

IMPROVEMENT ON OPERATOR AXIOMS AND FUNDAMENTAL OPERATOR FUNCTIONS

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ABSTRACT. The reference [2] constructs the Operator axioms to deduce number systems. In this paper, we slightly improve on the syntax of the Operator axioms and construct a semantics of the Operator axioms. Then on the basis of the improved Operator axioms, we define two fundamental operator functions to study the analytic properties of the Operator axioms. Finally, we prove two theorems about the fundamental operator functions.

1. INTRODUCTION

In [1], we distinguish the limit from the infinite sequence. Then in [2], we define the Operator axioms to extend the traditional real number system. the Operator axioms have shown typical features as follows:

1. The logical calculus provides a uniform frame for arithmetic axioms. Based on the same logical calculus, small number system can import new axioms to produce big number systems.

2. Consistent binary relation are the nature of number systems. The order relation and equivalence relation of each number system is consistent, so all numbers of each number system are layed in fixed positions of number line.

3. The number systems equate number with operation. The number '1' and various operators compose all numbers through operation. Any number except '1' is also an operation. For example, the number "[[1+[1+1]]- - - -[1+1]]" derives from the operation of the number "[1+[1+1]]", the number "[1+1]" and the real operator "- - -". So operator distinguishes different number systems and is the foremost component of number system.

4. The Operator axioms produces new real numbers with new operators. While producing new real numbers in arithmetic, the new operators certainly produce new equations and inequalities in algebra. So the Operator axioms not only extends real number system, but also extends equations and inequalities.

In conclusion, the Operator axioms forms a new arithmetic axiom. The [2, Definition 2.2] defines 'number' on the basis of the logical calculus $\{\Phi, \Psi\}$. The [2, TABLE 2] defines new operators according to the definition of number systems. Real operators naturally produce new equations such as $y = [x++++[1+1]]$, $y = [[1+1]- - - -x]$, $y = [x////[1+1]]$ and so on. In other words, real operators extend the traditional mathematical models which are selected to describe various scientific rules. So real operators help to describe complex scientific rules which are difficult described by traditional equations and have an enormous application potential.

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The paper is organized as follows. In Section 2, we construct the syntax and semantics of the improved Operator axioms. In Section 3, we define two fundamental operator functions and prove two theorems about the fundamental operator functions.

2. OPERATOR AXIOMS

The Operator axioms divides into syntax and semantics. The syntax aims at logical deduction, while the semantics aims at the objects mapped from the syntax deduction.

2.1. Syntax Of The Operator Axioms[2]. The syntax of the Operator axioms derives from [2] and is revised a little in this section. Table 1 translates the simple part of the syntax to natural language. The complex part of the syntax is difficult to be translated to natural language.

TABLE 1. [2, TABLE 2] Translation From Syntax To Natural Language.

Syntax	Natural Language
\wedge	and
\vee	or
\neg	not
\rightarrow	replaced by
\Rightarrow	imply
\Leftrightarrow	symmetrical imply
{	punctuation
}	punctuation
,	punctuation
(punctuation
)	punctuation
\emptyset	emptiness
\dots	omission
$\Phi\{\dots\}$	Denote Φ as a set of notations and particular axioms between { and }. Different logical calculus correspond to different notations and particular axioms.
$V\{\dots\}$	Denote V as a set of variables between { and }.
$C\{\dots\}$	Denote C as a set of constants between { and }.
$P\{\dots\}$	Denote P as a set of predicate symbols between { and }.
$V \circ C\{\dots\}$	Denote $V \circ C$ as a set of concatenations between V and C .
$C \circ C\{\dots\}$	Denote $C \circ C$ as a set of concatenations between C and C .
$V \circ C \circ P\{\dots\}$	Denote $V \circ C \circ P$ as a set of concatenations among V , C and P .
$(\dots \in \dots) \dots \Leftrightarrow \dots$	Define the binary predicate symbol \in .
$(\dots \in (V \circ C))$	Define a variable ranging over $V \circ C$.
$(\dots \in (V \circ C \circ P))$	Define a variable ranging over $V \circ C \circ P$.
$(\dots \subseteq \dots) \dots \Leftrightarrow \dots$	Define the binary predicate symbol \subseteq .
$\Psi\{\dots\}$	Denote Ψ as a set of general axioms between { and }. Different logical calculus correspond to the same general axioms.
$(\dots \rightarrow \dots) \wedge (\dots \rightarrow \dots) \Rightarrow \dots$	Define the binary predicate symbol \rightarrow .
$(\dots \dots) \dots \Rightarrow \dots$	Define the binary predicate symbol $ $.
$(\dots < \dots) \dots \Rightarrow \dots$	Define the binary predicate symbol $<$.
$(\dots = \dots)$	Define the binary predicate symbol $=$.
$(\dots = \dots) \dots \Rightarrow \dots$	Define the binary predicate symbol $=$.
$(\dots \leq \dots) \dots \Leftrightarrow \dots$	Define the binary predicate symbol \leq .
$(\dots \leq \dots) \dots \Rightarrow \dots$	Define the binary predicate symbol \leq .

Definition 2.1. [2, Definition 2.2] In a logical calculus $\{\Phi, \Psi\}$, if $\bar{a} \equiv true$, then \bar{a} is a number.

Definition 2.2. the Operator axioms is a logical calculus $R\{\Phi, \Psi\}$ such that:

- $$\Phi\{$$
- (2.1) $V\{\emptyset, a, b, c, d, e, f, g, h, i, j\},$
- (2.2) $C\{\emptyset, 1, +, [,], -, /\},$
- (2.3) $P\{\emptyset, \in, \subseteq, \rightarrow, |, =, <, \leq, \|\},$
- (2.4) $V \circ C\{\emptyset, a, b \cdots, 1, + \cdots, aa, ab \cdots, a1, a + \cdots, ba, bb \cdots, b1, b + \cdots, aaa, aab \cdots, aa1, aa + \cdots, baa, bab \cdots, ba1, ba + \cdots\},$
- (2.5) $C \circ C\{\emptyset, 1, + \cdots, 11, 1 + \cdots, 111, 11 + \cdots\},$
- (2.6) $V \circ C \circ P\{\emptyset, a, b \cdots, 1, + \cdots, \in, \subseteq \cdots, aa, ab \cdots, a1, a + \cdots, a \in, a \subseteq \cdots, ba, bb \cdots, b1, b + \cdots, b \in, b \subseteq \cdots, aaa, aab \cdots, aa1, aa + \cdots, aa \in, aa \subseteq \cdots, baa, bab \cdots, ba1, ba + \cdots, ba \in, ba \subseteq \cdots\},$
- (2.7) $(\hat{a} \in V) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots),$
- (2.8) $(\hat{a} \in C) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots),$
- (2.9) $(\hat{a} \in (V \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv 1) \cdots \vee (\hat{a} \equiv aa) \vee (\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a1) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa1) \cdots),$
- (2.10) $(\hat{a} \in (C \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots \vee (\hat{a} \equiv 11) \vee (\hat{a} \equiv 1+) \cdots \vee (\hat{a} \equiv 111) \vee (\hat{a} \equiv 11+) \cdots),$
- (2.11) $(\hat{a} \in (V \circ C \circ P)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv \in) \cdots \vee (\hat{a} \equiv aa) \vee (\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a \in) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa \in) \cdots),$
- (2.12) $(\bar{a} \in (V \circ C)) \wedge (\bar{b} \in (V \circ C)) \wedge (\bar{c} \in (V \circ C)) \wedge (\bar{d} \in (V \circ C)) \wedge (\bar{e} \in (V \circ C)) \wedge (\bar{f} \in (V \circ C)) \wedge (\bar{g} \in (V \circ C)) \wedge (\bar{h} \in (V \circ C)) \wedge (\bar{i} \in (V \circ C)) \wedge (\bar{j} \in (V \circ C)) \wedge (\bar{k} \in (V \circ C)) \wedge (\bar{l} \in (V \circ C)) \wedge (\bar{a} \in (V \circ C \circ P)) \wedge (\bar{b} \in (V \circ C \circ P)) \wedge (\bar{c} \in (V \circ C \circ P)),$
- (2.13) $((\bar{a} \subseteq \{\bar{b}, \bar{c}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}))) \wedge ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}))) \wedge ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}))) \wedge ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}))) \wedge ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}))) \wedge ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}))) \wedge ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}))) \wedge ((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}))),$

- (2.14) $(\bar{a}\bar{b}\bar{c} = \bar{d}\bar{b}\bar{e}\bar{f}\bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \wedge \neg(\bar{f} \subseteq \{\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{g}\}) \wedge$
 $((\bar{b} \rightarrow \bar{h}) \parallel (\bar{f} \rightarrow \bar{i})) \Rightarrow (\bar{a}\bar{h}\bar{c} = \bar{d}\bar{h}\bar{e}\bar{i}\bar{g}),$
- (2.15) $(\bar{a}\bar{b}\bar{c}\bar{d}\bar{e} = \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \wedge \neg(\bar{d} \subseteq \{\bar{a}, \bar{b}, \bar{c}, \bar{e}, \bar{f}\}) \wedge ((\bar{b} \rightarrow \bar{g}) \parallel (\bar{d} \rightarrow \bar{h}))$
 $\Rightarrow (\bar{a}\bar{g}\bar{c}\bar{h}\bar{e} = \bar{f}),$
- (2.16) $a \rightarrow 1[[aba],$
- (2.17) $b \rightarrow +|-,$
- (2.18) $c|d \rightarrow e|f|g,$
- (2.19) $e \rightarrow +|+e,$
- (2.20) $f \rightarrow -|-f,$
- (2.21) $g \rightarrow /|/g,$
- (2.22) $(h \rightarrow +) \parallel (i \rightarrow -),$
- (2.23) $(h \rightarrow +h) \parallel (i \rightarrow -i),$
- (2.24) $(i \rightarrow -) \parallel (h \rightarrow +),$
- (2.25) $(i \rightarrow -i) \parallel (h \rightarrow +h),$
- (2.26) $(h \rightarrow +) \parallel (j \rightarrow /),$
- (2.27) $(h \rightarrow +h) \parallel (j \rightarrow /j),$
- (2.28) $a < [1 + a],$
- (2.29) $(\bar{a} < \bar{b}) \wedge \bar{c} \Rightarrow (([\bar{a} + \bar{c}] < [\bar{b} + \bar{c}]) \wedge ([\bar{a} - \bar{c}] < [\bar{b} - \bar{c}]) \wedge ([\bar{c} - \bar{b}] < [\bar{c} - \bar{a}])),$
- (2.30) $([1 - 1] \leq \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1 - 1] < \bar{c}) \Rightarrow ([\bar{a} - -\bar{c}] < [\bar{b} - -\bar{c}]),$
- (2.31) $([1 - 1] < \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1 - 1] < \bar{c}) \Rightarrow ([\bar{c} - -\bar{b}] < [\bar{c} - -\bar{a}]),$
- (2.32) $(1 < \bar{a}) \wedge ([1 - 1] < \bar{b}) \Rightarrow (1 < [\bar{a} - -f\bar{b}]),$
- (2.33) $(1 < \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([1 - 1] < [\bar{a}/g\bar{b}]),$
- (2.34) $(1 < \bar{a}) \wedge (\bar{a} < \bar{b}) \Rightarrow (1 < [\bar{b}/g\bar{a}]),$
- (2.35) $(1 \leq \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1 - 1] < \bar{c}) \Rightarrow ([\bar{a}e\bar{c}] < [\bar{b}e\bar{c}]),$
- (2.36) $(1 \leq \bar{a}) \wedge (1 \leq \bar{b}) \wedge ([1 - 1] < \bar{c}) \wedge ([\bar{a}e\bar{c}] < [\bar{b}e\bar{c}]) \Rightarrow (\bar{a} < \bar{b}),$
- (2.37) $(1 < \bar{a}) \wedge ([1 - 1] \leq \bar{b}) \wedge (\bar{b} < \bar{c}) \Rightarrow ([\bar{a}e\bar{b}] < [\bar{a}e\bar{c}]),$
- (2.38) $(1 < \bar{a}) \wedge ([1 - 1] \leq \bar{b}) \wedge ([1 - 1] \leq \bar{c}) \wedge ([\bar{a}e\bar{b}] < [\bar{a}e\bar{c}]) \Rightarrow (\bar{b} < \bar{c}),$
- (2.39) $\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} - \bar{b}] = [\bar{a}/\bar{b}]),$
- (2.40) $\bar{a} \wedge \neg(\bar{b} = [1 - 1]) \Rightarrow ([\bar{a} - -\bar{b}] = [\bar{a}/\bar{b}]),$
- (2.41) $\bar{a} \Rightarrow ([\bar{a} - \bar{a}] = [1 - 1]),$
- (2.42) $\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} + \bar{b}] = [\bar{b} + \bar{a}]),$
- (2.43) $\bar{a} \wedge \bar{b} \Rightarrow ([[\bar{a} - \bar{b}] + \bar{b}] = \bar{a}),$
- (2.44) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([[\bar{a} - \bar{b}] + \bar{c}] = [[\bar{a} + \bar{c}] - \bar{b}]),$
- (2.45) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + [\bar{b} + \bar{c}]] = [[\bar{a} + \bar{b}] + \bar{c}]),$
- (2.46) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + [\bar{b} - \bar{c}]] = [[\bar{a} + \bar{b}] - \bar{c}]),$
- (2.47) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - [\bar{b} + \bar{c}]] = [[\bar{a} - \bar{b}] - \bar{c}]),$
- (2.48) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - [\bar{b} - \bar{c}]] = [[\bar{a} - \bar{b}] + \bar{c}]),$

