of the water volume/bottle capacity ratio.

[3] We use vagueness in a sense similar to ambiguity, considering it a different quality of 'fuzziness'.

In general, vagueness does not lead to the development of a specific logic, but to the need of understating statements into the context [physical, temporal and semantic] and in the sense they are stated.

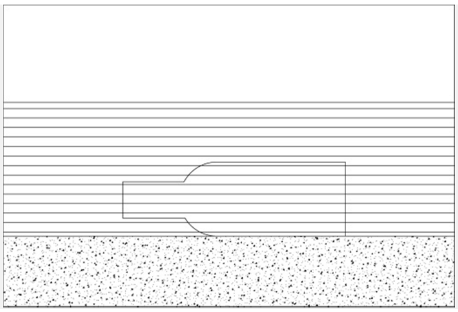

Figure 04. Is the bottle full or empty? We use the term 'full' in general to refer to the fact that we have filled it on purpose with some kind of fluid, which is different to that of its environment [e.g. a 'diver' considers his 'bottle' to be full when it contains air]. If 'full' referred only to the state of containing a fluid, then any bottle would always be full, as it is always full of liquid and/or air.

Truth values need to be assessed into a context/environment, and related to the sense in which statements are made.

The above issues have been developed by different authors, leading to re-enunciating/complementing the principles of classical logic. *Fuzziness, temporal variation and language vagueness shall be incorporated by fuzzy logic rules [Zadeh, 1965] and considering the contexts [physical, temporal and semantic] in which the assertions are made,* re-stating the above principles as:

- The *Law of Identity* shall be reformulated as there *'cannot be an object that is not identical to itself at the same point in time, context and in the same sense':*

01.1 $$Identity \rightarrow \forall I; I \equiv I \quad (5)$$

- *Truth Values* referring to qualities that can emerge in a continuous degree in objects may take any value in the range 0-1 [e.g., the bottle can full/empty any value in the range 0-1]:

02.1 $$Truth\ values \rightarrow \forall x; |x| \rightarrow [0,1] \quad (6)$$

- The *Law of non-contradiction* in fuzzy terms shall be reformulated as *'there can be no concept whose truth and falsity relating an object are -at the same moment in time, context and in the same sense- different from one'* [e.g., the amount the bottle is full and empty must add up to one]:

03.1 $$Non-contradiction \rightarrow \forall x; |x| = 1 - |\neg x| \quad (7)$$

- The *Law of excluded middle* remains valid but it requires *interpreting the sign '¬' as 'absolutely false' not as the 'no' of natural language*[4] [e.g., the bottle cannot be absolutely full and absolutely empty at the same time and in the same sense]:

04.1 $$Excluded\ middle \rightarrow \nexists y; y = x \wedge y = \neg x \quad (8)$$

[4] For a detailed explanation, refer to Peña, 1993, p. 5

COMPLEXITY LOGIC

Non classical logics have allowed us to incorporate some of the new features assumed by knowledge along the 20th century, yet two more issues have also arisen which radically change the way we look reality:

- The first relates to **Uncertainty**. From different perspectives, we find that our statements always imply some uncertainty related to their truth value. Uncertainty Principle [Heisenberg], Undecidability [Turing], Incompleteness Theorem [Gödel] and Unpredictability of Chaos Theory [Lorenz, Feigenbaum]... They all imply that our knowledge of reality can never be complete; there always exist some 'unknown' areas.
- The second relates to the **Observer**. No statement can be made without the intervention of an observer, which implies both the introduction of subjective issues/as well as choosing a point of view [von Bertalanffy; von Foerster, Maturana& Varela…]

While previous logic was mainly related to objects' properties [i.e., it searched 'objectivity'], 20th century brings the **observer** [subject] into the statements [i.e., introduces 'subjectivity'] generating a wide range of situations difficult to model/understand … *What if the bottle has a label which prevents us from being able to see the extent to which it is full or empty?*

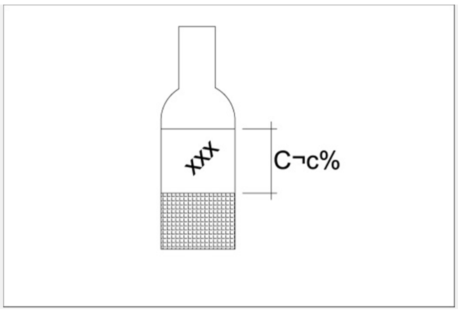

Figure 05. What happens behind the label?

