

Infinite Serie and Integral Representations for Bessel Function of the first kind $J_\nu(z)$

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Therefore also now, saith the LORD, turn ye even to me with all your heart, and with fasting, and with weeping, and with mourning. - Joel 2:12.

ABSTRACT. I prove one expansion of infinite serie and some integral representations for Bessel function of the first kind.

1. INTRODUCTION

In this paper, I demonstrated the expansions of infinite power series for Bessel function of the first kind:

$$\frac{2^v \sqrt{\pi}}{z^v} J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k + \frac{1}{2})}{\Gamma(v + k + 1)(2k)!} z^{2k}.$$

I, too, prove the news integral representations for Bessel function of the first kind:

$$2 \int_0^1 \frac{t^{2\nu} \cos(z \sqrt{1-t^2})}{\sqrt{1-t^2}} dt = \frac{2^v \sqrt{\pi}}{z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(z),$$

$$\int_0^{\pi/2} \sin^2\left(\frac{z \sin \varphi}{2}\right) \cos^{2\nu} \varphi d\varphi = \frac{\sqrt{\pi} \Gamma(v + \frac{1}{2})}{4\Gamma(v + 1)} - \frac{2^v \sqrt{\pi}}{4z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(z),$$

$$\begin{aligned} & \int_0^{\pi/2} \cos(z \sin \varphi) \sin^2\left(\frac{z \sin \varphi}{2}\right) \cos^{2\nu} \varphi d\varphi \\ &= -\frac{\sqrt{\pi} \Gamma(v + \frac{1}{2})}{8\Gamma(v + 1)} + \frac{2^v \sqrt{\pi}}{4z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(z) - \frac{\sqrt{\pi}}{8z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(2z) \end{aligned}$$

and

$$\begin{aligned} & \int_0^{\pi/2} \cos^2(z \sin \varphi) \sin^2\left(\frac{z \sin \varphi}{2}\right) \cos^{2\nu} \varphi d\varphi \\ &= \frac{\sqrt{\pi} \Gamma(v + \frac{1}{2})}{8\Gamma(v + 1)} - \frac{2^v 3 \sqrt{\pi}}{16 \cdot z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(z) \\ &+ \frac{\sqrt{\pi}}{8 \cdot z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(2z) - \frac{2^v \sqrt{\pi}}{16 \cdot 3^v \cdot z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(3z). \end{aligned}$$

2. ACCELERATION OF INFINITE SERIE FOR BESSEL FUNCTION OF THE FIRST KIND

Lemma 1. For $z \in \mathbb{R}$, then

$$e^{-z} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} \left[1 - \frac{z}{2k+1}\right],$$

where e^z denotes the exponential function and $k!$ denotes the factorial function.

Proof. I easily know that

$$e^{-z} = \sum_{k=0}^{\infty} \frac{\cos\left(\frac{\pi k}{2}\right)}{\Gamma\left(\frac{k}{2} + 1\right)} z^{k/2}. \quad (1)$$

On the other hand, I calculate

$$\cos\left(\frac{\pi k}{2}\right) = \begin{cases} 1, & k = 0, 4, 8, 12, 16, 20, \dots \\ 0, & k = 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, \dots \\ -1, & k = 2, 6, 10, 14, 18, \dots \end{cases} \quad (2)$$

From (1) and (2), it follows that

$$\begin{aligned} e^{-z} &= \sum_{k=0}^{\infty} \frac{\cos\left(\frac{\pi k}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)} z^{k/2} = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} - \sum_{k=0}^{\infty} \frac{z^{2k+1}}{\Gamma(2k+2)} \\ &= \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} - \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} \left[1 - \frac{z}{2k+1}\right], \end{aligned}$$

which is the desired result. \square

Theorem 2. For $\operatorname{Re}(v) > -\frac{1}{2}$ and $z \in \mathbb{R}$, then

$$\frac{2^v \sqrt{\pi}}{z^v} J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(k + \frac{1}{2}\right)}{\Gamma(v+k+1)(2k)!} z^{2k},$$

where $J_\nu(z)$ denotes the Bessel function of the first kind, $\Gamma(v)$ denotes the gamma function and $k!$ denotes the factorial function.