- (2.49) $\bar{a} \Rightarrow ([\bar{a} + +1] = \bar{a}) \wedge ([\bar{a} - -1] = \bar{a}),$
- (2.50) $\neg(\bar{a} = [1 - 1]) \Rightarrow ([\bar{a} - -\bar{a}] = 1),$
- (2.51) $\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} + +\bar{b}] = [\bar{b} + +\bar{a}]),$
- (2.52) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} + +\bar{c}]] = [[\bar{a} + +\bar{b}] + +\bar{c}]),$
- (2.53) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} + \bar{c}]] = [[\bar{a} + +\bar{b}] + [\bar{a} + +\bar{c}]]),$
- (2.54) $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} - \bar{c}]] = [[\bar{a} + +\bar{b}] - [\bar{a} + +\bar{c}]]),$
- (2.55) $\bar{a} \wedge \neg(\bar{b} = [1 - 1]) \Rightarrow ([[\bar{a} - -\bar{b}] + +\bar{b}] = \bar{a}),$
- (2.56) $\bar{a} \wedge \neg(\bar{b} = [1 - 1]) \wedge \bar{c} \Rightarrow ((([\bar{a} - -\bar{b}] + +\bar{c}] = [[\bar{a} + +\bar{c}] - -\bar{b}]) \wedge$
 $([[\bar{a} + \bar{c}] - -\bar{b}] = [[\bar{a} - -\bar{b}] + [\bar{c} - -\bar{b}]])) \wedge ([[\bar{a} - \bar{c}] - -\bar{b}] =$
 $[[\bar{a} - -\bar{b}] - [\bar{c} - -\bar{b}]])),$
- (2.57) $\bar{a} \wedge \neg(\bar{b} = [1 - 1]) \wedge \neg(\bar{c} = [1 - 1]) \Rightarrow (([\bar{a} + +[\bar{b} - -\bar{c}]] = [[\bar{a} + +\bar{b}] - -\bar{c}]) \wedge$
 $([\bar{a} - -[\bar{b} + +\bar{c}]] = [[\bar{a} - -\bar{b}] - -\bar{c}]) \wedge ([\bar{a} - -[\bar{b} - -\bar{c}]] = [[\bar{a} - -\bar{b}] + +\bar{c}])),$
- (2.58) $\bar{a} \Rightarrow ([\bar{a} + + + 1] = \bar{a}) \wedge ([\bar{a} - - - 1] = \bar{a}),$
- (2.59) $\bar{a} \Rightarrow ([1 + + + \bar{a}] = 1),$
- (2.60) $\neg(\bar{a} = [1 - 1]) \Rightarrow ([\bar{a} + + + [1 - 1]] = 1),$
- (2.61) $([1 - 1] < \bar{a}) \Rightarrow ([[1 - 1] + + + \bar{a}] = [1 - 1]),$
- (2.62) $([1 - 1] < \bar{a}) \wedge \neg(\bar{b} = [1 - 1]) \wedge \bar{c} \Rightarrow ((([\bar{a} - - - \bar{b}] + + + \bar{b}] = \bar{a}) \wedge$
 $([[\bar{a} - - - \bar{b}] + + + \bar{c}] = [[\bar{a} + + + \bar{c}] - - - \bar{b}]) \wedge ([\bar{a} + + + [\bar{c} - -\bar{b}]] =$
 $[[\bar{a} + + + \bar{c}] - - - \bar{b}))),$
- (2.63) $([1 - 1] < \bar{a}) \wedge ([1 - 1] < \bar{b}) \wedge \bar{c} \Rightarrow (([\bar{a} + + + [\bar{b} // \bar{a}]] = \bar{b}) \wedge$
 $([[\bar{a} + + + \bar{c}] // \bar{b}] = [\bar{c} + + [\bar{a} // \bar{b}]])) \wedge ([[\bar{a} - -\bar{b}] + + + \bar{c}] =$
 $[[\bar{a} + + + \bar{c}] - -[\bar{b} + + + \bar{c}]]),$
- (2.64) $([1 - 1] < \bar{a}) \wedge ([1 - 1] < \bar{b}) \wedge ([1 - 1] < \bar{c}) \Rightarrow ((([\bar{a} // \bar{c}] - -[\bar{b} // \bar{c}]] =$
 $[\bar{a} // \bar{b}]) \wedge ([[\bar{a} + +\bar{b}] // \bar{c}] = [[\bar{a} // \bar{c}] + [\bar{b} // \bar{c}]])) \wedge ([[\bar{a} - -\bar{b}] // \bar{c}] =$
 $[[\bar{a} // \bar{c}] - [\bar{b} // \bar{c}]]),$
- (2.65) $\neg(\bar{a} = [1 - 1]) \wedge \neg(\bar{b} = [1 - 1]) \wedge \neg(\bar{c} = [1 - 1]) \Rightarrow ((([\bar{a} + +\bar{b}] + + + \bar{c}] =$
 $[[\bar{a} + + + \bar{c}] + +[\bar{b} + + + \bar{c}]])) \wedge ([\bar{a} + + + [\bar{b} + +\bar{c}]] = [[\bar{a} + + + \bar{b}] + + + \bar{c}])$
 $\wedge ([\bar{a} + + + [\bar{b} + \bar{c}]] = [[\bar{a} + + + \bar{b}] + +[\bar{a} + + + \bar{c}]])) \wedge ([\bar{a} + + + [\bar{b} - \bar{c}]] =$
 $[[\bar{a} + + + \bar{b}] - -[\bar{a} + + + \bar{c}]]),$
- (2.66) $([1 - 1] < \bar{a}) \wedge \neg(\bar{b} = [1 - 1]) \wedge \neg(\bar{c} = [1 - 1]) \Rightarrow (([\bar{a} - - - [\bar{b} + +\bar{c}]] =$
 $[[\bar{a} - - - \bar{b}] - - - \bar{c}]) \wedge ([\bar{a} - - - [\bar{b} - -\bar{c}]] = [[\bar{a} - - - \bar{b}] + + + \bar{c}]),$
- (2.67) $([1 - 1] < \bar{a}) \wedge ([1 - 1] < \bar{b}) \wedge \neg(\bar{c} = [1 - 1]) \Rightarrow ((([\bar{a} + +\bar{b}] - - - \bar{c}] =$
 $[[\bar{a} - - - \bar{c}] + +[\bar{b} - - - \bar{c}]])) \wedge ([[\bar{a} - -\bar{b}] - - - \bar{c}] =$
 $[[\bar{a} - - - \bar{c}] - -[\bar{b} - - - \bar{c}]]),$
- (2.68) $(1 \leq \bar{a}) \Rightarrow ([\bar{a} + e1] = \bar{a}),$
- (2.69) $(1 \leq \bar{a}) \Rightarrow ([\bar{a} + +e[1 - 1]] = 1),$
- (2.70) $(1 \leq \bar{a}) \Rightarrow ([\bar{a} - f1] = \bar{a}),$

- (2.71) $([1 - 1] \leq \bar{a}) \Rightarrow ([1 + +e\bar{a}] = 1),$
 (2.72) $([1 - 1] < \bar{a}) \Rightarrow ([1 - -f\bar{a}] = 1),$
 (2.73) $(1 < \bar{a}) \Rightarrow ([1//g\bar{a}] = [1 - 1]),$
 (2.74) $(1 < \bar{a}) \Rightarrow ([\bar{a}/g\bar{a}] = 1),$
 (2.75) $(1 \leq \bar{a}) \wedge ([1 - 1] < \bar{b}) \Rightarrow ([[\bar{a}i\bar{b}]h\bar{b}] = \bar{a}),$
 (2.76) $(1 \leq \bar{a}) \wedge ([1 - 1] < \bar{b}) \Rightarrow ([[\bar{a}h\bar{b}]i\bar{b}] = \bar{a}),$
 (2.77) $(1 \leq \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([\bar{b}h[\bar{a}j\bar{b}]] = \bar{a}),$
 (2.78) $(1 \leq \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([[\bar{b}h\bar{a}]j\bar{b}] = \bar{a}),$
 (2.79) $(1 \leq \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([\bar{a} + + + e\bar{b}] = [\bar{a} + + + e[\bar{a} + + + e[\bar{b} - 1]]]),$
 (2.80) $(1 \leq \bar{a}) \wedge ([1 - 1] < \bar{b}) \wedge (\bar{b} < 1) \Rightarrow ([\bar{a} + + + e\bar{b}] = [\bar{a} + + + \bar{b}]),$
 $\},$
 $\Psi\{$
 (2.81) $(\bar{a} \subseteq \bar{b}) \Leftrightarrow (\bar{b} = \bar{c}\bar{a}\bar{d}),$
 (2.82) $(\bar{a} \rightarrow \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} \rightarrow \bar{e}) \Rightarrow (\bar{a} \rightarrow \bar{b}\bar{e}\bar{d}),$
 (2.83) $(\bar{a} \rightarrow \bar{b}|\bar{c}) \Rightarrow ((\bar{a} \rightarrow \bar{b}) \wedge (\bar{a} \rightarrow \bar{c})),$
 (2.84) $(\bar{a}|\bar{b} \rightarrow \bar{c}) \Rightarrow ((\bar{a} \rightarrow \bar{c}) \wedge (\bar{b} \rightarrow \bar{c})),$
 (2.85) $(\bar{a} < \bar{b}) \Rightarrow \neg(\bar{b} < \bar{a}),$
 (2.86) $(\bar{a} < \bar{b}) \Rightarrow \neg(\bar{a} = \bar{b}),$
 (2.87) $(\bar{a} < \bar{b}) \wedge (\bar{b} < \bar{c}) \Rightarrow (\bar{a} < \bar{c}),$
 (2.88) $(\bar{a} < \bar{b}) \wedge (\bar{a} \in (C \circ C)) \wedge (\bar{b} \in (C \circ C)) \Rightarrow (\bar{a} \wedge \bar{b}),$
 (2.89) $(\bar{a} < \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} = \bar{e}) \Rightarrow (\bar{a} < \bar{b}\bar{e}\bar{d}),$
 (2.90) $(\bar{a}\bar{b}\bar{c} < \bar{d}) \wedge (\bar{b} = \bar{e}) \Rightarrow (\bar{a}\bar{e}\bar{c} < \bar{d}),$
 (2.91) $(\bar{a} < \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} \rightarrow \bar{e}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}\}) \Rightarrow (\bar{a} < \bar{b}\bar{e}\bar{d}),$
 (2.92) $(\bar{a} < \bar{b}\bar{c}\bar{d}\bar{c}\bar{e}) \wedge (\bar{c} \rightarrow \bar{f}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a} < \bar{b}\bar{f}\bar{d}\bar{f}\bar{e}),$
 (2.93) $(\bar{a}\bar{b}\bar{c} < \bar{d}) \wedge (\bar{b} \rightarrow \bar{e}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}\}) \Rightarrow (\bar{a}\bar{e}\bar{c} < \bar{d}),$
 (2.94) $(\bar{a}\bar{b}\bar{c} < \bar{d}\bar{b}\bar{e}) \wedge (\bar{b} \rightarrow \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a}\bar{f}\bar{c} < \bar{d}\bar{f}\bar{e}),$
 (2.95) $(\bar{a}\bar{b}\bar{c} < \bar{d}\bar{b}\bar{e}\bar{b}\bar{f}) \wedge (\bar{b} \rightarrow \bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Rightarrow (\bar{a}\bar{g}\bar{c} < \bar{d}\bar{g}\bar{e}\bar{g}\bar{f}),$
 (2.96) $(\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}) \wedge (\bar{b} \rightarrow \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a}\bar{f}\bar{c}\bar{f}\bar{d} < \bar{e}),$
 (2.97) $(\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}\bar{b}\bar{f}) \wedge (\bar{b} \rightarrow \bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Rightarrow (\bar{a}\bar{g}\bar{c}\bar{g}\bar{d} < \bar{e}\bar{g}\bar{f}),$
 (2.98) $(\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}\bar{b}\bar{f}\bar{b}\bar{g}) \wedge (\bar{b} \rightarrow \bar{h}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Rightarrow (\bar{a}\bar{h}\bar{c}\bar{h}\bar{d} < \bar{e}\bar{h}\bar{f}\bar{h}\bar{g}),$
 (2.99) $\bar{a} = \bar{a},$
 (2.100) $(\bar{a} = \bar{b}) \Rightarrow (\bar{b} = \bar{a}),$
 (2.101) $(\bar{a} = \bar{b}) \Rightarrow \neg(\bar{a} < \bar{b}),$
 (2.102) $(\bar{a} = \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} = \bar{e}) \Rightarrow (\bar{a} = \bar{b}\bar{e}\bar{d}),$
 (2.103) $(\bar{a}\bar{b}\bar{c}) \wedge (\bar{b} = \bar{d}) \Rightarrow (\bar{a}\bar{b}\bar{c} = \bar{a}\bar{d}\bar{c}),$
 (2.104) $(\bar{a} = \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} \rightarrow \bar{e}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}\}) \Rightarrow (\bar{a} = \bar{b}\bar{e}\bar{d}),$

$$\begin{aligned}
(2.105) \quad & (\bar{a} = \bar{b}\bar{c}\bar{d}\bar{c}\bar{e}) \wedge (\bar{c} \rightarrow \bar{f}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a} = \bar{b}\bar{f}\bar{d}\bar{f}\bar{e}), \\
(2.106) \quad & (\bar{a}\bar{b}\bar{c} = \bar{d}\bar{b}\bar{e}) \wedge (\bar{b} \rightarrow \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a}\bar{f}\bar{c} = \bar{d}\bar{f}\bar{e}), \\
(2.107) \quad & (\bar{a}\bar{b}\bar{c} = \bar{d}\bar{b}\bar{e}\bar{f}) \wedge (\bar{b} \rightarrow \bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Rightarrow (\bar{a}\bar{g}\bar{c} = \bar{d}\bar{g}\bar{e}\bar{g}\bar{f}), \\
(2.108) \quad & (\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} = \bar{e}\bar{b}\bar{f}\bar{b}\bar{g}) \wedge (\bar{b} \rightarrow \bar{h}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Rightarrow (\bar{a}\bar{h}\bar{c}\bar{h}\bar{d} = \bar{e}\bar{h}\bar{f}\bar{h}\bar{g}), \\
(2.109) \quad & (\bar{a} \leq \bar{b}) \Leftrightarrow ((\bar{a} < \bar{b}) \vee (\bar{a} = \bar{b})), \\
(2.110) \quad & (\bar{a} \leq \bar{b}) \wedge (\bar{b} \leq \bar{c}) \Rightarrow (\bar{a} \leq \bar{c}) \\
& \}.
\end{aligned}$$

In the following, we will deduce some numbers and equalities as examples.