The existence of an observer/subject implies a certain point of view [i.e., subjectivity], which in turn implies the possibility of 'hidden areas' to the observer … The bottle is still full and empty to some extent, but we cannot fully know it, and therefore we cannot state it as an absolute truth.

We know the bottle is full to a certain extent, but the label creates an uncertain zone, preventing us from being able to assert the degree of truth of the statement 'the bottle is full'. Behind the label the bottle can be full or empty; totally or to a certain extent.

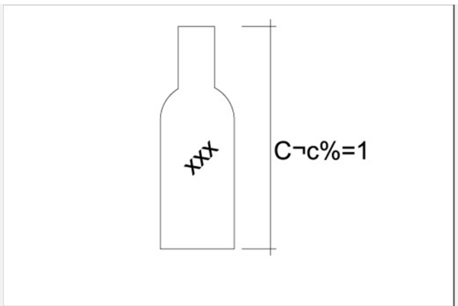

Figure 06: Is the bottle full or empty?

As bottles start to have 'labels', we no longer are able to state with complete certainty to which extent a bottle is full or empty, in a degree which relates to the size of the label. And if a bottle had a label totally covering it, then our degree of uncertainty in relation to such bottle would be complete, approaching us to the paradox of the 'Hooded man' [Eubulides of Miletus, IV BCE]

It is necessary to highlight that uncertainty is not a quality of the object itself, but of the object in relation to an observer. It appears as a result of a complex relation [the interaction object-observer] hence we designate it as **Logic of Complexity**.

But uncertainty also relates to temporal logic. **Unpredictability** of Chaos and systems' decision making ability makes our uncertainty in relation to the future state of things usually to increase as prediction interval increases. It equates saying that in the absence of the unusual, the label of the bottle increases over time...

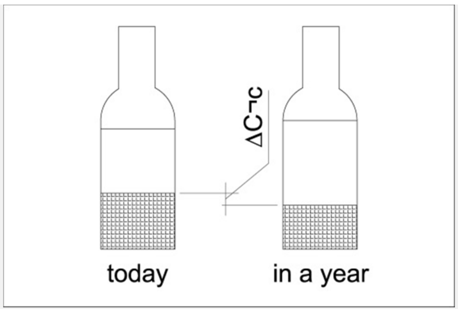

Figure 07. Uncertainty increases with the prediction interval. If we want to describe bottle's state in the future, the size of the label will be increased.

$$\Delta C_{\neg c}[x] \geq 0$$

Uncertainty forces us to complement/re-enunciate the former last two principles of logic. When objects have a degree of membership to class uncertainty [when bottles have labels], we shall complement the last three principles stated above.

There are the classes of indeterminate, undecidable and uncertain objects; which do not allow our statements[5] *to comply neither with the Law of non-contradiction nor with the Law of Excluded Middle,* and...

- Truth values must be supplemented by *Certainty values* which may take any value in the range 0-1.

2.2 $$Certainty\ values\ \ \rightarrow \forall I; C_c[I]_{\%} \rightarrow [0,1] \quad (9)$$

- The *Law of non-contradiction* shall be complemented in terms of uncertainty as *'there can be no concept about which -when referred to an object- our certainty and uncertainty degree at one time, context and in one sense are different from one'*

3.2.a $$Non-contradiction\ \ \rightarrow \forall I; C_c[I]_{\%} = 1 - C_{\neg c}[I]_{\%} \quad (10)$$

And we have said that uncertainty degree reduces the truth degree we can state, hence...

3.2.b $$Non-contradiction\ \ \rightarrow \forall I, x{:}\, |x|' = 1 - C_{\neg c}[I]_{\%} - |\neg x|' \quad (11)$$

Being |x|' and |¬x|' the degree of truth/falsity we can assert.

- The validity of the *Law of Excluded Middle* requires interpreting that this principle shall be met except by those concepts / objects that belong entirely to *Indeterminacy, Uncertainty and Undecidability* classes.