Proof. Let $z \rightarrow izt$ in Lemma 1, therefore,

$$e^{-izt} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \left[t^{2k} - iz \frac{t^{2k+1}}{2k+1} \right]. \quad (3)$$

I multiply (3) by $(1-t^2)^{v-\frac{1}{2}}$ and integrate from -1 at 1 with respect to t , thus

$$\begin{aligned} \int_{-1}^1 (1-t^2)^{v-\frac{1}{2}} e^{-izt} dt &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \left[\int_{-1}^1 (1-t^2)^{v-\frac{1}{2}} t^{2k} dt \right. \\ &\quad \left. - \frac{iz}{2k+1} \int_{-1}^1 (1-t^2)^{v-\frac{1}{2}} t^{2k+1} dt \right] = \Gamma\left(v + \frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(k + \frac{1}{2}\right)}{\Gamma(v+k+1)(2k)!} z^{2k}. \end{aligned} \quad (4)$$

But, I know [1, p. 48] that

$$\int_{-1}^1 (1-t^2)^{v-\frac{1}{2}} e^{-izt} dt = \frac{2^v \Gamma\left(v + \frac{1}{2}\right) \sqrt{\pi}}{z^v} J_\nu(z). \quad (5)$$

From (4) and (5), it follows that

$$\frac{2^v \Gamma\left(v + \frac{1}{2}\right) \sqrt{\pi}}{z^v} J_\nu(z) = \Gamma\left(v + \frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(k + \frac{1}{2}\right)}{\Gamma(v+k+1)(2k)!} z^{2k}.$$

Eliminate $\Gamma\left(v + \frac{1}{2}\right)$ in both the members of the equation above; this completes the proof. \square

3. CONSEQUENCES OF INFINITE SERIE FOR BESSEL FUNCTION OF THE FIRST KIND

3.1. Integral Representations for Bessel Function of the first kind.

Theorem 3. For $\operatorname{Re}(v) > -\frac{1}{2}$ and $z \in \mathbb{R}$, then

$$\begin{aligned} (i) \frac{2^v \sqrt{\pi}}{z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(z) &= 2 \int_0^{\pi/2} \cos(z \sin \varphi) \cos^{2v} \varphi d\varphi, \\ (ii) \frac{2^v \sqrt{\pi}}{z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(z) &= 2 \int_0^{\pi/2} \cos(z \cos \varphi) \sin^{2v} \varphi d\varphi, \\ (iii) \frac{2^v \sqrt{\pi}}{z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(z) &= 2 \int_0^1 \cos(zt) (1-t^2)^{v-\frac{1}{2}} dt, \\ (iv) \frac{2^v \sqrt{\pi}}{z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(z) &= 2 \int_0^1 \frac{t^{2v} \cos(z \sqrt{1-t^2})}{\sqrt{1-t^2}} dt, \end{aligned}$$

where $\Gamma(v)$ denotes the gamma function, $J_v(z)$ denotes the Bessel function of the first kind, $\cos(\varphi)$ denotes the cosine function and $\sin(\varphi)$ denotes the sine function.

Proof. Multiply both members of the Theorem 2 by $\Gamma(v + \frac{1}{2})$

$$\frac{2^v \Gamma(v + \frac{1}{2}) \sqrt{\pi}}{z^v} J_v(z) = \Gamma\left(v + \frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k + \frac{1}{2})}{\Gamma(v + k + 1) (2k)!} z^{2k}. \quad (6)$$

I know [2, page 899, formula 8.384.1] that

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = B(b, a), \quad (7)$$

for $\text{Re}(a) > 0$ and $\text{Re}(b) > 0$.

Setting $a = k + \frac{1}{2}$ and $b = v + \frac{1}{2}$ in (7), I encounter

$$B\left(k + \frac{1}{2}, v + \frac{1}{2}\right) = \frac{\Gamma(k + \frac{1}{2})\Gamma(v + \frac{1}{2})}{\Gamma(v + k + 1)} = B\left(v + \frac{1}{2}, k + \frac{1}{2}\right). \quad (8)$$

From (6) and (8), I obtain

$$\frac{2^v \Gamma(v + \frac{1}{2}) \sqrt{\pi}}{z^v} J_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} B\left(k + \frac{1}{2}, v + \frac{1}{2}\right) z^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} B\left(v + \frac{1}{2}, k + \frac{1}{2}\right) z^{2k}. \quad (9)$$

I know [2, page 898, formula 8.380.2] that

$$B(a, b) = 2 \int_0^{\pi/2} \sin^{2a-1} \varphi \cos^{2b-1} \varphi d\varphi. \quad (10)$$