$$\begin{aligned}
(A1) \quad & (a \rightarrow 1|[aba]) \Rightarrow (a \rightarrow 1) && \text{by(2.16), (2.83)} \\
(A2) \quad & \Rightarrow (a \rightarrow [aba]) && \text{by(2.16), (2.83)} \\
(A3) \quad & \Rightarrow (a \rightarrow [1ba]) && \text{by(A2), (A1), (2.82)} \\
(A4) \quad & \Rightarrow (a \rightarrow [1b1]) && \text{by(A3), (A1), (2.82)} \\
(A5) \quad & (b \rightarrow +|-) \Rightarrow (b \rightarrow -) && \text{by(2.17), (2.83)} \\
(A6) \quad & \Rightarrow (a \rightarrow [1 - 1]) && \text{by(A4), (A5), (2.82)} \\
(A7) \quad & (a < [1 + a]) \Rightarrow (1 < [1 + 1]) && \text{by(2.28), (A1), (2.94)} \\
(A8) \quad & \Rightarrow 1 && \text{by(2.88)} \\
(A9) \quad & \Rightarrow [1 + 1] && \text{by(A7), (2.88)} \\
(A10) \quad & (a < [1 + a]) \Rightarrow ([1 - 1] < [1 + [1 - 1]]) && \text{by(2.28), (A6), (2.94)} \\
(A11) \quad & \Rightarrow ([1 - 1] < [[1 - 1] + 1]) && \text{by(A10), (2.42), (2.89)} \\
(A12) \quad & \Rightarrow ([1 - 1] < 1) && \text{by(A11), (2.43), (2.89)} \\
(A13) \quad & \Rightarrow [1 - 1] && \text{by(2.88)} \\
(A14) \quad & (b \rightarrow +|-) \Rightarrow (b \rightarrow +) && \text{by(2.17), (2.83)} \\
(A15) \quad & \Rightarrow (a \rightarrow [1 + 1]) && \text{by(A4), (A14), (2.82)} \\
(A16) \quad & (a < [1 + a]) \Rightarrow ([1 + 1] < [1 + [1 + 1]]) && \text{by(2.28), (A15), (2.94)} \\
(A17) \quad & \Rightarrow ([1 - 1] < [1 + 1]) && \text{by(A12), (A7), (2.87)} \\
(A18) \quad & \Rightarrow \neg([1 + 1] = [1 - 1]) && \text{by(A17), (2.86)} \\
(A19) \quad & \Rightarrow ([[1 - 1] - -[1 + 1]] < [1 - -[1 + 1]]) && \text{by(A12), (A17), (2.30)} \\
(A20) \quad & \Rightarrow [1 - -[1 + 1]] && \text{by(2.88)} \\
(A21) \quad & \Rightarrow ([[1 - 1] - -[1 + 1]] = [[1 - -[1 + 1]] - [1 - -[1 + 1]]) && \text{by(A6), (A18), (2.56)} \\
(A22) \quad & \Rightarrow ([[1 - -[1 + 1]] - [1 - -[1 + 1]]) = [1 - 1]) && \text{by(A20), (2.41)} \\
(A23) \quad & \Rightarrow ([[1 - 1] - -[1 + 1]] = [1 - 1]) && \text{by(A21), (A22), (2.102)} \\
(A24) \quad & \Rightarrow ([1 - 1] < [1 - -[1 + 1]]) && \text{by(A19), (A23), (2.90)} \\
(A25) \quad & \Rightarrow ([1 - -[1 + 1]] < [[1 + 1] - -[1 + 1]]) && \text{by(A12), (A7),} \\
& && \text{(A17), (2.30)} \\
(A26) \quad & \Rightarrow ([[1 + 1] - -[1 + 1]] = 1) && \text{by(A18), (2.50)} \\
(A27) \quad & \Rightarrow ([1 - -[1 + 1]] < 1) && \text{by(A25), (A26), (2.89)}
\end{aligned}$$

$$(A28) \quad ([[1 - 1] + 1] < [[1 - -[1 + 1]] + 1]) \quad \text{by}(A24), (A8),$$

(2.29)

$$(A29) \quad (1 < [[1 - -[1 + 1]] + 1]) \quad \text{by}(A28), (2.43),$$

(2.90)

⋮

⋮

⋮

Then according to Definition 2.1, we can deduce from $R\{\Phi, \Psi\}$ the numbers as follows:

$$\{1, [1 + 1], [1 - 1], [1 + [1 + 1]], [1 - [1 + 1]], [1 - -[1 + [1 + 1]]], \\ [[1 + [1 + 1]] - - - [1 + 1]], [[1 + [1 + 1]] - -[1 + 1]] \cdots \}$$

$$(B1) \quad ([[1 + 1] + + + e[1 - -[1 + 1]]] = [[1 + 1] + + + [1 - -[1 + 1]]) \quad \text{by}(A7), (A24),$$

(A27), (2.80)

$$(B2) \Rightarrow ([[1 + 1] + + + [1 - -[1 + 1]]) = [[1 + 1] + + + e[1 - -[1 + 1]]) \quad \text{by}(2.100)$$

$$(B3) \quad (e \rightarrow + | + e) \Rightarrow (e \rightarrow +) \quad (2.19), (2.83)$$

$$(B4) \Rightarrow ([[1 + 1] + + + [1 - -[1 + 1]]) = [[1 + 1] + + + + [1 - -[1 + 1]]) \quad \text{by}(B2), (B3),$$

(2.104)

$$(B5) \Rightarrow ([[1 + 1] + + + + [1 - -[1 + 1]]) = [[1 + 1] + + + [1 - -[1 + 1]]) \quad \text{by}(2.100)$$

$$(B6) \quad ([[1 - 1] + [1 + 1]] < [[1 - -[1 + 1]] + [1 + 1]]) \quad \text{by}(A24), (A9),$$

(2.29)

$$(B7) \quad ([[1 - 1] + [1 + 1]] = [[1 + 1] + [1 - 1]]) \quad \text{by}(A13), (A9),$$

(2.42)

$$(B8) \quad ([[1 + 1] + [1 - 1]] = [[[1 + 1] + 1] - 1]) \quad \text{by}(2.46)$$

$$(B9) \Rightarrow ([[1 + 1] + [1 - 1]] = [[[1 + 1] - 1] + 1]) \quad \text{by}(2.44), (2.100)$$

$$(B10) \quad \Rightarrow ([[1 + 1] + [1 - 1]] = [1 + 1]) \quad \text{by}(2.43)$$

$$(B11) \quad \Rightarrow ([[1 - 1] + [1 + 1]] = [1 + 1]) \quad \text{by}(B7), (B10),$$

(2.102)

$$(B12) \quad \Rightarrow ([1 + 1] < [[1 - -[1 + 1]] + [1 + 1]]) \quad \text{by}(B6), (B11),$$

(2.90)

$$(B13) \quad \Rightarrow (1 < [[1 - -[1 + 1]] + [1 + 1]]) \quad \text{by}(A7), (B12),$$

(2.87)

$$(B14) \quad ([[1 + 1] + + + e[[1 - -[1 + 1]] + [1 + 1]]) = \\ [[1 + 1] + + e[[1 + 1] + + + e[[[1 - -[1 + 1]] + [1 + 1]] - 1]]) \quad \text{by}(A7), (B13),$$

(2.79)

$$(B15) \quad \Rightarrow ([[1 + 1] + + + + [[1 - -[1 + 1]] + [1 + 1]]) = \\ [[1 + 1] + + + [[1 + 1] + + + + [[[1 - -[1 + 1]] + [1 + 1]] - 1]]) \quad \text{by}(B14), (B3),$$

(2.107)

$$(B16) \quad ([[1 + 1] - 1] = [[1 - 1] + 1]) \quad \text{by}(A8), (2.44)$$

$$(B17) \quad \Rightarrow ([[1 + 1] - 1] = 1) \quad \text{by}(A8), (2.43)$$

$$\begin{aligned}
(B18) \quad & ((([1 - -[1 + 1]] + [1 + 1]) - 1) = [[1 - -[1 + 1]] + [[1 + 1] - 1]]) && \text{by}(A20), (A9), \\
& && (A8), (2.46) \\
(B19) \quad & \Rightarrow ((([1 - -[1 + 1]] + [1 + 1]) - 1) = [[1 - -[1 + 1]] + 1]) && \text{by}(A18), (A17), \\
& && (2.102) \\
(B20) \quad & \Rightarrow ((([1 + 1] + + + + [[1 - -[1 + 1]] + [1 + 1]]) = && \\
& \quad [[1 + 1] + + + + [[1 + 1] + + + + [[1 - -[1 + 1]] + 1]]]) && \text{by}(B15), (B19), \\
& && (2.102) \\
(B21) \quad & ((([1 + 1] + + + + e[[1 - -[1 + 1]] + 1]) = && \\
& \quad [[1 + 1] + + + + e[[1 + 1] + + + + e[[[1 - -[1 + 1]] + 1] - 1]]]) && \text{by}(A7), (A29), \\
& && (2.79) \\
(B22) \quad & \Rightarrow ((([1 + 1] + + + + [[1 - -[1 + 1]] + 1]) = && \\
& \quad [[1 + 1] + + + + [[1 + 1] + + + + [[1 - -[1 + 1]] + 1] - 1]]]) && \text{by}(B21), (B3), \\
& && (2.107) \\
(B23) \quad & ((([1 - -[1 + 1]] + 1) - 1) = [[[1 - -[1 + 1]] - 1] + 1]) && \text{by}(A20), (A8), \\
& && (2.44) \\
(B24) \quad & \Rightarrow ((([1 - -[1 + 1]] + 1) - 1) = [1 - -[1 + 1]]) && \text{by}(B23), (2.43), \\
& && (2.102) \\
(B25) \quad & \Rightarrow ((([1 + 1] + + + + [[1 - -[1 + 1]] + 1]) = && \\
& \quad [[1 + 1] + + + + [[1 + 1] + + + + [1 - -[1 + 1]]]]) && \text{by}(B22), (B24), \\
& && (2.102) \\
(B26) \quad & \Rightarrow ((([1 + 1] + + + + [[1 - -[1 + 1]] + 1]) = && \\
& \quad [[1 + 1] + + + + [[1 + 1] + + + + [1 - -[1 + 1]]]]) && \text{by}((B25), (B5), \\
& && (2.102) \\
(B27) \quad & \Rightarrow ((([1 + 1] + + + + [[1 - -[1 + 1]] + [1 + 1]]) = && \\
& \quad [[1 + 1] + + + + [[1 + 1] + + + + [[1 + 1] + + + + [1 - -[1 + 1]]]]) && \text{by}(B20), (B26), \\
& && (2.102) \\
& \vdots && \vdots && \vdots
\end{aligned}$$

Then we can deduce from $R\{\Phi, \Psi\}$ the equalities as follows:

$$\begin{aligned}
[[1 + 1] + + + + [[1 + 1] - - - [1 + 1]]] &= [[[1 + 1] - - - [1 + 1]] + + + [1 + 1]], \\
[[1 + 1] + + + + [[1 - -[1 + 1]] + 1]] &= [[1 + 1] + + + + [[1 + 1] + + + + [1 - -[1 + 1]]]], \\
&\vdots \quad \vdots \quad \vdots
\end{aligned}$$

The deduced numbers correspond to real numbers as follows:

$$\begin{aligned}
&\vdots \quad \vdots \quad \vdots, \\
[[[1 + [1 + 1]] - - - - [1 + 1]] + + + [1 + 1]] &\equiv \\
&\vdots \quad \vdots \quad \vdots,
\end{aligned}$$

$$\begin{aligned}
 [1 + [1 + 1]] &\equiv 3, \\
 &\vdots \quad \vdots \quad \vdots, \\
 [[1 + 1] + [1 - -[1 + 1]]] &\equiv \frac{5}{2}, \\
 &\vdots \quad \vdots \quad \vdots, \\
 [1 + 1] &\equiv 2, \\
 &\vdots \quad \vdots \quad \vdots, \\
 [[1 + [1 + 1]] - - - -[1 + 1]] &\equiv \\
 &\vdots \quad \vdots \quad \vdots, \\
 [[1 + [1 + 1]] /// [1 + 1]] &\equiv \log_2 3, \\
 &\vdots \quad \vdots \quad \vdots, \\
 [1 + [1 - -[1 + 1]]] &\equiv \frac{3}{2}, \\
 &\vdots \quad \vdots \quad \vdots, \\
 [[1 + 1] - - - [1 + 1]] &\equiv \sqrt[2]{2}, \\
 &\vdots \quad \vdots \quad \vdots, \\
 [[1 + 1] - - - [1 + [1 + 1]]] &\equiv \sqrt[3]{2}, \\
 &\vdots \quad \vdots \quad \vdots, \\
 1 &\equiv 1, \\
 &\vdots \quad \vdots \quad \vdots, \\
 [[1 + 1] /// [1 + [1 + 1]]] &\equiv \log_3 2, \\
 &\vdots \quad \vdots \quad \vdots, \\
 [1 - -[1 + 1]] &\equiv \frac{1}{2}, \\
 &\vdots \quad \vdots \quad \vdots, \\
 [1 - 1] &\equiv 0, \\
 &\vdots \quad \vdots \quad \vdots, \\
 [[1 - 1] - [1 - -[1 + 1]]] &\equiv -\frac{1}{2}, \\
 &\vdots \quad \vdots \quad \vdots, \\
 [[1 - 1] - 1] &\equiv -1, \\
 &\vdots \quad \vdots \quad \vdots, \\
 [[1 - 1] - [[1 + 1] - - - [1 + [1 + 1]]]] &\equiv -\sqrt[3]{2}, \\
 &\vdots \quad \vdots \quad \vdots,
 \end{aligned}$$

$$\begin{aligned}
[[1 - 1] - [[1 + 1] - - - [1 + 1]]] &\equiv -\sqrt[2]{2}, \\
&\vdots \quad \vdots \quad \vdots, \\
[[1 - 1] - [[1 + [1 + 1]] - - - - [1 + 1]]] &\equiv \\
&\vdots \quad \vdots \quad \vdots.
\end{aligned}$$

The equalities on deduced numbers correspond to addition, subtraction, multiplication, division, exponentiation operation, root-extraction operation, logarithm operation and more other operations in real number system. These deduced numbers hold the consist order relation and equivalence relation, so the logical calculus $R\{\Phi, \Psi\}$ is a consist axiom.

Since $Q\{\Phi, \Psi\}$ is dense in $R\{\Phi, \Psi\}$ (with its standard topology), every deduced number has rational numbers arbitrary close to it. So every deduced number holds only one position in number line.