4.2 $$\forall x; [x = A\ \wedge x = \neg A] \leftrightarrow [C_{\neg c}[x]_{\%} = C_{\neg c}[\neg x]_{\%} = 1] \quad (12)$$

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[5] It is important to differentiate between objects and statements [conceptual objects]. While most objects still comply with the principles of logic, it is our statements which may no longer comply with them since the entry of uncertainty. The bottle is full or empty, but we cannot assert to what extent. The 'hooded man' is or is not our brother, but we cannot assert either of them.

This statement is false

Figure 08. The Liar Paradox [Eubulides of Miletus IV BCE], is a statement that belongs completely to indeterminacy class, hence it does not comply with the Law of Excluded Middle. Both its truth |x| as well as its falsity |¬x| are undecidable, therefore:

$$|x| = |\neg x| = undecidable$$

Contradiction disappears if we re-state it as 'This statement is undecidable' [we cannot state its truth or falsity]

3_MEASURING THE UNCERTAINTY DEGREE

We continually assign 'truth values to different statements'. Each time we hold a conversation, we read a book, we watch a movie… we assign truth values to the different 'messages' we receive. This truth value assignment appears as a precondition for our ability to understand. And we see that uncertainty has a major influence on our ability to assign a truth value to such statements. *If we cannot assign a truth value to a statement, we cannot decide whether it is true or not, and [measuring and understanding] Uncertainty acquires a special relevance.*

If we consider we have one bottle, we can estimate our uncertainty degree, by several ways [to simplify it, let us consider it equates to simply measuring the 'size of the label']...

But what happens when we need to calculate our uncertainty degree in relation to several bottles? When we watch a movie, read a book, participate into a conversation… statements are usually interacting[6] and so do their truth values as well as their uncertainty degrees…

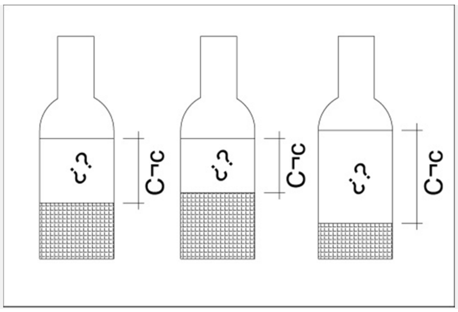

Figure 09. How can we add our uncertainty degree in relation to different bottles?

Actually in the case of the bottles, is one of the rare cases [volume or mass measurements] we could just add them up [arithmetic mean]. But let us remind that bottles are just a metaphor so each bottle represents a concept x_i and the degree to which it is full represents its truth value. And let us consider that the interaction of such truth value with the truth values of other x_i concepts [other bottles] determines the truth value of a global concept X.

In such case, their arithmetic mean usually is not a measure of our uncertainty degree, and to find a solution we develop a specific formulation based on Entropy formulation.

[6] It largely relates to the idea of L-Fuzzy Set [Goguen, 1967]

Formulation Based On Entropy And Mutual Information

Prior to explaining the formulation, let us review a couple of concepts / formulas from Communication Theory, that we adapt considering that I is an object [factual or conceptual] and X is a concept [or conceptual object].

Communication Theory states that we can measure the amount of information provided by a description of each symbol 's' emitted by a source I using **Entropy's** formula [Shannon, 1948]:

$$H[s] = -K \sum_{i=1}^{n} p_i * log_2 p_i \tag{13}$$

Being H[s] _ symbol's entropy, p_i_ the probability of each of the possible values for that symbol and K_ a constant depending on the chosen measuring unit.

The Entropy formula is an approach to measuring the information provided by the reception of a message from the maximum uncertainty or ignorance it is possible to have regarding its contents[7]. It builds on *Classical Logic* and *Probability Theory*, considering two possible values for each symbol *[true or false]* and the probability of occurrence of each of them.

And to measure the amount of shared information between two objects, Communication Theory proposes the **Mutual Information** which can be conceptualized as "the average reduction of uncertainty of [a source] due to knowledge of another" [Crutchfield & Feldman, 2001, p. 5] and it is:

Mutual Information

$$I[I;X] = H[X] - H_X[I] \tag{14}$$

H[X] is the maximum Entropy of X; i.e.: the maximum amount of ignorance that we may have about X [which therefore coincides with the maximum amount of information that we can acquire in relation to X] and its value depends on the range of possible symbols in X information/description.