From (9) and (10), I get

$$\begin{aligned} \frac{2^v \Gamma(v + \frac{1}{2}) \sqrt{\pi}}{z^v} J_v(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left[2 \int_0^{\pi/2} \sin^{2k} \varphi \cos^{2v} \varphi d\varphi \right] z^{2k} \\ &= 2 \int_0^{\pi/2} \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \sin^{2k} \varphi z^{2k} \right] \cos^{2v} \varphi d\varphi = 2 \int_0^{\pi/2} \cos(z \sin \varphi) \cos^{2v} \varphi d\varphi \end{aligned}$$

and

$$\begin{aligned} \frac{2^v \Gamma(v + \frac{1}{2}) \sqrt{\pi}}{z^v} J_v(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left[2 \int_0^{\pi/2} \sin^{2v} \varphi \cos^{2k} \varphi d\varphi \right] z^{2k} \\ &= 2 \int_0^{\pi/2} \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \cos^{2k} \varphi z^{2k} \right] \sin^{2v} \varphi d\varphi = 2 \int_0^{\pi/2} \cos(z \cos \varphi) \sin^{2v} \varphi d\varphi. \end{aligned}$$

I know [2, page 898, formula 8.380.1] that

$$B(a, b) = 2 \int_0^1 t^{2a-1} (1-t^2)^{b-1} dt. \quad (11)$$

From (9) and (11), I find

$$\begin{aligned} \frac{2^v \Gamma(v + \frac{1}{2}) \sqrt{\pi}}{z^v} J_v(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left[2 \int_0^1 t^{2k} (1-t^2)^{v-\frac{1}{2}} dt \right] z^{2k} \\ &= 2 \int_0^1 \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k} z^{2k} \right] (1-t^2)^{v-\frac{1}{2}} dt = 2 \int_0^1 \cos(zt) (1-t^2)^{v-\frac{1}{2}} dt \end{aligned}$$

and

$$\begin{aligned} \frac{2^v \Gamma(v + \frac{1}{2}) \sqrt{\pi}}{z^v} J_v(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left[2 \int_0^1 t^{2v} (1-t^2)^{k-\frac{1}{2}} dt \right] z^{2k} \\ &= 2 \int_0^1 \frac{t^{2v}}{\sqrt{1-t^2}} \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (1-t^2)^k z^{2k} \right] dt = 2 \int_0^1 \frac{t^{2v} \cos(z \sqrt{1-t^2})}{\sqrt{1-t^2}} dt. \end{aligned}$$

This completes the proof. \square

3.2. Integral Representations for Sine Function.

Corollary 4. For $z \in \mathbb{R}$, then

$$\begin{aligned} (i) \frac{\sin(z)}{z} &= \int_0^{\pi/2} \cos(z \sin \varphi) \cos \varphi d\varphi, \\ (ii) \frac{\sin(z)}{z} &= \int_0^{\pi/2} \cos(z \cos \varphi) \sin \varphi d\varphi, \\ (iii) \frac{\sin(z)}{z} &= \int_0^1 \cos(zt) dt, \\ (iv) \frac{\sin(z)}{z} &= \int_0^1 \frac{t \cos(z \sqrt{1-t^2})}{\sqrt{1-t^2}} dt, \end{aligned}$$

where $\cos(z)$ denotes the cosine function and $\sin(z)$ denotes the sine function.

Proof. I know [2, page 914, formula 8.464.1] that

$$\sqrt{\frac{2\pi}{z}} J_{1/2}(z) = \frac{2 \sin(z)}{z}. \quad (12)$$

On the other hand, I put v into $1/2$ in Theorem 3, Eqs. (i), (ii), (iii), (iv), and obtain

$$\begin{aligned} \sqrt{\frac{2\pi}{z}} J_{1/2}(z) &= 2 \int_0^{\pi/2} \cos(z \sin \varphi) \cos \varphi d\varphi, \\ \sqrt{\frac{2\pi}{z}} J_{1/2}(z) &= 2 \int_0^{\pi/2} \cos(z \cos \varphi) \sin \varphi d\varphi, \\ \sqrt{\frac{2\pi}{z}} J_{1/2}(z) &= 2 \int_0^1 \cos(zt) dt, \\ \sqrt{\frac{2\pi}{z}} J_{1/2}(z) &= 2 \int_0^1 \frac{t \cos(z \sqrt{1-t^2})}{\sqrt{1-t^2}} dt. \end{aligned} \quad (13)$$

I substitute (12) into (13) and eliminate 2 in both members; this concludes the proof. \square