Note that in the correspondence above, some deduced numbers such as $[[[1 + [1 + 1]] - - - - [1 + 1]] + + [1 + 1]]$, $[[1 + [1 + 1]] - - - - [1 + 1]]$ and $[[1 - 1] - [[1 + [1 + 1]] - - - - [1 + 1]]]$ do not correspond to any known real numbers. So the logical calculus $R\{\Phi, \Psi\}$ can deduce more real numbers than before.

In fact, the logical calculus $R\{\Phi, \Psi\}$ not only deduces more real numbers than before, but also makes its deduced numbers join in algebraical operations. So the logical calculus $R\{\Phi, \Psi\}$ intuitively and logically denote real number system.

2.2. Semantics Of The Operator Axioms. The syntax of the Operator axioms may map to many semantics, but only one semantics is shown here to explain the syntax of the Operator axioms. As shown in Figure 1, one semantics of the Operator axioms is a line. We will explain the semantics according to Figure 1 as follows.

The symbol ‘O’ in Figure 1 stand for “the origin”. The number ‘1’ maps to “a point move up unit length from the origin”. The symbol ‘>’ maps to “upon”. The symbol ‘<’ maps to “under”. The symbol ‘=’ maps to “overlap”. The numbers such as “[1 + 1]”, “[[1 + [1 + 1]] - - - - [1 + 1]]”, “[[[1 + 1] - - - [1 + [1 + 1]]]]”, “[1 - 1]” map to the points arranged in the line in Figure 1. The number “[1 - 1]” overlap “the origin O”.

Supposing that ‘ \check{a} ’, ‘ \check{b} ’, “[$\check{a} + \check{b}$]” are numbers, then “[$\check{a} + \check{b}$]” maps to “a point move up \check{b} units from the point \check{a} ”. The symbol “++” maps to “iteratively +”. The symbol “+++” maps to “iteratively ++”. And so on, the symbols “++++”, “+++++”, “++++++”, \dots can map to similar semantics.

Supposing that ‘ \check{a} ’, ‘ \check{b} ’, “[$\check{a} - \check{b}$]” are numbers, then “[$\check{a} - \check{b}$]” maps to “a point can move to the point \check{a} by the operation “[$\check{a} - \check{b}$] + \check{b}]”. Supposing that ‘ \check{a} ’, ‘ \check{b} ’, “[$\check{a} - - \check{b}$]” are numbers, then “[$\check{a} - - \check{b}$]” maps to “a point can move to the point \check{a} by the operation “[$\check{a} - - \check{b}$] + + \check{b}]”. Supposing that ‘ \check{a} ’, ‘ \check{b} ’, “[$\check{a} - - - \check{b}$]” are numbers, then “[$\check{a} - - - \check{b}$]” maps to “a point can move to the point \check{a} by the operation “[$\check{a} - - - \check{b}$] + + + \check{b}]”. And so on, the symbols “- - -”, “- - - - -”, “- - - - - - -”, \dots can map to similar semantics.

Supposing that ‘ \check{a} ’, ‘ \check{b} ’, “[\check{a}/\check{b}]” are numbers, then “[\check{a}/\check{b}]” maps to “a point can move to the point \check{a} by the operation “[$\check{b} + [\check{a}/\check{b}]$]”. Supposing that ‘ \check{a} ’, ‘ \check{b} ’, “[$\check{a}//\check{b}$]” are numbers, then “[$\check{a}//\check{b}$]” maps to “a point can move to the point \check{a} by the operation “[$\check{b} + + [\check{a}//\check{b}]$]”. Supposing that ‘ \check{a} ’, ‘ \check{b} ’, “[$\check{a}///\check{b}$]” are numbers, then “[$\check{a}///\check{b}$]” maps to “a point can move

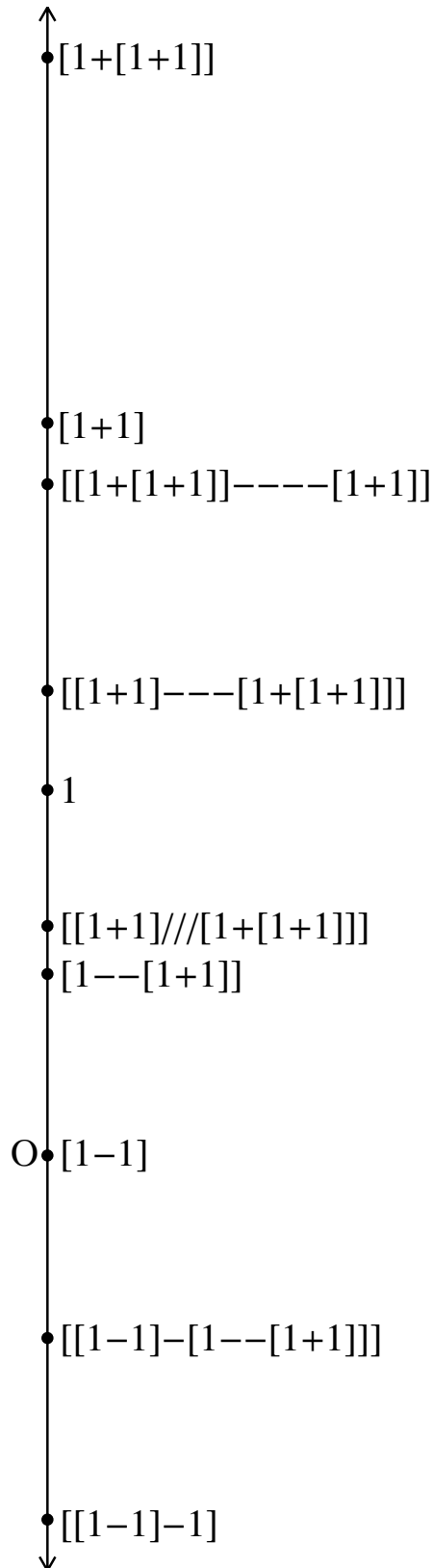


FIGURE 1. The semantics of the Operator axioms

to the point \check{a} by the operation $[\check{b}+++[\check{a}///\check{b}]]$ ". And so on, the symbols "////", "/////", "//////", \dots can map to similar semantics.

3. FUNDAMENTAL OPERATOR FUNCTIONS

On the basis of the Operator axioms, we define two fundamental operator functions in this section to study the analytic properties of the Operator axioms. It is supposed that the constant $\check{d} \in R$ with $[1 - 1] < \check{d}$. It is supposed that the constant $\check{e} \in R$ with $1 < \check{e}$.

Definition 3.1. *VE Function* is the function $\check{f} : [1, +\infty) \rightarrow R$ defined by $\check{f}(\check{x}) = [\check{x} + + + e\check{d}]$.

Definition 3.2. *EV Function* is the function $\check{f} : [[1 - 1], +\infty) \rightarrow R$ defined by $\check{f}(\check{x}) = [\check{e} + + + e\check{x}]$.

Definition 3.3. *Fundamental operator functions* are VE Function and EV Function.

Theorem 3.4. *The VE Function $\check{f}(\check{x}) = [\check{x} + + + e\check{d}]$ is continuous, unbounded and strictly increasing.*

Proof. According to (2.19), the symbol ‘ e ’ represents some successive ‘+’—“+...+”. According to (2.20), the symbol ‘ f ’ represents some successive ‘-’—“-...-”. According to (2.21), the symbol ‘ g ’ represents some successive ‘/’—“/.../”.

- (A1) Supposing that $\check{x}_1, \check{x}_2 \in [1, +\infty)$ with $\check{x}_1 < \check{x}_2$.
- (A2) $\Rightarrow (1 \leq \check{x}_1) \wedge (\check{x}_1 < \check{x}_2)$
- (A3) $[1 - 1] < \check{d}$ by (Premise)
- (A4) $\Rightarrow [\check{x}_1 + + + e\check{d}] < [\check{x}_2 + + + e\check{d}]$ by (A2),(A3),(2.35)
- (A5) $\check{f}(\check{x}_1) = [\check{x}_1 + + + e\check{d}]$ by (Premise)
- (A6) $\Rightarrow \check{f}(\check{x}_1) < [\check{x}_2 + + + e\check{d}]$ by (A4),(A5),(2.90)
- (A7) $\check{f}(\check{x}_2) = [\check{x}_2 + + + e\check{d}]$ by (Premise)
- (A8) $\Rightarrow \check{f}(\check{x}_1) < \check{f}(\check{x}_2)$ by (A6),(A7),(2.89)

(A1)~(A8) derive that $\check{f}(\check{x})$ is strictly increasing.

For any number $\check{x}_0 \in [1, +\infty)$ and any number $[1 - 1] < \varepsilon$, we can always construct δ_0 as follows:

- (B1) $[1 + + + e\check{d}] \leq [\check{x}_0 + + + e\check{d}]$ by (2.35),(2.99)
- (B2) $\Rightarrow 1 \leq [\check{x}_0 + + + e\check{d}]$ by (2.71),(2.90)
- (B3) $\Rightarrow [1 - 1] \leq [[\check{x}_0 + + + e\check{d}] - 1]$ by (2.29)
- (B4) $[1 - 1] < \varepsilon$ by (Premise)
- (B5) $\Rightarrow [[1 - 1] - -[1 + 1]] < [\varepsilon - -[1 + 1]]$ by (2.30)
- (B6) $\Rightarrow [[1 - -[1 + 1]] - [1 - -[1 + 1]]] < [\varepsilon - -[1 + 1]]$ by (2.56), (2.90)
- (B7) $\Rightarrow [1 - 1] < [\varepsilon - -[1 + 1]]$ by (2.41), (2.90)
- (B8) $\Rightarrow [[1 - 1] + [\varepsilon - -[1 + 1]]] < [[\varepsilon - -[1 + 1]] + [\varepsilon - -[1 + 1]]]$ by (2.29)
- (B9) $\Rightarrow [[1 - 1] + [\varepsilon - -[1 + 1]]] < [[\varepsilon + \varepsilon] - -[1 + 1]]$ by (2.56), (2.89)

$$\begin{aligned}
 (B10) & \Rightarrow [[\varepsilon - -[1 + 1]] + [1 - 1]] < [[\varepsilon + \varepsilon] - -[1 + 1]] && \text{by (2.42), (2.90)} \\
 (B11) & \Rightarrow [[[\varepsilon - -[1 + 1]] + 1] - 1] < [[\varepsilon + \varepsilon] - -[1 + 1]] && \text{by (2.46), (2.90)} \\
 (B12) & \Rightarrow [[[\varepsilon - -[1 + 1]] - 1] + 1] < [[\varepsilon + \varepsilon] - -[1 + 1]] && \text{by (2.44), (2.90)} \\
 (B13) & \Rightarrow [\varepsilon - -[1 + 1]] < [[\varepsilon + \varepsilon] - -[1 + 1]] && \text{by (2.43), (2.90)} \\
 (B14) & \Rightarrow [\varepsilon - -[1 + 1]] < [[\varepsilon + +[1 + 1]] - -[1 + 1]] && \text{by (2.53),(2.49),} \\
 & & & \text{(2.89)} \\
 (B15) & \Rightarrow [\varepsilon - -[1 + 1]] < [\varepsilon + +[[1 + 1] - -[1 + 1]]] && \text{by (2.57),(2.89)} \\
 (B16) & \Rightarrow [\varepsilon - -[1 + 1]] < [\varepsilon + +1] && \text{by (2.50),(2.89)} \\
 (B17) & \Rightarrow [\varepsilon - -[1 + 1]] < \varepsilon && \text{by (2.49),(2.89)} \\
 (B18) & \delta_0 = [\varepsilon - -[1 + 1]]
 \end{aligned}$$

We construct δ according to $[[\check{x}_0 + + + e\check{d}] - \delta_0]$.

$$(1) \quad 1 \leq [[\check{x}_0 + + + e\check{d}] - \delta_0].$$

We construct δ as follows:

$$\begin{aligned}
 (C1) \quad \delta_1 &= [\check{x}_0 - [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}]] \\
 (C2) \quad \delta_2 &= [[[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0]] \\
 (C3) & \quad \delta = \min \{ \delta_1, \delta_2 \}
 \end{aligned}$$

$$\begin{aligned}
 (D1) & \quad \delta_0 = [\varepsilon - -[1 + 1]] && \text{by (B18)} \\
 (D2) & \quad \Rightarrow [1 - 1] < \delta_0 && \text{by (B7),(D1),} \\
 & & & \text{(2.89)} \\
 (D3) & \quad [\varepsilon - -[1 + 1]] < \varepsilon && \text{by (B17)} \\
 (D4) & \quad \Rightarrow \delta_0 < \varepsilon && \text{by (D3),(D1),} \\
 & & & \text{(2.90)} \\
 (D5) & \quad \delta = \min \{ \delta_1, \delta_2 \} && \text{by (C3)} \\
 (D6) & \quad \Rightarrow \delta \leq \delta_1 \\
 (D7) & \quad \Rightarrow \delta \leq \delta_2 \\
 (D8) & \quad \delta_1 = [\check{x}_0 - [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}]] && \text{by (C1)} \\
 (D9) & \quad \Rightarrow [\check{x}_0 - \delta_1] = [\check{x}_0 - [\check{x}_0 - [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}]]] && \text{by (2.103)} \\
 (D10) & \quad \Rightarrow [\check{x}_0 - \delta_1] = [[\check{x}_0 - \check{x}_0] + [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}]] && \text{by (2.48),(2.102)} \\
 (D11) & \quad \Rightarrow [\check{x}_0 - \delta_1] = [[[[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] + [\check{x}_0 - \check{x}_0]]] && \text{by (2.42),(2.102)} \\
 (D12) & \quad \Rightarrow [\check{x}_0 - \delta_1] = [[[[[[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] + \check{x}_0] - \check{x}_0]]] && \text{by (2.46),(2.102)} \\
 (D13) & \quad \Rightarrow [\check{x}_0 - \delta_1] = [[[[[[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] - \check{x}_0] + \check{x}_0]]] && \text{by (2.44),(2.102)} \\
 (D14) & \quad \Rightarrow [\check{x}_0 - \delta_1] = [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] && \text{by (2.43),(2.102)} \\
 (D15) & \quad 1 \leq [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] && \text{by (Premise),(2.32),} \\
 & & & \text{(2.72)} \\
 (D16) & \quad \Rightarrow 1 \leq [\check{x}_0 - \delta_1] && \text{by (D14),(D15),} \\
 & & & \text{(2.89)}
 \end{aligned}$$