H_x[I] is the **Conditional Entropy** of I which is a measure of "how uncertain we are on I when we know X" [Shannon 1948, p. 12], and its formula is:

Conditional Entropy

$$H_X[I] = -\sum_{i,j} p(i,j) * log_2\, p_i(j) \tag{15}$$

The three above formulas and concepts, allow us to pose our formulation of Certainty/Uncertainty degrees that we review next.

[7] "Shannon's H [entropy] measures the degree of ignorance of the communication engineer when he designs the technical equipment in the channel" [Jaynes 1978, p. 25], and this conceptualization of *Entropy as Ignorance or Uncertainty* advances us that it also will allow us to measure *Certainty* as complementary value.

We want to propose a formulation on the degree of certainty/uncertainty we can have on the degree that a meaning [or concept] X is true referred to an object I[8], which requires reviewing two issues:

- *Measuring the 'mutual information'* between concept X and Object I
- Measuring the *meaning transformation* introduced by aggregating the truth values of concepts x_i.

And to calculate it, we are going to measure both issues for two special concepts:

- X = 'certainty' [C], which allows us calculating our *Certainty Degree* $C_c[I]_{\%}$ related to I.
- ¬X = 'uncertainty' [¬C], which allows us calculating our *Uncertainty Degree* $C_{\neg c}[I]_{\%}$ related to I.

Both measures will report the *Certainty/Uncertainty Degree* that, known the global state of an object I [truth value of X referred to object I], we have in relation to its *detailed status* [truth values of concepts x_i referred to I] and we use the following codes:

- $|x_i|$ _ truth value when X=certainty
- $|\neg x_i|$ _ truth value when ¬X=uncertainty

First we review the formulation for the *Uncertainty Degree:*

MODELING FOR ¬X = 'UNCERTAINTY'

Let us assume that the description is complete so X's truth value can be fully determined from the truth values of some x_i sub-concepts, by means of a formulation that requires four steps:

STEP 1: MEASURING MUTUAL 'KNOWLEDGE' BETWEEN I AND ¬C

We can define the degree of truth of a concept X referred to an object I, as the extent to which the rules that define membership to class X are satisfied by such objet, and we can model it for ¬C based on the formula of the *Mutual Information*, i.e.:

$$I[I;\neg c] = H[\neg c] - H_{\neg c}[I]^{9} \tag{16}$$

However, it is necessary to make some clarifications in relation to the formulation:

H[¬c] is the maximum possible truth value of concept ¬C [which for now we equate to its information content], and we can state two issues in relation to the term:

- it is the maximum value for $H_{\neg c}[I]_{max}$ to be reached when every $|\neg x_i|$ reaches value '1'

$$\forall i: |\neg x_i| = 1 \rightarrow H[\neg c]_{max} = -\sum_{i=1}^{n} P_i * log_m P_i \tag{17}$$

[8] It is equivalent to calculating its Membership Grade to classes Certainty and Uncertainty.

[9] It is the maximum Certainty we can have relating I, less the Uncertainty we have about I once ¬C is known.

- it must match $H_c[I]_{max}$[10]:

$$H_{\neg c}[I]_{max} = H_c[I]_{max} = -\sum_{i=1}^{n} P_i * log_m P_i \quad (18)$$

And $H_I[\neg c]$ is the Conditional Entropy which incorporates a fundamental issue. **It comes to measuring the ignorance or lack of knowledge in terms of the measured concept and therefore the formula does not incorporate the truth values related to ¬x_i sub-concepts |¬x_i| but their complementary |x_i|, i.e.:**

$$H_I[\neg c] = -\sum_{i=1}^{n}[1 - |\neg x_i|] * P_i * log_m P_i = -\sum_{i=1}^{n}|x_i| * P_i * log_m P_i \quad (19)$$

|x_i|_ are the values of the membership functions to the opposite concept to ¬c, i.e. to 'c'[11] and p_i_ is assigned by the probabilities structure of the *logical decomposition* of ¬c.