3.3. Integral Representations Arising out of a Recurrence Relation.

Theorem 5. For $\operatorname{Re}(v) > -\frac{1}{2}$ and $n \in \mathbb{N}_{>2}$, then

$$\begin{aligned} (i) \int_0^{\pi/2} \cos(\sin \varphi) \cos[(n-1)\sin \varphi] \cos^{2v} \varphi d\varphi &= 2^{v-1} \sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right) \left[\frac{J_v(n)}{n^v} + \frac{J_v(n-2)}{(n-2)^v} \right], \\ (ii) \int_0^{\pi/2} \cos(\cos \varphi) \cos[(n-1)\cos \varphi] \sin^{2v} \varphi d\varphi &= 2^{v-1} \sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right) \left[\frac{J_v(n)}{n^v} + \frac{J_v(n-2)}{(n-2)^v} \right], \end{aligned}$$

where $\Gamma(v)$ denotes the gamma function, $J_v(z)$ denotes the Bessel function of the first kind, $\cos(\varphi)$ denotes the cosine function and $\sin(\varphi)$ denotes the sine function.

Proof. It is well know that

$$\cos n\theta = 2 \cos \theta \cos(n-1)\theta - \cos(n-2)\theta. \quad (14)$$

Let $z \rightarrow n$ in Theorem Eqs. (i) and (ii), therefore

$$\begin{aligned} (i) \frac{2^v \sqrt{\pi}}{n^v} \Gamma\left(v + \frac{1}{2}\right) J_v(n) &= 2 \int_0^{\pi/2} \cos(n \sin \varphi) \cos^{2v} \varphi d\varphi, \\ (ii) \frac{2^v \sqrt{\pi}}{n^v} \Gamma\left(v + \frac{1}{2}\right) J_v(n) &= 2 \int_0^{\pi/2} \cos(n \cos \varphi) \sin^{2v} \varphi d\varphi. \end{aligned} \quad (15)$$

Now, I set $\theta \rightarrow \sin \varphi$ in (14), multiply by $2 \cos^{2v} \varphi$ and integrate from 0 at $\pi/2$ with respect to φ , thus

$$\begin{aligned} 2 \int_0^{\pi/2} \cos(n \sin \varphi) \cos^{2v} \varphi d\varphi &= 2 \int_0^{\pi/2} \cos(\sin \varphi) \cos[(n-1) \sin \varphi] \cos^{2v} \varphi d\varphi \\ &\quad - 2 \int_0^{\pi/2} \cos[(n-2) \sin \varphi] \cos^{2v} \varphi d\varphi, \end{aligned} \quad (16)$$

consequently, from (15.i) and (16), I find

$$\int_0^{\pi/2} \cos(\sin \varphi) \cos[(n-1) \sin \varphi] \cos^{2v} \varphi d\varphi = 2^{v-1} \sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right) \left[\frac{J_v(n)}{n^v} + \frac{J_v(n-2)}{(n-2)^v} \right].$$

Again, I set $\theta \rightarrow \cos \varphi$ in (14), multiply by $2 \sin^{2v} \varphi$ and integrate from 0 at $\pi/2$ with respect to φ , thus

$$\begin{aligned} 2 \int_0^{\pi/2} \cos(n \cos \varphi) \sin^{2v} \varphi d\varphi &= 2 \int_0^{\pi/2} \cos(\cos \varphi) \cos[(n-1) \cos \varphi] \sin^{2v} \varphi d\varphi \\ &\quad - 2 \int_0^{\pi/2} \cos[(n-2) \cos \varphi] \sin^{2v} \varphi d\varphi, \end{aligned} \quad (17)$$

consequently, from (15.ii) and (17), I encounter

$$\int_0^{\pi/2} \cos(\cos \varphi) \cos[(n-1) \cos \varphi] \sin^{2v} \varphi d\varphi = 2^{v-1} \sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right) \left[\frac{J_v(n)}{n^v} + \frac{J_v(n-2)}{(n-2)^v} \right].$$

This concludes the proof. \square

3.4. Special Cases.

Theorem 6. For $\operatorname{Re}(v) > -\frac{1}{2}$ and $z \in \mathbb{R}$, then

$$\int_0^{\pi/2} \cos^2(\sin \varphi) \cos^{2v} \varphi d\varphi = \frac{\sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right) J_v(2)}{4} + \frac{\sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right)}{4\Gamma(v+1)}$$

and

$$\int_0^{\pi/2} \cos(z \sin \varphi) \cos(2\varphi) d\varphi = \frac{\pi}{z} J_1(z) - \frac{\pi}{2} J_0(z),$$

where $\Gamma(v)$ denotes the gamma function, $J_v(z)$ denotes the Bessel function of the first kind, $\cos(\varphi)$ denotes the cosine function and $\sin(\varphi)$ denotes the sine function.