$$\begin{aligned}
(D17) \quad & \Rightarrow [\check{x}_0 - \delta_1] \leq [\check{x}_0 - \delta] && \text{by (D6),(2.29)} \\
(D18) \quad & \Rightarrow [[\check{x}_0 - \delta_1] + + + h\check{d}] \leq [[\check{x}_0 - \delta] + + + h\check{d}] && \text{by (D17),(2.35),} \\
& && (2.103) \\
(D19) \quad & \Rightarrow [[[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] + + + h\check{d}] \leq && \\
& \quad \quad \quad [[\check{x}_0 - \delta] + + + h\check{d}] && \text{by (D14),(2.90)} \\
(D20) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] - \delta_0] \leq [[\check{x}_0 - \delta] + + + h\check{d}] && \text{by (2.75),(2.90)} \\
(D21) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] - \varepsilon] < [[\check{x}_0 + + + h\check{d}] - \delta_0] && \text{by (D4),(2.29)} \\
(D22) \quad & \Rightarrow [[\check{x}_0 + + + e\check{d}] - \varepsilon] < [[\check{x}_0 - \delta] + + + e\check{d}] && \text{by (D21),(D20),} \\
& && (2.87) \\
(D23) \quad & \Rightarrow [\check{f}(\check{x}_0) - \varepsilon] < \check{f}([\check{x}_0 - \delta]) && \text{by (Premise)} \\
(D24) \quad & \delta_2 = [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0] && \text{by (C2)} \\
(D25) \quad & \Rightarrow [\check{x}_0 + \delta_2] = [\check{x}_0 + [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0]] && \text{by (2.103)} \\
(D26) \quad & \Rightarrow [\check{x}_0 + \delta_2] = [[[[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0] + \check{x}_0]] && \text{by (2.42),(2.102)} \\
(D27) \quad & \Rightarrow [\check{x}_0 + \delta_2] = [[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] && \text{by (2.43),(2.102)} \\
(D28) \quad & [1 - 1] < \delta_0 && \text{by (D2)} \\
(D29) \quad & \Rightarrow [[1 - 1] + 1] < [\delta_0 + 1] && \text{by (D28),(2.29)} \\
(D30) \quad & \Rightarrow 1 < [\delta_0 + 1] && \text{by (2.43),(2.90)} \\
(D31) \quad & \Rightarrow 1 < [1 + \delta_0] && \text{by (2.42),(2.89)} \\
(D32) \quad & 1 \leq [\check{x}_0 + + + h\check{d}] && \text{by (B2)} \\
(D33) \quad & \Rightarrow [1 + \delta_0] \leq [[\check{x}_0 + + + h\check{d}] + \delta_0] && \text{by (2.29)} \\
(D34) \quad & \Rightarrow 1 < [[\check{x}_0 + + + h\check{d}] + \delta_0] && \text{by (D31),(D33),} \\
& && (2.87) \\
(D35) \quad & \Rightarrow 1 < [[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] && \text{by (2.32)} \\
(D36) \quad & \Rightarrow [[\check{x}_0 + \delta_2] + + + h\check{d}] = && \\
& \quad \quad \quad [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] + + + h\check{d}] && \text{by (D27),(D35),} \\
& && (2.103) \\
(D37) \quad & \Rightarrow [[\check{x}_0 + \delta_2] + + + h\check{d}] = [[\check{x}_0 + + + h\check{d}] + \delta_0] && \text{by (2.75),(2.102)} \\
(D38) \quad & \delta_0 < \varepsilon && \text{by (D4)} \\
(D39) \quad & \Rightarrow [\delta_0 + [\check{x}_0 + + + h\check{d}]] < [\varepsilon + [\check{x}_0 + + + h\check{d}]] && \text{by (2.29)} \\
(D40) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] + \delta_0] < [\varepsilon + [\check{x}_0 + + + h\check{d}]] && \text{by (2.42),(2.90)} \\
(D41) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] + \delta_0] < [[\check{x}_0 + + + h\check{d}] + \varepsilon] && \text{by (2.42),(2.89)} \\
(D42) \quad & \Rightarrow [[\check{x}_0 + \delta_2] + + + h\check{d}] < [[\check{x}_0 + + + h\check{d}] + \varepsilon] && \text{by (D41),(D37),} \\
& && (2.90) \\
(D43) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] - \delta_0] < [[\check{x}_0 + + + h\check{d}] - [1 - 1]] && \text{by (D2),(2.29)} \\
(D44) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] - \delta_0] < [[[\check{x}_0 + + + h\check{d}] - 1] + 1] && \text{by (2.48),(2.89)}
\end{aligned}$$

$$\begin{aligned}
 (D45) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] - \delta_0] < [\check{x}_0 + + + h\check{d}] && \text{by (2.43),(2.89)} \\
 (D46) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] - \delta_0] = \\
 & \quad [[[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] + + + h\check{d}] && \text{by (Premise),(2.75)} \\
 (D47) \quad & \Rightarrow [\check{x}_0 + + + h\check{d}] = \\
 & \quad [[[\check{x}_0 + + + h\check{d}] - - - i\check{d}] + + + h\check{d}] && \text{by (B2),(2.75)} \\
 (D48) \quad & \Rightarrow [[[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] + + + h\check{d}] < \\
 & \quad [[[\check{x}_0 + + + h\check{d}] - - - i\check{d}] + + + h\check{d}] && \text{by (D45),(D46),} \\
 & && \text{(D47),(2.89),} \\
 & && \text{(2.90)} \\
 (D49) \quad & \Rightarrow [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] < [[\check{x}_0 + + + h\check{d}] - - - i\check{d}] && \text{by (D15),(D48),} \\
 & && \text{(2.32),(2.36)} \\
 (D50) \quad & \Rightarrow [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] < \check{x}_0 && \text{by (Premise),(2.76),} \\
 & && \text{(2.89)} \\
 (D51) \quad & \Rightarrow [\check{x}_0 - \check{x}_0] < [\check{x}_0 - [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}]] && \text{by (2.29)} \\
 (D52) \quad & \Rightarrow [\check{x}_0 - \check{x}_0] < \delta_1 && \text{by (C1),(2.89)} \\
 (D53) \quad & \Rightarrow [1 - 1] < \delta_1 && \text{by (2.41),(2.90)} \\
 (D54) \quad & \Rightarrow [[1 - 1] + [\check{x}_0 + + + h\check{d}]] < [\delta_0 + [\check{x}_0 + + + h\check{d}]] && \text{by (D2),(2.29)} \\
 (D55) \quad & \Rightarrow [[1 - 1] + [\check{x}_0 + + + h\check{d}]] < [[\check{x}_0 + + + h\check{d}] + \delta_0] && \text{by (2.42),(2.89)} \\
 (D56) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] + [1 - 1]] < [[\check{x}_0 + + + h\check{d}] + \delta_0] && \text{by (2.42),(2.90)} \\
 (D57) \quad & \Rightarrow [[[\check{x}_0 + + + h\check{d}] + 1] - 1] < [[\check{x}_0 + + + h\check{d}] + \delta_0] && \text{by (2.46),(2.90)} \\
 (D58) \quad & \Rightarrow [[[\check{x}_0 + + + h\check{d}] - 1] + 1] < [[\check{x}_0 + + + h\check{d}] + \delta_0] && \text{by (2.44),(2.90)} \\
 (D59) \quad & \Rightarrow [\check{x}_0 + + + h\check{d}] < [[\check{x}_0 + + + h\check{d}] + \delta_0] && \text{by (2.43),(2.90)} \\
 (D60) \quad & \Rightarrow [\check{x}_0 + + + h\check{d}] = \\
 & \quad [[[\check{x}_0 + + + h\check{d}] - - - i\check{d}] + + + h\check{d}] && \text{by (B2),(2.75)} \\
 (D61) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] + \delta_0] = \\
 & \quad [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] + + + h\check{d}] && \text{by (D34),(2.75)} \\
 (D62) \quad & \Rightarrow [[[\check{x}_0 + + + h\check{d}] - - - i\check{d}] + + + h\check{d}] < \\
 & \quad [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] + + + h\check{d}] && \text{by (D59),(D60),} \\
 & && \text{(D61),(2.89),} \\
 & && \text{(2.90)} \\
 (D63) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] - - - i\check{d}] < [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}]] && \text{by (D35),(2.75),} \\
 & && \text{(D62),(2.36)} \\
 (D64) \quad & \Rightarrow \check{x}_0 < [[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] && \text{by (Premise),(2.76),} \\
 & && \text{(2.90)} \\
 (D65) \quad & \Rightarrow [\check{x}_0 - \check{x}_0] < [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0] && \text{by (2.29)}
 \end{aligned}$$

$$\begin{aligned}
(D66) \quad & \Rightarrow [1 - 1] < [[[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0] && \text{by (2.41),(2.90)} \\
(D67) \quad & \Rightarrow [1 - 1] < \delta_2 && \text{by (C2),(2.89)} \\
(D68) \quad & [1 - 1] < \delta && \text{by (D5),(D53),} \\
& && \text{(D67)} \\
(D69) \quad & \Rightarrow [[1 - 1] + 1] < [\delta + 1] && \text{by (2.29)} \\
(D70) \quad & \Rightarrow 1 < [\delta + 1] && \text{by (2.43),(2.90)} \\
(D71) \quad & \Rightarrow 1 < [1 + \delta] && \text{by (2.42),(2.89)} \\
(D72) \quad & 1 \leq \check{x}_0 && \text{by (Premise)} \\
(D73) \quad & \Rightarrow [1 + \delta] \leq [\check{x}_0 + \delta] && \text{by (2.29)} \\
(D74) \quad & \Rightarrow 1 < [\check{x}_0 + \delta] && \text{by (D71),(D73),} \\
& && \text{(2.87)} \\
(D75) \quad & \Rightarrow [\delta + \check{x}_0] \leq [\delta_2 + \check{x}_0] && \text{by (D7),(2.29)} \\
(D76) \quad & \Rightarrow [\check{x}_0 + \delta] \leq [\delta_2 + \check{x}_0] && \text{by (2.42),(2.90)} \\
(D77) \quad & \Rightarrow [\check{x}_0 + \delta] \leq [\check{x}_0 + \delta_2] && \text{by (2.42),(2.89)} \\
(D78) \quad & \Rightarrow [[\check{x}_0 + \delta] + + + h\check{d}] \leq [[\check{x}_0 + \delta_2] + + + h\check{d}] && \text{by (D74),(D77),} \\
& && \text{(2.35)} \\
(D79) \quad & \Rightarrow [[\check{x}_0 + \delta] + + + e\check{d}] < [[\check{x}_0 + + + e\check{d}] + \varepsilon] && \text{by (D78),(D42),} \\
& && \text{(2.87)} \\
(D80) \quad & \Rightarrow \check{f}([\check{x}_0 + \delta]) < [\check{f}(\check{x}_0) + \varepsilon] && \text{by (Premise)}
\end{aligned}$$

Since $\check{f}(\check{x})$ is strictly increasing, $\check{f}(\check{x}_0 - \delta) < \check{f}(\check{x})$ and $\check{f}(\check{x}) < \check{f}(\check{x}_0 + \delta)$ hold for all $\check{x} \in (\check{x}_0 - \delta, \check{x}_0 + \delta)$. (D23) and (D80) derive that $[\check{f}(\check{x}_0) - \varepsilon] < \check{f}(\check{x})$ and $\check{f}(\check{x}) < [\check{f}(\check{x}_0) + \varepsilon]$ hold for all $\check{x} \in (\check{x}_0 - \delta, \check{x}_0 + \delta)$.

$$(2) \quad [[\check{x}_0 + + + e\check{d}] - \delta_0] < 1.$$

We construct δ as follows:

$$(E1) \quad \delta = [[[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0]$$

$$\begin{aligned}
(F1) \quad & a \rightarrow 1|[aba] && \text{by (2.16)} \\
(F2) \quad & \Rightarrow (a \rightarrow 1) && \text{by (2.83)} \\
(F3) \quad & \Rightarrow (a \rightarrow [aba]) && \text{by (F1),(2.83)} \\
(F4) \quad & \Rightarrow (a \rightarrow [1ba]) && \text{by (F3),(F2),(2.82)} \\
(F5) \quad & \Rightarrow (a \rightarrow [1b1]) && \text{by (F3),(F2),(2.82)} \\
(F6) \quad & b \rightarrow +|- && \text{by (2.17)} \\
(F7) \quad & \Rightarrow (b \rightarrow -) && \text{by (2.83)} \\
(F8) \quad & \Rightarrow (a \rightarrow [1 - 1]) && \text{by (F5),(F7),(2.82)} \\
(F9) \quad & a < [1 + a] && \text{by (2.28)} \\
(F10) \quad & \Rightarrow 1 < [1 + 1] && \text{by (F9),(F2),(2.94)} \\
(F11) \quad & \Rightarrow 1 && \text{by (2.88)}
\end{aligned}$$