And by replacing the two terms in the formula of the Mutual Information we have:

$$I[I, \neg c] = -\sum_{i=1}^{n} P_i * log_m P_i + \sum_{i=1}^{n}|x_i| * P_i * log_m P_i \quad (20)$$

However, *adding the values of the description involves introducing uncertainty in the obtained value.* Known the truth value of X referred to I, our *uncertainty* related to object 'I' is greater than if we know the truth values of concepts x_i; i.e., a truth value of X may be describing different combinations of truth values of x_i.

The *real meaning* of the added value is more 'uncertain' than that of the individual values. By adding the truth values of concepts x_i our uncertainty regarding the object increases[12].

It is necessary to review the meaning of the two terms in the above equation, because when comparing an object I with any concept X we find some relevant differences:

- The first term refers to 'certainty'; to the maximum amount of *knowledge* [or *ignorance*] that X [or ¬X] and 'I' may share, and **its value is determined by the logical decomposition of X / ¬X** [it does not depend on 'I' but of the concept X].
- The second term refers to 'uncertainty'; to the amount of *knowledge* that 'I' and ¬X do not share, and **its value is determined by x_i sub-concepts' truth values; therefore it depends on object I's features.**

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[10] The maximum amount of knowledge that may involve a concept is the same as its opposite concept may imply, since any relevant variable for C will also be relevant for ¬C. Thus,$H_{\neg c}[I]_{max}$= $H_c[I]_{max}$.

[11] That is, the membership functions of I to concept 'x=certainty'. The reason is that the truth values express knowledge [certainty] in relation to the revised concept, but what we want to measure is its lack of knowledge [ignorance], and thus in the formula we use the complementary values.

[12] The truth degree of concept 'uncertainty' increases and truth degree of concept 'certainty' is reduced in relation to I.

However, as they refer to the concept ¬X='uncertainty', the meaning of the two terms is going to change:

- The first implies *certainty* regarding concept 'uncertainty', i.e., an assertion of a negation, and therefore *uncertainty [or denial].*
- The second implies *uncertainty* regarding the concept 'uncertainty', i.e., a double negation, and therefore *certainty [or affirmation].*

And this means that **by aggregating the truth values, the second term shall decrease its value on a percentage depending on the aggregated values,** which we calculate below.

STEP 2: CALCULATION OF UNCERTAINTY INCREASE DUE TO TRUTH VALUES AGGREGATION

We know that, in most situations, aggregating truth values involves loss of 'certainty'. We usually have greater certainty about the state of an object when we also know the state of such object in relation to x_i concepts [their truth values], that when we only know the truth value of X, and there are two limiting cases:

- A situation of *zero uncertainty increase,* which happens if the *truth value of all sub-concepts x_i is equal,* and therefore X's truth value is equal to every one of them.

$$\forall i \in I, x: |x_i| = k \rightarrow \forall i \in I, x: |x_i| = |X| = C_c[I]_{\%} \tag{21}$$

- A situation of *maximum uncertainty increase,* which occurs if half of the truth values take the minimum value, and the other half takes the maximum value, reaching the added value the highest possible differentiation with each truth value.

$$\begin{gathered} \forall i \in I, c: |x_i| = min\left[|x_i|_{i=1}^{n}\right] \;\wedge\; |x_{i+1}| = max\left[|x_i|_{i=1}^{n}\right] \\ \vee \\ \forall i \in I, c: |x_i| = max\left[|x_i|_{i=1}^{n}\right] \;\wedge\; |x_{i+1}| = min\left[|x_i|_{i=1}^{n}\right] \end{gathered} \tag{22}$$

And in the latter case, the greatest loss of certainty is reached when the *minimum* and *maximum* value of the truth values are 0 and 1 respectively.

$$\forall i \in I, c: |x_i| = 0 \vee 1 \quad \wedge \quad |x_{i+1}| = 1 - |x_i| \tag{23}$$

Since we know the maximum added value, we can calculate the uncertainty introduced by any distribution of x_i sub-concepts' truth values as a measure of the Mutual Information for each truth value related to the added value [I], namely[13]:

$$I_c[I, |x_i|] = -\sum_{i=1}^{n} p_i * |x_i| * log_2 |x_i|^{14} \tag{24}$$

Being I_c[I, |x_i|] the *'common knowledge'* to I and |x_i|.