Proof. In [2, page 34, formula 1.335.1], I have

$$\cos(2x) = 2\cos^2 x - 1. \quad (18)$$

Change x into $\sin \varphi$ in (18), multiply by $2 \cos^{2v} \varphi$ and integrate from 0 at $\pi/2$ with respect to φ , thus

$$2 \int_0^{\pi/2} \cos(2 \sin \varphi) \cos^{2v} \varphi d\varphi = 4 \int_0^{\pi/2} \cos^2(\sin \varphi) \cos^{2v} \varphi d\varphi - 2 \int_0^{\pi/2} \cos^{2v} \varphi d\varphi, \quad (19)$$

therefrom, I easily calculate that

$$\begin{aligned} \sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right) J_v(2) &= 4 \int_0^{\pi/2} \cos^2(\sin \varphi) \cos^{2v} \varphi d\varphi - \sqrt{\pi} \frac{\Gamma\left(v + \frac{1}{2}\right)}{\Gamma(v+1)} \\ \Rightarrow \int_0^{\pi/2} \cos^2(\sin \varphi) \cos^{2v} \varphi d\varphi &= \frac{\sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right) J_v(2)}{4} + \frac{\sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right)}{4\Gamma(v+1)}. \end{aligned} \quad (20)$$

Let $x = \varphi$ in (18), multiply by $2 \cos(z \sin \varphi)$ and integrate from 0 at $\pi/2$ with respect to φ , thus

$$2 \int_0^{\pi/2} \cos(z \sin \varphi) \cos(2\varphi) d\varphi = 4 \int_0^{\pi/2} \cos(z \sin \varphi) \cos^2 \varphi d\varphi - 2 \int_0^{\pi/2} \cos(z \sin \varphi) d\varphi, \quad (21)$$

thereafter, I easily calculate

$$\int_0^{\pi/2} \cos(z \sin \varphi) \cos(2\varphi) d\varphi = \frac{\pi}{z} J_1(z) - \frac{\pi}{2} J_0(z),$$

which is the desired result. \square

3.5. More Integral Representations.

Theorem 7. For $\operatorname{Re}(v) > -\frac{1}{2}$ and $z \in \mathbb{R}$, then

$$\int_0^{\pi/2} \sin^2\left(\frac{z \sin \varphi}{2}\right) \cos^{2v} \varphi d\varphi = \frac{\sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right)}{4\Gamma(v+1)} - \frac{2^v \sqrt{\pi}}{4z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(z),$$

$$\begin{aligned} & \int_0^{\pi/2} \cos(z \sin \varphi) \sin^2\left(\frac{z \sin \varphi}{2}\right) \cos^{2v} \varphi d\varphi \\ &= -\frac{\sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right)}{8\Gamma(v+1)} + \frac{2^v \sqrt{\pi}}{4z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(z) - \frac{\sqrt{\pi}}{8z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(2z) \end{aligned}$$

and

$$\begin{aligned} & \int_0^{\pi/2} \cos^2(z \sin \varphi) \sin^2\left(\frac{z \sin \varphi}{2}\right) \cos^{2v} \varphi d\varphi \\ &= \frac{\sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right)}{8\Gamma(v+1)} - \frac{2^v 3\sqrt{\pi}}{16 \cdot z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(z) \\ &+ \frac{\sqrt{\pi}}{8 \cdot z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(2z) - \frac{2^v \sqrt{\pi}}{16 \cdot 3^v \cdot z^v} \Gamma\left(v + \frac{1}{2}\right) J_\nu(3z), \end{aligned}$$

where $\Gamma(v)$ denotes the gamma function, $J_\nu(z)$ denotes the Bessel function of the first kind, $\cos(\varphi)$ denotes the cosine function and $\sin(\varphi)$ denotes the sine function.