$$\begin{aligned}
 (F12) \quad & \Rightarrow [1 - 1] < [1 + [1 - 1]] && \text{by (F9),(F8),(2.94)} \\
 (F13) \quad & \Rightarrow [1 - 1] < [[1 - 1] + 1] && \text{by (2.42),(2.89)} \\
 (F14) \quad & \Rightarrow [1 - 1] < 1 && \text{by (F11),(2.43),} \\
 & && \text{(F13),(2.89)} \\
 (F15) \quad & \Rightarrow [1 - 1] < [\check{x}_0 + + + h\check{d}] && \text{by (F14),(B2),} \\
 & && \text{(2.89),(2.87)} \\
 (F16) \quad & \Rightarrow [1 - 1] && \text{by (2.88)} \\
 (F17) \quad & \Rightarrow [\check{x}_0 + + + h\check{d}] && \text{by (F15),(2.88)} \\
 (F18) \quad & \delta_0 = [\varepsilon - - [1 + 1]] && \text{by (B18)} \\
 (F19) \quad & \Rightarrow [1 - 1] < \delta_0 && \text{by (B7),(F18),} \\
 & && \text{(2.89)} \\
 (F20) \quad & \Rightarrow [[1 - 1] + [\check{x}_0 + + + h\check{d}]] < [\delta_0 + [\check{x}_0 + + + h\check{d}]] && \text{by (F19),(F17),(2.29)} \\
 (F21) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] + [1 - 1]] < [\delta_0 + [\check{x}_0 + + + h\check{d}]] && \text{by (F16),(F17),} \\
 & && \text{(2.42),(2.90)} \\
 (F22) \quad & \Rightarrow [[[\check{x}_0 + + + h\check{d}] + 1] - 1] < [\delta_0 + [\check{x}_0 + + + h\check{d}]] && \text{by (2.46),(2.90)} \\
 (F23) \quad & \Rightarrow [[[\check{x}_0 + + + h\check{d}] - 1] + 1] < [\delta_0 + [\check{x}_0 + + + h\check{d}]] && \text{by (2.44),(2.90)} \\
 (F24) \quad & \Rightarrow [\check{x}_0 + + + h\check{d}] < [\delta_0 + [\check{x}_0 + + + h\check{d}]] && \text{by (2.43),(2.90)} \\
 (F25) \quad & \Rightarrow [\check{x}_0 + + + h\check{d}] < [[\check{x}_0 + + + h\check{d}] + \delta_0] && \text{by (2.42),(2.89)} \\
 (F26) \quad & \Rightarrow 1 < [[\check{x}_0 + + + h\check{d}] + \delta_0] && \text{by (B2),(F25),} \\
 & && \text{(2.90),(2.87)} \\
 (F27) \quad & \Rightarrow [\check{x}_0 + + + h\check{d}] = [[[\check{x}_0 + + + h\check{d}] - - - i\check{d}] + + + h\check{d}] && \text{by (B2),(2.75),(2.89)} \\
 (F28) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] + \delta_0] = && \\
 & \quad [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] + + + h\check{d}] && \text{by (F26),(2.75),(2.89)} \\
 (F29) \quad & \Rightarrow [[[\check{x}_0 + + + h\check{d}] - - - i\check{d}] + + + h\check{d}] < && \\
 & \quad [[\check{x}_0 + + + h\check{d}] + \delta_0] && \text{by (F25),(F27),(2.90)} \\
 (F30) \quad & \Rightarrow [[[\check{x}_0 + + + h\check{d}] - - - i\check{d}] + + + h\check{d}] < && \\
 & \quad [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] + + + h\check{d}] && \text{by (F29),(F28),(2.89)} \\
 (F31) \quad & \Rightarrow 1 \leq [[\check{x}_0 + + + h\check{d}] - - - i\check{d}] && \text{by (B2),(2.32),(2.72)} \\
 (F32) \quad & \Rightarrow 1 < [[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] && \text{by (F26),(2.32)} \\
 (F33) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] - - - i\check{d}] < && \\
 & \quad [[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] && \text{by (F31),(F32),} \\
 & && \text{(F30),(2.36)} \\
 (F34) \quad & \Rightarrow \check{x}_0 < [[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] && \text{by (2.76),(2.90)} \\
 (F35) \quad & \Rightarrow [\check{x}_0 - \check{x}_0] < [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0] && \text{by (2.29)} \\
 (F36) \quad & \Rightarrow [1 - 1] < [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0] && \text{by (2.41),(F35),(2.90)} \\
 (F37) \quad & \Rightarrow [1 - 1] < \delta && \text{by (F36),(E1),(2.89)}
 \end{aligned}$$

$$\begin{array}{ll}
(F38) & [[\check{x}_0 + + + h\check{d}] - \delta_0] < 1 & \text{by (Premise)} \\
(F39) & \Rightarrow [[\check{x}_0 + + + h\check{d}] - [\varepsilon - -[1 + 1]]] < 1 & \text{by (B18),(2.90)} \\
(F40) & [1 - 1] < \varepsilon & \text{by (Premise)} \\
(F41) & \Rightarrow [\varepsilon - -[1 + 1]] < [\varepsilon - -1] & \text{by (F40),(F10),(F14),(2.31)} \\
(F42) & [\varepsilon - -1] = \varepsilon & \text{by (2.49)} \\
(F43) & \Rightarrow [\varepsilon - -[1 + 1]] < \varepsilon & \text{by (F41),(F42),(2.89)} \\
(F44) & \Rightarrow [[\check{x}_0 + + + h\check{d}] - \varepsilon] < & \\
& [[\check{x}_0 + + + h\check{d}] - [\varepsilon - -[1 + 1]]] & \text{by (2.29)} \\
(F45) & \Rightarrow [[\check{x}_0 + + + e\check{d}] - \varepsilon] < 1 & \text{by (F44),(F39),(2.87)} \\
(F46) & \Rightarrow [\check{f}(\check{x}_0) - \varepsilon] < 1 & \\
(F47) & \check{f}(1) = [1 + + + e\check{d}] & \\
(F48) & \Rightarrow \check{f}(1) = 1 & \text{by (2.71)} \\
(F49) & \Rightarrow 1 \leq \check{f}(\check{x}) & \text{by (A1)~(A8),(2.90)} \\
(F50) & \Rightarrow [\check{f}(\check{x}_0) - \varepsilon] < \check{f}(\check{x}) & \text{by (F46),(F49),(2.87)} \\
(F51) & \delta_0 = [\varepsilon - -[1 + 1]] & \text{by (B18)} \\
(F52) & \Rightarrow [1 - 1] < \delta_0 & \text{by (B7),(F51),} \\
& & \text{(2.89)} \\
(F53) & [\varepsilon - -[1 + 1]] < \varepsilon & \text{by (D9)} \\
(F54) & \Rightarrow \delta_0 < \varepsilon & \text{by (F53),(F51),} \\
& & \text{(2.90)} \\
(F55) & \delta = [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0] & \text{by (E1)} \\
(F56) & \Rightarrow [\check{x}_0 + \delta] = [\check{x}_0 + [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0]] & \text{by (2.103)} \\
(F57) & \Rightarrow [\check{x}_0 + \delta] = [[[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0] + \check{x}_0] & \text{by (2.42),(2.102)} \\
(F58) & \Rightarrow [\check{x}_0 + \delta] = [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] & \text{by (2.43),(2.102)} \\
(F59) & [1 - 1] < \delta_0 & \text{by (F52)} \\
(F60) & \Rightarrow [[1 - 1] + 1] < [\delta_0 + 1] & \text{by (F59),(2.29)} \\
(F61) & \Rightarrow 1 < [\delta_0 + 1] & \text{by (2.43),(2.90)} \\
(F62) & \Rightarrow 1 < [1 + \delta_0] & \text{by (2.42),(2.89)} \\
(F63) & 1 \leq [\check{x}_0 + + + h\check{d}] & \text{by (B2)} \\
(F64) & \Rightarrow [1 + \delta_0] \leq [[\check{x}_0 + + + h\check{d}] + \delta_0] & \text{by (2.29)} \\
(F65) & \Rightarrow 1 < [[\check{x}_0 + + + h\check{d}] + \delta_0] & \text{by (F62),(F64),} \\
& & \text{(2.110)} \\
(F66) & \Rightarrow 1 < [[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] & \text{by (2.32)} \\
(F67) & \Rightarrow [[\check{x}_0 + \delta] + + + h\check{d}] = & \\
& [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] + + + h\check{d}] & \text{by (F58),(F66),} \\
& & \text{(2.103)}
\end{array}$$

$$\begin{aligned}
 (F68) \quad & \Rightarrow [[\check{x}_0 + \delta] + + + h\check{d}] = [[\check{x}_0 + + + h\check{d}] + \delta_0] && \text{by (2.75),(2.102)} \\
 (F69) \quad & \delta_0 < \varepsilon && \text{by (F54)} \\
 (F70) \quad & \Rightarrow [\delta_0 + [\check{x}_0 + + + h\check{d}]] < [\varepsilon + [\check{x}_0 + + + h\check{d}]] && \text{by (2.29)} \\
 (F71) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] + \delta_0] < [\varepsilon + [\check{x}_0 + + + h\check{d}]] && \text{by (2.42),(2.90)} \\
 (F72) \quad & \Rightarrow [[\check{x}_0 + + + h\check{d}] + \delta_0] < [[\check{x}_0 + + + h\check{d}] + \varepsilon] && \text{by (2.42),(2.89)} \\
 (F73) \quad & \Rightarrow [[\check{x}_0 + \delta] + + + e\check{d}] < [[\check{x}_0 + + + e\check{d}] + \varepsilon] && \text{by (F72),(F68),} \\
 & & & (2.90) \\
 (F74) \quad & \Rightarrow \check{f}([\check{x}_0 + \delta]) < [\check{f}(\check{x}_0) + \varepsilon] && \text{by (Premise)}
 \end{aligned}$$

Since $\check{f}(\check{x})$ is strictly increasing, $\check{f}(1) \leq \check{f}(\check{x})$ and $\check{f}(\check{x}) < \check{f}(\check{x}_0 + \delta)$ hold for all $\check{x} \in (\check{x}_0 - \delta, \check{x}_0 + \delta)$. (F17) and (F41) derive that $[\check{f}(\check{x}_0) - \varepsilon] < \check{f}(\check{x})$ and $\check{f}(\check{x}) < [\check{f}(\check{x}_0) + \varepsilon]$ hold for all $\check{x} \in (\check{x}_0 - \delta, \check{x}_0 + \delta)$.

Items 1~2 derive that $\check{f}(\check{x})$ is continuous.

(F16) derives that $1 \leq \check{f}(\check{x})$ holds on the domain $[1, +\infty)$. For any number $1 \leq \varepsilon$, (2.32) and (2.72) always derive that $[[\varepsilon - - - i\check{d}] + 1] \in [1, +\infty)$. So there always exists $\check{x}_0 = [[\varepsilon - - - i\check{d}] + 1]$ on the domain $[1, +\infty)$.

$$\begin{aligned}
 (G1) \quad & [1 - 1] < 1 && \text{by (F14)} \\
 (G2) \quad & \Rightarrow [[1 - 1] + [\varepsilon - - - i\check{d}]] < [1 + [\varepsilon - - - i\check{d}]] && \text{by (2.29)} \\
 (G3) \quad & \Rightarrow [[\varepsilon - - - i\check{d}] + [1 - 1]] < [1 + [\varepsilon - - - i\check{d}]] && \text{by (2.42),(2.90)} \\
 (G4) \quad & \Rightarrow [[[\varepsilon - - - i\check{d}] + 1] - 1] < [1 + [\varepsilon - - - i\check{d}]] && \text{by (2.46),(2.90)} \\
 (G5) \quad & \Rightarrow [[[\varepsilon - - - i\check{d}] - 1] + 1] < [1 + [\varepsilon - - - i\check{d}]] && \text{by (2.44),(2.90)} \\
 (G6) \quad & \Rightarrow [\varepsilon - - - i\check{d}] < [1 + [\varepsilon - - - i\check{d}]] && \text{by (2.43),(2.90)} \\
 (G7) \quad & \Rightarrow [\varepsilon - - - i\check{d}] < [[\varepsilon - - - i\check{d}] + 1] && \text{by (2.42),(2.89)} \\
 (G8) \quad & \check{x}_0 = [[\varepsilon - - - i\check{d}] + 1] && \text{by (Premise)} \\
 (G9) \quad & \Rightarrow [\varepsilon - - - i\check{d}] < \check{x}_0 && \text{by (G7),(G8),(2.89)} \\
 (G10) \quad & \Rightarrow [[\varepsilon - - - i\check{d}] + + + h\check{d}] < [\check{x}_0 + + + h\check{d}] && \text{by (2.32),(2.72),(2.35)} \\
 (G11) \quad & \Rightarrow \varepsilon < [\check{x}_0 + + + e\check{d}] && \text{by (Premise),(2.75),(2.90)} \\
 (G12) \quad & \Rightarrow \varepsilon < \check{f}(\check{x}_0) && \text{by (Premise)}
 \end{aligned}$$

(G1)~(G12) derive that $\check{f}(\check{x})$ is unbounded. □

Theorem 3.5. *The EV Function $\check{f}(\check{x}) = [\check{e} + + + e\check{x}]$ is continuous, unbounded and strictly increasing.*

Proof. According to (2.19), the symbol ‘e’ represents some successive ‘+’—“+...+”. According to (2.20), the symbol ‘f’ represents some successive ‘-’—“-...-”. According

- (D1) $\delta_0 = [\varepsilon - -[1 + 1]]$ by (B18)
- (D2) $\Rightarrow [1 - 1] < \delta_0$ by (B7),(D1),
(2.89)
- (D3) $[\varepsilon - -[1 + 1]] < \varepsilon$ by (B17)
- (D4) $\Rightarrow \delta_0 < \varepsilon$ by (D3),(D1),
(2.90)
- (D5) $\delta = \min \{\delta_1, \delta_2\}$ by (C3)
- (D6) $\Rightarrow \delta \leq \delta_1$
- (D7) $\Rightarrow \delta \leq \delta_2$
- (D8) $\delta_1 = [\check{x}_0 - [[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}]]$ by (C1)
- (D9) $\Rightarrow [\check{x}_0 - \delta_1] = [\check{x}_0 - [\check{x}_0 - [[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}]]]$ by (2.103)
- (D10) $\Rightarrow [\check{x}_0 - \delta_1] = [[\check{x}_0 - \check{x}_0] + [[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}]]$ by (2.48),(2.102)
- (D11) $\Rightarrow [\check{x}_0 - \delta_1] = [[[[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}] + [\check{x}_0 - \check{x}_0]]]$ by (2.42),(2.102)
- (D12) $\Rightarrow [\check{x}_0 - \delta_1] = [[[[[[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}] + \check{x}_0] - \check{x}_0]]]$ by (2.46),(2.102)
- (D13) $\Rightarrow [\check{x}_0 - \delta_1] = [[[[[[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}] - \check{x}_0] + \check{x}_0]]]$ by (2.44),(2.102)
- (D14) $\Rightarrow [\check{x}_0 - \delta_1] = [[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}]$ by (2.43),(2.102)
- (D15) $[1 - 1] \leq [[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}]$ by (Premise),(2.34)
- (D16) $\Rightarrow [1 - 1] < [\check{x}_0 - \delta_1]$ by (D14),(D15),
(2.89)
- (D17) $\Rightarrow [\check{x}_0 - \delta_1] \leq [\check{x}_0 - \delta]$ by (D6),(2.29)
- (D18) $\Rightarrow [\check{e} + + + h[\check{x}_0 - \delta_1]] \leq [\check{e} + + + h[\check{x}_0 - \delta]]$ by (D17),(2.37),
(2.103)
- (D19) $\Rightarrow [\check{e} + + + h[[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}]] \leq$
 $[\check{e} + + + h[\check{x}_0 - \delta]]$ by (D14),(2.90)
- (D20) $\Rightarrow [[[\check{e} + + + h\check{x}_0] - \delta_0] \leq [\check{e} + + + h[\check{x}_0 - \delta]]]$ by (2.77),(2.90)
- (D21) $\Rightarrow [[[\check{e} + + + h\check{x}_0] - \varepsilon] < [[[\check{e} + + + h\check{x}_0] - \delta_0]]]$ by (D4),(2.29)
- (D22) $\Rightarrow [[[\check{e} + + + e\check{x}_0] - \varepsilon] < [\check{e} + + + e[\check{x}_0 - \delta]]]$ by (D21),(D20),
(2.87)
- (D23) $\Rightarrow [f(\check{x}_0) - \varepsilon] < f([\check{x}_0 - \delta])$ by (Premise)
- (D24) $\delta_2 = [[[[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{h}] - \check{x}_0]]$ by (C2)
- (D25) $\Rightarrow [\check{x}_0 + \delta_2] = [\check{x}_0 + [[[[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{h}] - \check{x}_0]]]$ by (2.103)
- (D26) $\Rightarrow [\check{x}_0 + \delta_2] = [[[[[[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{h}] - \check{x}_0] + \check{x}_0]]]$ by (2.42),(2.102)
- (D27) $\Rightarrow [\check{x}_0 + \delta_2] = [[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{h}]$ by (2.43),(2.102)
- (D28) $[1 - 1] < \delta_0$ by (D2)
- (D29) $\Rightarrow [[1 - 1] + 1] < [\delta_0 + 1]$ by (D28),(2.29)
- (D30) $\Rightarrow 1 < [\delta_0 + 1]$ by (2.43),(2.90)