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[13] Before we calculate the 'common information' between concept X and object I. Now we want to calculate the common information between x_i sub-concepts' truth values and X's truth value, which inform us of the uncertainty introduced by adding them.

[14] Though we have departed from classical logic, we maintain 2 as the base for the logarithms for greater similarity to the Communication Theory since the result will be independent of such base.

Once 'I' is known, the maximum certainty as to the value of '$|x_i|$ ' is reached when every $|x_i|$ is equal to the arithmetic mean of the sub-concepts x_i truth values weighted by their probabilities, i.e.:

$$\forall i \in I: |x_i| = \overline{|x_\iota| * p_I} \leftrightarrow I_c[I, |x_i|]_{max} = \overline{p_\iota} * \overline{|x_\iota| * p_i} * log_2 \overline{|x_\iota| * p_i} \quad (25)$$

This means that aggregating object's truth values, introduces a reduction of certainty that can be deducted from the distribution of their values, and that is for each truth value:

$$I_c[I, |x_i|]_{\%} = \frac{I_c[I, |x_i|]}{I_c[I, |x_i|]_{max}} = \frac{p_i * |x_i| * log_2 |x_i|}{\overline{p_\iota} * \overline{|x_\iota| * p_i} * log_2 \overline{|x_\iota| * p_i}} \quad (26)$$

And for the set of all the sub-concepts:

$$I_c[I]_{\%} = \sum_{I=1}^{n} I_c[I, |x_i|]_{\%} = \sum_{i=1}^{n} \frac{p_i * |x_i| * log_2 |x_i|}{\overline{p_\iota} * \overline{|x_\iota| * p_i} * log_2 \overline{|x_\iota| * p_i}} \quad (27)$$

STEP 3: CALCULATING I'S UNCERTAINTY/ UNCERTAINTY DEGREE

The formula for the Uncertainty can be derived from the above formulas:

$$C_{\neg c}[I] = H_c[I]_{max} * I_c[I]_{max} - H_I[\neg c] * I_c[I]_{\%} \quad (28)$$

$I_c[I]_{max}$ corresponds to the situation where all truth values are 1 and consequently its value is equal to 1, and therefore the uncertainty is[15]:

$$C_{\neg c}[I] = H_c[I]_{max} - H_I[\neg c] * I_c[I]_{\%} \quad (29)$$

This confirms that there is a limit to the maximum value of the uncertainty related to I, which is reached when the second term is zero, i.e.:

$$C_{\neg c}[I]_{max} = C[\neg c] = C[c] = H_c[I]_{max} \quad (30)$$

The calculation of the *Uncertainty Degree* $C_{\neg c}[I]_{\%}$ can also be deducted from previous formulas:

$$C_{\neg c}[I]_{\%} = 1 - \frac{H_I[\neg c]}{H_c[I]_{max}} * I_c[I]_{\%} \quad (31)$$

[15] By referring to ¬C, the first term is 'uncertainty' and it does not reduce with the aggregation, while the second term is 'certainty' and therefore reduces when adding the information

And we can express it as:

$$C_{\neg c}[I]_{\%} = 1 - H_I[\neg c]_{\%} * I_c[I]_{\%} \tag{32}$$

And because…

$$H_I[\neg c]_{\%} = \frac{\sum_{i=1}^{n} |x_i| * P_i * log_2 P_i}{\sum_{i=1}^{n} P_i * log_2 P_i} \tag{33}$$

The Uncertainty Degree is:

$$C_{\neg c}[I]_{\%} = 1 - \frac{\sum_{i=1}^{n} |x_i| * P_i * log_2 P_i}{\sum_{i=1}^{n} P_i * log_2 P_i} * \sum_{i=1}^{n} \frac{p_i * |x_i| * log_2 |x_i|}{\overline{\overline{p_\iota} * |x_\iota| * p_\iota} * log_2 \overline{|x_\iota| * p_\iota}} \tag{34}$$

However, usually the truth values of sub-concepts x_i have equal relevance for the truth value of X, which equates considering them to be equally likely, allowing a considerable simplification of the above formulation:

STEP 4: SIMPLIFICATION FOR EQUALLY LIKELY [RELEVANT] SUB-CONCEPTS' TRUTH VALUES

In most situations we can understand that *truth values are equally likely*[16] and this allows us to considerably simplify the above formulas, because:

$$\forall i \in I: p_i = \overline{p_\iota} \tag{35}$$

Hence, the *certainty* produced by truth values' distribution is:

$$H_I[\neg c]_{\%} = \frac{1}{n} * \sum_{i=1}^{n} |x_i| = \overline{|x_i|} \tag{36}$$

While the certainty reduction due to their aggregation is:

$$I_c[I]_{\%} = \sum_{I=1}^{N} I_c[|x_i|, I]_{\%} = \sum_{i=1}^{n} \frac{|x_i| * log_2 |x_i|}{\overline{\overline{|x_\iota|} * log_2 \overline{|x_\iota|}}} \tag{37}$$

And therefore, the *Uncertainty Degree* is:

$$C_{\neg c}[I]_{\%} = 1 - \overline{|x_i|} * \sum_{i=1}^{n} \frac{|x_i| * log_2 |x_i|}{\overline{\overline{|x_i|} * log_2 \overline{|x_i|}}} \tag{38}$$

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[16] A truth value of X may be generated by many different combinations of truth values of concepts x_i, in which the range of influence of each sub-concept' truth value is the same.

MODELING FOR X='CERTAINTY'

Let us now calculate the Certainty Degree as the opposite case to the above, i.e., when x='certainty', which allows us to highlight some interesting differences, starting from the formula of Common Information:

$$I[I;c] = H[I] - H_I[c] \tag{39}$$

And the difference appears when incorporating the complementary values into the Conditional Entropy formula, being:

$$H_I[c] = \sum_{i=1}^{n} [1 - |x_i|] * P_i * log_2 P_i \tag{40}$$

We can develop as:

$$H_I[c] = -\sum_{i=1}^{n} P_i * log_2 P_i + \sum_{i=1}^{n} |x_i| * P_i * log_2 P_i \tag{41}$$

And by substituting into the formula of the Mutual Information, we obtain:

$$I[I,c] = -\sum_{i=1}^{n} P_i * log_m P_i + \sum_{i=1}^{n} P_i * log_m P_i - \sum_{i=1}^{n} |x_i| * P_i * log_m P_i \tag{42}$$

Being annulled the first and second terms, and thus…

$$I[I,c] = -\sum_{i=1}^{n} |x_i| * P_i * log_2 P_i \tag{43}$$

We see, therefore, that I[I;c] is the complementary value to that obtained for x='uncertainty':

$$I[I;c] = H_x[I]_{max} - I[I;\neg c] \tag{44}$$

That gives us the following formula expressed as a percentage:

$$I[I;c]_{\%} = 1 - I[I;\neg c]_{\%} \tag{45}$$

The value of the term $I_c[I]_{\%}$ *Uncertainty increase in the aggregation* refers to the same term, hence does not change its calculation procedure, and the overall *certainty* is:

$$C_c[I] = I[I;c] * I_c[I]_{\%} \tag{46}$$

And the obtained formula is again the complement of that proposed above for 'Uncertainty'; i.e., when x='uncertainty':

$$C_c[I] = C_c[I]_{max} - C_{\neg c}[I] \tag{47}$$

Being $C_c[I]$_ our certainty about I's microscopic state once we know its global status; $C_c[I]_{max}$_ the maximum certainty we may have; $C_{\neg c}[I]$ _ the uncertainty we have.

And the *Certainty Degree* $C_c[I]_{\%}$ is:

$$C_c[I]_{\%} = I[I;c]_{\%} * I_c[I]_{\%} \tag{48}$$

That is also the complementary value of the *Uncertainty Degree*, i.e.:

$$C_c[I]_{\%} = 1 - -C_{\neg c}[I]_{\%} \tag{49}$$

If we consider the situation of *equally likely truth values* then:

$$I[I,c]_{\%} = \frac{1}{n} * \sum_{i=1}^{n} |x_i| = \overline{|x_i|} \tag{50}$$

Therefore the *Certainty Degree* is:

$$C_c[I]_{\%} = \overline{f_c[i]} * \sum_{i=1}^{n} \frac{|x_i| * log_2 |x_i|}{\overline{|x_i| * log_2 |x_i|}} \tag{51}$$

4_ CONCLUSIONS

The importance of modeling both Certainty Degree and Uncertainty degree is significant for several reasons.