Proof. I know that

$$\sin^2\left(\frac{x}{2}\right) = \frac{1}{2}[1 - \cos(x)], \quad (22)$$

$$\cos(x) \sin^2\left(\frac{x}{2}\right) = \frac{1}{4}[-1 + 2\cos(x) - \cos(2x)] \quad (23)$$

and

$$\cos^2(x) \sin^2\left(\frac{x}{2}\right) = \frac{1}{8}[2 - 3\cos(x) + 2\cos(2x) - \cos(3x)]. \quad (24)$$

Let $x = z \sin \varphi$ in (22)-(24), multiply by $2\cos^{2v} \varphi$ and integrate from 0 at $\pi/2$ with respect to φ , thus

$$2 \int_0^{\pi/2} \sin^2\left(\frac{z \sin \varphi}{2}\right) \cos^{2v} \varphi d\varphi = \int_0^{\pi/2} \cos^{2v} \varphi d\varphi - \int_0^{\pi/2} \cos(z \sin \varphi) \cos^{2v} \varphi d\varphi, \quad (25)$$

$$\begin{aligned} & 4 \int_0^{\pi/2} \cos(z \sin \varphi) \sin^2\left(\frac{z \sin \varphi}{2}\right) \cos^{2v} \varphi d\varphi \\ &= -\int_0^{\pi/2} \cos^{2v} \varphi d\varphi + 2 \int_0^{\pi/2} \cos(z \sin \varphi) \cos^{2v} \varphi d\varphi - \int_0^{\pi/2} \cos(2z \sin \varphi) \cos^{2v} \varphi d\varphi \end{aligned} \quad (26)$$

and

$$\begin{aligned} & 8 \int_0^{\pi/2} \cos^2(z \sin \varphi) \sin^2\left(\frac{z \sin \varphi}{2}\right) \cos^{2v} \varphi d\varphi \\ &= 2 \int_0^{\pi/2} \cos^{2v} \varphi d\varphi - 3 \int_0^{\pi/2} \cos(z \sin \varphi) \cos^{2v} \varphi d\varphi \\ &+ 2 \int_0^{\pi/2} \cos(2z \sin \varphi) \cos^{2v} \varphi d\varphi - \int_0^{\pi/2} \cos(3z \sin \varphi) \cos^{2v} \varphi d\varphi \end{aligned} \quad (27)$$

From Theorem 3 and (25) at (27), I obtain easily

$$\int_0^{\pi/2} \sin^2\left(\frac{z \sin \varphi}{2}\right) \cos^{2v} \varphi d\varphi = \frac{\sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right)}{4\Gamma(v+1)} - \frac{2^v \sqrt{\pi}}{4z^v} \Gamma\left(v + \frac{1}{2}\right) J_v(z), \quad (28)$$

$$\begin{aligned} & \int_0^{\pi/2} \cos(z \sin \varphi) \sin^2\left(\frac{z \sin \varphi}{2}\right) \cos^{2v} \varphi d\varphi \\ &= -\frac{\sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right)}{8\Gamma(v+1)} + \frac{2^v \sqrt{\pi}}{4z^v} \Gamma\left(v + \frac{1}{2}\right) J_v(z) - \frac{\sqrt{\pi}}{8z^v} \Gamma\left(v + \frac{1}{2}\right) J_v(2z) \end{aligned} \quad (29)$$

and

$$\begin{aligned} & 8 \int_0^{\pi/2} \cos^2(z \sin \varphi) \sin^2\left(\frac{z \sin \varphi}{2}\right) \cos^{2v} \varphi d\varphi \\ &= \frac{\sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right)}{\Gamma(v+1)} - \frac{2^v 3 \sqrt{\pi}}{2 \cdot z^v} \Gamma\left(v + \frac{1}{2}\right) J_v(z) \\ &+ \frac{\sqrt{\pi}}{z^v} \Gamma\left(v + \frac{1}{2}\right) J_v(2z) - \frac{2^v \sqrt{\pi}}{2 \cdot 3^v \cdot z^v} \Gamma\left(v + \frac{1}{2}\right) J_v(3z) \\ &\Rightarrow \int_0^{\pi/2} \cos^2(z \sin \varphi) \sin^2\left(\frac{z \sin \varphi}{2}\right) \cos^{2v} \varphi d\varphi \\ &= \frac{\sqrt{\pi} \Gamma\left(v + \frac{1}{2}\right)}{8\Gamma(v+1)} - \frac{2^v 3 \sqrt{\pi}}{16 \cdot z^v} \Gamma\left(v + \frac{1}{2}\right) J_v(z) \\ &+ \frac{\sqrt{\pi}}{8 \cdot z^v} \Gamma\left(v + \frac{1}{2}\right) J_v(2z) - \frac{2^v \sqrt{\pi}}{16 \cdot 3^v \cdot z^v} \Gamma\left(v + \frac{1}{2}\right) J_v(3z), \end{aligned} \quad (30)$$

and this completes the proof. \square

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