$$\begin{aligned}
(D31) \quad & \Rightarrow 1 < [1 + \delta_0] && \text{by (2.42),(2.89)} \\
(D32) \quad & 1 \leq [\check{e} + + + h\check{x}_0] && \text{by (B2)} \\
(D33) \quad & \Rightarrow [1 + \delta_0] \leq [[\check{e} + + + h\check{x}_0] + \delta_0] && \text{by (2.29)} \\
(D34) \quad & \Rightarrow 1 < [[\check{e} + + + h\check{x}_0] + \delta_0] && \text{by (D31),(D33),} \\
& && (2.87) \\
(D35) \quad & \Rightarrow [1 - 1] < [[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{h}] && \text{by (2.33)} \\
(D36) \quad & \Rightarrow [\check{e} + + + h[\check{x}_0 + \delta_2]] = && \\
& [\check{e} + + + h[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{h}] && \text{by (D27),(D35),} \\
& && (2.103) \\
(D37) \quad & \Rightarrow [\check{e} + + + h[\check{x}_0 + \delta_2]] = [[\check{e} + + + h\check{x}_0] + \delta_0] && \text{by (2.77),(2.102)} \\
(D38) \quad & \delta_0 < \varepsilon && \text{by (D4)} \\
(D39) \quad & \Rightarrow [\delta_0 + [\check{e} + + + h\check{x}_0]] < [\varepsilon + [\check{e} + + + h\check{x}_0]] && \text{by (2.29)} \\
(D40) \quad & \Rightarrow [[\check{e} + + + h\check{x}_0] + \delta_0] < [\varepsilon + [\check{e} + + + h\check{x}_0]] && \text{by (2.42),(2.90)} \\
(D41) \quad & \Rightarrow [[\check{e} + + + h\check{x}_0] + \delta_0] < [[\check{e} + + + h\check{x}_0] + \varepsilon] && \text{by (2.42),(2.89)} \\
(D42) \quad & \Rightarrow [\check{e} + + + h[\check{x}_0 + \delta_2]] < [[\check{e} + + + h\check{x}_0] + \varepsilon] && \text{by (D41),(D37),} \\
& && (2.90) \\
(D43) \quad & \Rightarrow [[\check{e} + + + h\check{x}_0] - \delta_0] < [[\check{e} + + + h\check{x}_0] - [1 - 1]] && \text{by (D2),(2.29)} \\
(D44) \quad & \Rightarrow [[\check{e} + + + h\check{x}_0] - \delta_0] < [[[\check{e} + + + h\check{x}_0] - 1] + 1] && \text{by (2.48),(2.89)} \\
(D45) \quad & \Rightarrow [[\check{e} + + + h\check{x}_0] - \delta_0] < [\check{e} + + + h\check{x}_0] && \text{by (2.43),(2.89)} \\
(D46) \quad & \Rightarrow [[\check{e} + + + h\check{x}_0] - \delta_0] = && \\
& [\check{e} + + + h[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}] && \text{by (Premise),(2.77)} \\
(D47) \quad & \Rightarrow [\check{e} + + + h\check{x}_0] = && \\
& [\check{e} + + + h[[\check{e} + + + h\check{x}_0]///j\check{h}]] && \text{by (B2),(2.77)} \\
(D48) \quad & \Rightarrow [\check{e} + + + h[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}] < && \\
& [\check{e} + + + h[[\check{e} + + + h\check{x}_0]///j\check{h}]] && \text{by (D45),(D46),} \\
& && (D47),(2.89), \\
& && (2.90) \\
(D49) \quad & \Rightarrow [[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}] < [[\check{e} + + + h\check{x}_0]///j\check{h}] && \text{by (D15),(D48),} \\
& && (2.33),(2.38) \\
(D50) \quad & \Rightarrow [[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}] < \check{x}_0 && \text{by (Premise),(2.78),} \\
& && (2.89) \\
(D51) \quad & \Rightarrow [\check{x}_0 - \check{x}_0] < [\check{x}_0 - [[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}]] && \text{by (2.29)} \\
(D52) \quad & \Rightarrow [\check{x}_0 - \check{x}_0] < \delta_1 && \text{by (C1),(2.89)} \\
(D53) \quad & \Rightarrow [1 - 1] < \delta_1 && \text{by (2.41),(2.90)} \\
(D54) \quad & \Rightarrow [[1 - 1] + [\check{e} + + + h\check{x}_0]] < [\delta_0 + [\check{e} + + + h\check{x}_0]] && \text{by (D2),(2.29)} \\
(D55) \quad & \Rightarrow [[1 - 1] + [\check{e} + + + h\check{x}_0]] < [[\check{e} + + + h\check{x}_0] + \delta_0] && \text{by (2.42),(2.89)}
\end{aligned}$$

$$\begin{aligned}
(D56) \quad & \Rightarrow [[\check{e} + + + h\check{x}_0] + [1 - 1]] < [[\check{e} + + + h\check{x}_0] + \delta_0] && \text{by (2.42),(2.90)} \\
(D57) \quad & \Rightarrow [[[\check{e} + + + h\check{x}_0] + 1] - 1] < [[\check{e} + + + h\check{x}_0] + \delta_0] && \text{by (2.46),(2.90)} \\
(D58) \quad & \Rightarrow [[[\check{e} + + + h\check{x}_0] - 1] + 1] < [[\check{e} + + + h\check{x}_0] + \delta_0] && \text{by (2.44),(2.90)} \\
(D59) \quad & \Rightarrow [\check{e} + + + h\check{x}_0] < [[\check{e} + + + h\check{x}_0] + \delta_0] && \text{by (2.43),(2.90)} \\
(D60) \quad & \Rightarrow [\check{e} + + + h\check{x}_0] = && \\
& \quad \quad \quad [\check{e} + + + h[[\check{e} + + + h\check{x}_0]///j\check{h}]] && \text{by (B2),(2.77)} \\
(D61) \quad & \Rightarrow [[\check{e} + + + h\check{x}_0] + \delta_0] = && \\
& \quad \quad \quad [\check{e} + + + h[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{h}]] && \text{by (D34),(2.77)} \\
(D62) \quad & \Rightarrow [\check{e} + + + h[[\check{e} + + + h\check{x}_0]///j\check{h}]] < && \\
& \quad \quad \quad [\check{e} + + + h[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{h}]] && \text{by (D59),(D60),} \\
& \quad \quad \quad && \text{(D61),(2.89),} \\
& \quad \quad \quad && \text{(2.90)} \\
(D63) \quad & \Rightarrow [[\check{e} + + + h\check{x}_0]///j\check{h}] < [[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{h}] && \text{by (D35),(D62),} \\
& \quad \quad \quad && \text{(2.33),(2.38)} \\
(D64) \quad & \Rightarrow \check{x}_0 < [[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{h}] && \text{by (Premise),(2.78),} \\
& \quad \quad \quad && \text{(2.90)} \\
(D65) \quad & \Rightarrow [\check{x}_0 - \check{x}_0] < [[[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{h}] - \check{x}_0] && \text{by (2.29)} \\
(D66) \quad & \Rightarrow [1 - 1] < [[[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{h}] - \check{x}_0] && \text{by (2.41),(2.90)} \\
(D67) \quad & \Rightarrow [1 - 1] < \delta_2 && \text{by (C2),(2.89)} \\
(D68) \quad & \Rightarrow [1 - 1] < \delta && \text{by (D5),(D53),} \\
& \quad \quad \quad && \text{(D67)} \\
(D69) \quad & \Rightarrow [1 - 1] \leq \check{x}_0 && \text{by (Premise)} \\
(D70) \quad & \Rightarrow [[1 - 1] + \delta] \leq [\check{x}_0 + \delta] && \text{by (2.29)} \\
(D71) \quad & \Rightarrow [\delta + [1 - 1]] \leq [\check{x}_0 + \delta] && \text{by (2.42),(2.90)} \\
(D72) \quad & \Rightarrow [[\delta + 1] - 1] \leq [\check{x}_0 + \delta] && \text{by (2.46),(2.90)} \\
(D73) \quad & \Rightarrow [[\delta - 1] + 1] \leq [\check{x}_0 + \delta] && \text{by (2.44),(2.90)} \\
(D74) \quad & \Rightarrow \delta \leq [\check{x}_0 + \delta] && \text{by (2.43),(2.90)} \\
(D75) \quad & \Rightarrow [1 - 1] < [\check{x}_0 + \delta] && \text{by (D68),(D74),} \\
& \quad \quad \quad && \text{(2.87)} \\
(D76) \quad & \Rightarrow [\delta + \check{x}_0] \leq [\delta_2 + \check{x}_0] && \text{by (D7),(2.29)} \\
(D77) \quad & \Rightarrow [\check{x}_0 + \delta] \leq [\delta_2 + \check{x}_0] && \text{by (2.42),(2.90)} \\
(D78) \quad & \Rightarrow [\check{x}_0 + \delta] \leq [\check{x}_0 + \delta_2] && \text{by (2.42),(2.89)} \\
(D79) \quad & \Rightarrow [\check{e} + + + h[\check{x}_0 + \delta]] \leq [\check{e} + + + h[\check{x}_0 + \delta_2]] && \text{by (D75),(D78),} \\
& \quad \quad \quad && \text{(2.37)} \\
(D80) \quad & \Rightarrow [\check{e} + + + e[\check{x}_0 + \delta]] < [[\check{e} + + + e\check{x}_0] + \varepsilon] && \text{by (D79),(D42),} \\
& \quad \quad \quad && \text{(2.87)}
\end{aligned}$$

$$(D81) \quad \Rightarrow \check{f}([\check{x}_0 + \delta]) < [\check{f}(\check{x}_0) + \varepsilon] \quad \text{by (Premise)}$$

Since $\check{f}(\check{x})$ is strictly increasing, $\check{f}(\check{x}_0 - \delta) < \check{f}(\check{x})$ and $\check{f}(\check{x}) < \check{f}(\check{x}_0 + \delta)$ hold for all $\check{x} \in (\check{x}_0 - \delta, \check{x}_0 + \delta)$. (D23) and (D81) derive that $[\check{f}(\check{x}_0) - \varepsilon] < \check{f}(\check{x})$ and $\check{f}(\check{x}) < [\check{f}(\check{x}_0) + \varepsilon]$ hold for all $\check{x} \in (\check{x}_0 - \delta, \check{x}_0 + \delta)$.

$$(2) \quad [[\check{e} + + + e\check{x}_0] - \delta_0] < 1.$$

We construct δ as follows:

$$(E1) \quad \delta = [[[[[\check{e} + + + h\check{x}_0] + \delta_0] // j\check{e}] - \check{x}_0]$$

$$\begin{aligned} (F1) & \quad a \rightarrow 1[aba] && \text{by (2.16)} \\ (F2) & \quad \Rightarrow (a \rightarrow 1) && \text{by (2.83)} \\ (F3) & \quad \Rightarrow (a \rightarrow [aba]) && \text{by (F1),(2.83)} \\ (F4) & \quad \Rightarrow (a \rightarrow [1ba]) && \text{by (F3),(F2),(2.82)} \\ (F5) & \quad \Rightarrow (a \rightarrow [1b1]) && \text{by (F3),(F2),(2.82)} \\ (F6) & \quad b \rightarrow +|- && \text{by (2.17)} \\ (F7) & \quad \Rightarrow (b \rightarrow -) && \text{by (2.83)} \\ (F8) & \quad \Rightarrow (a \rightarrow [1 - 1]) && \text{by (F5),(F7),(2.82)} \\ (F9) & \quad a < [1 + a] && \text{by (2.28)} \\ (F10) & \quad \Rightarrow 1 < [1 + 1] && \text{by (F9),(F2),(2.94)} \\ (F11) & \quad \Rightarrow 1 && \text{by (2.88)} \\ (F12) & \quad \Rightarrow [1 - 1] < [1 + [1 - 1]] && \text{by (F9),(F8),(2.94)} \\ (F13) & \quad \Rightarrow [1 - 1] < [[1 - 1] + 1] && \text{by (2.42),(2.89)} \\ (F14) & \quad \Rightarrow [1 - 1] < 1 && \text{by (F11),(2.43),} \\ & & & \text{(F13),(2.89)} \\ (F15) & \quad \Rightarrow [1 - 1] < [\check{e} + + + h\check{x}_0] && \text{by (F14),(B2),} \\ & & & \text{(2.89),(2.87)} \\ (F16) & \quad \Rightarrow [1 - 1] && \text{by (2.88)} \\ (F17) & \quad \Rightarrow [\check{e} + + + h\check{x}_0] && \text{by (F15),(2.88)} \\ (F18) & \quad \delta_0 = [\varepsilon - -[1 + 1]] && \text{by (B18)} \\ (F19) & \quad \Rightarrow [1 - 1] < \delta_0 && \text{by (B7),(F18),} \\ & & & \text{(2.89)} \\ (F20) & \quad \Rightarrow [[1 - 1] + [\check{e} + + + h\check{x}_0]] < [\delta_0 + [\check{e} + + + h\check{x}_0]] && \text{by (F19),(F17),(2.29)} \\ (F21) & \quad \Rightarrow [[\check{e} + + + h\check{x}_0] + [1 - 1]] < [\delta_0 + [\check{e} + + + h\check{x}_0]] && \text{by (F16),(F17),} \\ & & & \text{(2.42),(2.90)} \\ (F22) & \quad \Rightarrow [[[\check{e} + + + h\check{x}_0] + 1] - 1] < [\delta_0 + [\check{e} + + + h\check{x}_0]] && \text{by (2.46),(2.90)} \\ (F23) & \quad \Rightarrow [[[\check{e} + + + h\check{x}_0] - 1] + 1] < [\delta_0 + [\check{e} + + + h\check{x}_0]] && \text{by (2.44),(2.90)} \\ (F24) & \quad \Rightarrow [\check{e} + + + h\check{x}_0] < [\delta_0 + [\check{e} + + + h\check{x}_0]] && \text{by (2.43),(2.90)} \\ (F25) & \quad \Rightarrow [\check{e} + + + h\check{x}_0] < [[\check{e} + + + h\check{x}_0] + \delta_0] && \text{by (2.42),(2.89)} \\ (F26) & \quad \Rightarrow 1 < [[\check{e} + + + h\check{x}_0] + \delta_0] && \text{by (B2),(F25),} \end{aligned}$$