Epistemology tells us that **almost every statement we can make implies some uncertainty** [e.g., currently 'almost every bottle has a label'], and we have to understand how it relates to the truth values we can assign to our statements relating objects.

Almost all objects have 'uncertain' areas apparently leading us to the impossibility of stating truth/falsity values adding up to one.

However, if we assume an equivalence relationship between synonym concepts, the uncertainty degree will increase the truth value of concepts synonym to 'uncertain', while it will decrease the truth value of concepts antonym to uncertain [synonym to 'certain'].

If we review it for concepts whose truth value depends on the aggregation of other concepts x_i, then we differentiate two situations:

- When a concept X shares some meaning with concept 'Certainty' [i.e.; it involves Certainty], its truth value will be between the arithmetic mean of truth value of concepts

x_i and the *Certainty Degree* $C_c[I]_\%$, approaching $C_c[I]_\%$ the higher the degree of shared meaning.

$$C_c[I]_\% \sim |X| \le \overline{|x_i|} \tag{52}$$

- When a concept ¬X shares some meaning with concept 'uncertainty' [i.e.; it involves Uncertainty], its truth value will be between the arithmetic mean of truth value of concepts $\neg x_i$ and the *Uncertainty Degree* $C_{\neg c}[I]_\%$, approaching $C_{\neg c}[I]_\%$ the higher the degree of shared meaning.

$$\overline{|\neg x_i|} \le |\neg X| \sim C_{\neg c}[I]_\% \tag{53}$$

And this enables us to propose the following equations which can be applied to any possible X related to any I[17]:

Certainty

$$|X| \sim C_c[I]_\% = \overline{|x_i|} * \sum_{i=1}^{n} \frac{|x_i| * log_2 |x_i|}{\overline{|x_i|} * log_2 \overline{|x_i|}} \tag{54}$$

Uncertainty

$$|\neg X| \sim C_{\neg c}[I]_\% = 1 - \overline{|x_i|} * \sum_{i=1}^{n} \frac{|x_i| * log_2 |x_i|}{\overline{|x_i|} * log_2 \overline{|x_i|}} \tag{55}$$

What we can graphically review as:

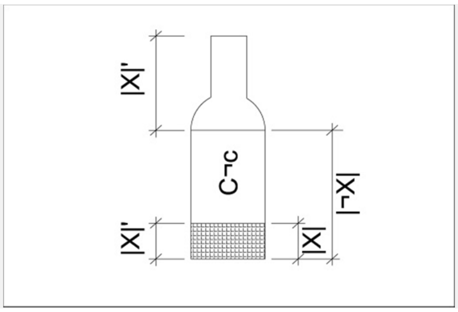

Figure 10. The uncertain area adds up to the truth value or not depending on the meaning of the concept.

If a concept involves Certainty:

$$|x| \cong |x|' \cong 1 - C_{\neg c}[I]_\% - |\neg x|'$$

If a concept involves Uncertainty:

$$|\neg x| \cong 1 - |x|'$$

However, the above formulations pose some indeterminacy in the limiting cases, so the formulations to be used are slightly different[18]:

Certainty

$$|X| \sim C_c[I]_\% = \frac{1}{n} * \sum_{i=1}^{n} \left[|x_i| * \left[1 + \overline{|x_i|_{i=1}^{n}} - |x_i|\right]\right] \tag{56}$$

Uncertainty

$$|\neg X| \sim C_{\neg c}[I]_\% = \frac{1}{n} * \sum_{i=1}^{n} \left[|x_i| * \left[1 - \overline{|x_i|_{i=1}^{n}} + |x_i|\right]\right] \tag{57}$$

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[17] Some applications of this issue are provided in Alvira 2014b and Alvira 2014c.

[18] The indeterminacy of the formulation based on Shannon's Entropy as well as the rationale of these formulations is explained in Alvira, 2014a and Alvira 2014c. It is noteworthy that these formulations are only valid in the case no sub-concept x_i exists such that its total falsity implies the total falsity of concept X.

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