$$\begin{aligned}
 & \hspace{15em} (2.90), (2.87) \\
 (F27) \quad & \Rightarrow [\check{e} + + + h\check{x}_0] = [\check{e} + + + h[[\check{e} + + + h\check{x}_0]///j\check{e}]] \quad \text{by (B2), (2.77), (2.89)} \\
 (F28) \quad & \Rightarrow [[\check{e} + + + h\check{x}_0] + \delta_0] = \\
 & \quad [\check{e} + + + h[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{e}]] \quad \text{by (F26), (2.77), (2.89)} \\
 (F29) \quad & \Rightarrow [\check{e} + + + h[[\check{e} + + + h\check{x}_0]///j\check{e}]] < [[\check{e} + + + h\check{x}_0] + \delta_0] \quad \text{by (F25), (F27), (2.90)} \\
 (F30) \quad & \Rightarrow [\check{e} + + + h[[\check{e} + + + h\check{x}_0]///j\check{e}]] < \\
 & \quad [\check{e} + + + h[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{e}]] \quad \text{by (F29), (F28), (2.89)} \\
 (F31) \quad & \Rightarrow [1 - 1] \leq [[\check{e} + + + h\check{x}_0]///j\check{e}] \quad \text{by (B2), (2.33), (2.73)} \\
 (F32) \quad & \Rightarrow [1 - 1] < [[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{e}] \quad \text{by (F26), (2.33)} \\
 (F33) \quad & \Rightarrow [[\check{e} + + + h\check{x}_0]///j\check{e}] < \\
 & \quad [[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{e}] \quad \text{by (F31), (F32),} \\
 & \hspace{15em} (F30), (2.38) \\
 (F34) \quad & \Rightarrow \check{x}_0 < [[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{e}] \quad \text{by (2.78), (2.90)} \\
 (F35) \quad & \Rightarrow [\check{x}_0 - \check{x}_0] < [[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{e}] - \check{x}_0 \quad \text{by (2.29)} \\
 (F36) \quad & \Rightarrow [1 - 1] < [[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{e}] - \check{x}_0 \quad \text{by (2.41), (F35), (2.90)} \\
 (F37) \quad & \Rightarrow [1 - 1] < \delta \quad \text{by (F36), (E1), (2.89)} \\
 (F38) \quad & \quad [[\check{e} + + + h\check{x}_0] - \delta_0] < 1 \quad \text{by (Premise)} \\
 (F39) \quad & \Rightarrow [[\check{e} + + + h\check{x}_0] - [\varepsilon - -[1 + 1]]] < 1 \quad \text{by (B18), (2.90)} \\
 (F40) \quad & \quad [1 - 1] < \varepsilon \quad \text{by (Premise)} \\
 (F41) \quad & \Rightarrow [\varepsilon - -[1 + 1]] < [\varepsilon - -1] \quad \text{by (F40), (F10), (F14), (2.31)} \\
 (F42) \quad & \quad [\varepsilon - -1] = \varepsilon \quad \text{by (2.49)} \\
 (F43) \quad & \Rightarrow [\varepsilon - -[1 + 1]] < \varepsilon \quad \text{by (F41), (F42), (2.89)} \\
 (F44) \quad & \Rightarrow [[\check{e} + + + h\check{x}_0] - \varepsilon] < \\
 & \quad [[\check{e} + + + h\check{x}_0] - [\varepsilon - -[1 + 1]]] \quad \text{by (2.29)} \\
 (F45) \quad & \Rightarrow [[\check{e} + + + e\check{x}_0] - \varepsilon] < 1 \quad \text{by (F44), (F39), (2.87)} \\
 (F46) \quad & \Rightarrow [\check{f}(\check{x}_0) - \varepsilon] < 1 \\
 (F47) \quad & \quad \check{f}([1 - 1]) = [\check{e} + + + e[1 - 1]] \\
 (F48) \quad & \Rightarrow \check{f}([1 - 1]) = 1 \quad \text{by (2.69)} \\
 (F49) \quad & \Rightarrow 1 \leq \check{f}(\check{x}) \quad \text{by (A1)~(A8), (2.90)} \\
 (F50) \quad & \Rightarrow [\check{f}(\check{x}_0) - \varepsilon] < \check{f}(\check{x}) \quad \text{by (F46), (F49), (2.87)} \\
 (F51) \quad & \quad \delta_0 = [\varepsilon - -[1 + 1]] \quad \text{by (B18)} \\
 (F52) \quad & \Rightarrow [1 - 1] < \delta_0 \quad \text{by (B7), (F51),} \\
 & \hspace{15em} (2.89) \\
 (F53) \quad & \quad [\varepsilon - -[1 + 1]] < \varepsilon \quad \text{by (D9)} \\
 (F54) \quad & \Rightarrow \delta_0 < \varepsilon \quad \text{by (F53), (F51),} \\
 & \hspace{15em} (2.90)
 \end{aligned}$$

$$\begin{aligned}
(F55) \quad & \delta = [\![\![\![\check{e} + + + h\check{x}_0] + \delta_0] \!/\!/\! / j\check{h}] - \check{x}_0] && \text{by (E1)} \\
(F56) \quad & \Rightarrow [\check{x}_0 + \delta] = [\check{x}_0 + [\![\![\![\check{e} + + + h\check{x}_0] + \delta_0] \!/\!/\! / j\check{h}] - \check{x}_0]] && \text{by (2.103)} \\
(F57) \quad & \Rightarrow [\check{x}_0 + \delta] = [\![\![\![\![\check{e} + + + h\check{x}_0] + \delta_0] \!/\!/\! / j\check{h}] - \check{x}_0] + \check{x}_0] && \text{by (2.42),(2.102)} \\
(F58) \quad & \Rightarrow [\check{x}_0 + \delta] = [\![\check{e} + + + h\check{x}_0] + \delta_0] \!/\!/\! / j\check{h}] && \text{by (2.43),(2.102)} \\
(F59) \quad & [1 - 1] < \delta_0 && \text{by (F52)} \\
(F60) \quad & \Rightarrow [[1 - 1] + 1] < [\delta_0 + 1] && \text{by (F59),(2.29)} \\
(F61) \quad & \Rightarrow 1 < [\delta_0 + 1] && \text{by (2.43),(2.90)} \\
(F62) \quad & \Rightarrow 1 < [1 + \delta_0] && \text{by (2.42),(2.89)} \\
(F63) \quad & 1 \leq [\check{e} + + + h\check{x}_0] && \text{by (B2)} \\
(F64) \quad & \Rightarrow [1 + \delta_0] \leq [\![\check{e} + + + h\check{x}_0] + \delta_0] && \text{by (2.29)} \\
(F65) \quad & \Rightarrow 1 < [\![\check{e} + + + h\check{x}_0] + \delta_0] && \text{by (F62),(F64),} \\
& & & \text{(2.87)} \\
(F66) \quad & \Rightarrow [1 - 1] < [\![\check{e} + + + h\check{x}_0] + \delta_0] \!/\!/\! / j\check{h}] && \text{by (2.33)} \\
(F67) \quad & \Rightarrow [\check{e} + + + h[\check{x}_0 + \delta]] = && \\
& \quad [\check{e} + + + h[\![\check{e} + + + h\check{x}_0] + \delta_0] \!/\!/\! / j\check{h}]] && \text{by (F58),(F66),} \\
& & & \text{(2.103)} \\
(F68) \quad & \Rightarrow [\check{e} + + + h[\check{x}_0 + \delta]] = [\![\check{e} + + + h\check{x}_0] + \delta_0] && \text{by (2.77),(2.102)} \\
(F69) \quad & \delta_0 < \varepsilon && \text{by (F54)} \\
(F70) \quad & \Rightarrow [\delta_0 + [\check{e} + + + h\check{x}_0]] < [\varepsilon + [\check{e} + + + h\check{x}_0]] && \text{by (2.29)} \\
(F71) \quad & \Rightarrow [\![\check{e} + + + h\check{x}_0] + \delta_0] < [\varepsilon + [\check{e} + + + h\check{x}_0]] && \text{by (2.42),(2.90)} \\
(F72) \quad & \Rightarrow [\![\check{e} + + + h\check{x}_0] + \delta_0] < [\![\check{e} + + + h\check{x}_0] + \varepsilon] && \text{by (2.42),(2.89)} \\
(F73) \quad & \Rightarrow [\check{e} + + + e[\check{x}_0 + \delta]] < [\![\check{e} + + + e\check{x}_0] + \varepsilon] && \text{by (F72),(F68),} \\
& & & \text{(2.90)} \\
(F74) \quad & \Rightarrow \check{f}([\check{x}_0 + \delta]) < [\check{f}(\check{x}_0) + \varepsilon] && \text{by (Premise)}
\end{aligned}$$

Since $\check{f}(\check{x})$ is strictly increasing, $\check{f}(1) \leq \check{f}(\check{x})$ and $\check{f}(\check{x}) < \check{f}(\check{x}_0 + \delta)$ hold for all $\check{x} \in (\check{x}_0 - \delta, \check{x}_0 + \delta)$. (F17) and (F41) derive that $[\check{f}(\check{x}_0) - \varepsilon] < \check{f}(\check{x})$ and $\check{f}(\check{x}) < [\check{f}(\check{x}_0) + \varepsilon]$ hold for all $\check{x} \in (\check{x}_0 - \delta, \check{x}_0 + \delta)$.

Items 1~2 derive that $\check{f}(\check{x})$ is continuous.

(F16) derives that $1 \leq \check{f}(\check{x})$ holds on the domain $[1, +\infty)$. For any number $1 \leq \varepsilon$, (2.33) and (2.73) always derive that $[\varepsilon \!/\!/\! / j\check{e}] \in [[1 - 1], +\infty)$. So there always exists $\check{x}_0 = [[\varepsilon \!/\!/\! / j\check{e}] + 1]$ on the domain $[1, +\infty)$.

$$\begin{aligned}
(G1) \quad & [1 - 1] < 1 && \text{by (F14)} \\
(G2) \quad & \Rightarrow [[1 - 1] + [\varepsilon \!/\!/\! / j\check{e}]] < [1 + [\varepsilon \!/\!/\! / j\check{e}]] && \text{by (2.29)} \\
(G3) \quad & \Rightarrow [[\varepsilon \!/\!/\! / j\check{e}] + [1 - 1]] < [1 + [\varepsilon \!/\!/\! / j\check{e}]] && \text{by (2.42),(2.90)} \\
(G4) \quad & \Rightarrow [[[\varepsilon \!/\!/\! / j\check{e}] + 1] - 1] < [1 + [\varepsilon \!/\!/\! / j\check{e}]] && \text{by (2.46),(2.90)} \\
(G5) \quad & \Rightarrow [[[\varepsilon \!/\!/\! / j\check{e}] - 1] + 1] < [1 + [\varepsilon \!/\!/\! / j\check{e}]] && \text{by (2.44),(2.90)} \\
(G6) \quad & \Rightarrow [\varepsilon \!/\!/\! / j\check{e}] < [1 + [\varepsilon \!/\!/\! / j\check{e}]] && \text{by (2.43),(2.90)}
\end{aligned}$$

$$\begin{aligned}
 (G7) & \quad \Rightarrow [\varepsilon///j\check{e}] < [[\varepsilon///j\check{e}] + 1] && \text{by (2.42),(2.89)} \\
 (G8) & \quad \check{x}_0 = [[\varepsilon///j\check{e}] + 1] && \text{by (Premise)} \\
 (G9) & \quad \Rightarrow [\varepsilon///j\check{e}] < \check{x}_0 && \text{by (G7),(G8),(2.89)} \\
 (G10) & \Rightarrow [\check{e} + + + h[\varepsilon///j\check{e}]] < [\check{e} + + + h\check{x}_0] && \text{by (2.33),(2.73),(2.37)} \\
 (G11) & \quad \Rightarrow \varepsilon < [\check{e} + + + e\check{x}_0] && \text{by (Premise),(2.77),(2.90)} \\
 (G12) & \quad \Rightarrow \varepsilon < \check{f}(\check{x}_0) && \text{by (Premise)}
 \end{aligned}$$

(G1)~(G12) derive that $\check{f}(\check{x})$ is unbounded. □

After further studies with Theorem 3.4 and Theorem 3.5, we pose the conjectures as follows.

Conjecture 3.6. *For a function $\check{f} : (1, +\infty) \rightarrow R$ defined by $\check{f}(\check{x}) = [\check{x} + + + e\check{d}]$, it is smooth.*

Conjecture 3.7. *For a function $\check{f} : ([1 - 1], +\infty) \rightarrow R$ defined by $\check{f}(\check{x}) = [\check{e} + + + e\check{x}]$, it is smooth.*

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