Relevant First-Order Logic $LP^{\#}$ and Curry's Paradox Resolution

Jaykov Foukzon

Israel Institute of Technology, Department of Mathematics, Haifa, Israel

Email address:
jaykovfoukzon@list.ru

Abstract: In recent years there has been a revitalised interest in non-classical solutions to the semantic paradoxes. In this article the non-classical resolution of Curry's Paradox and Shaw-Kwei's paradox without rejection any contraction postulate is proposed.

Keywords: Curry's paradox, Shaw-Kwei's paradox, relevance logics, Łukasiewicz logic, abelian logic,

---

I. Introduction

In 1942 Haskell B. Curry presented what is now called *Curry's paradox* [1]. The paradox I have in mind can be found in a logic independently of its stand on negation. The deduction appeals to no particular principles of negation, as it is negation-free. Any deduction must use some inferential principles.

Here are the principles needed to derive the paradox.

A transitive relation of consequence: we write this by $\vdash$ and take $\vdash$ to be a relation between statements, and we require that it be transitive: if $\boldsymbol{A} \vdash \boldsymbol{B}$ and $\boldsymbol{B} \vdash \boldsymbol{C}$ then $\boldsymbol{A} \vdash \boldsymbol{C}$.

Conjunction and implication: we require that the conjunction operator $\wedge$ be a greatest lower bound with respect to $\vdash$. That is, $\boldsymbol{A} \vdash \boldsymbol{B}$ and $\boldsymbol{A} \vdash \boldsymbol{C}$ if and only if $\boldsymbol{A} \vdash \boldsymbol{B} \wedge \boldsymbol{C}$.

Furthermore, we require that there be a residual for conjunction: a connective $\rightarrow$ such that $\boldsymbol{A} \wedge \boldsymbol{B} \vdash \boldsymbol{C}$ if and only if $\mathbf{A} \vdash \mathbf{B} \rightarrow \mathbf{C}$.

Unrestricted Modus Ponens rule :

$$\boldsymbol{A}, \boldsymbol{A} \rightarrow \boldsymbol{B} \vdash \boldsymbol{B}. \tag{1.1}$$

Unrestricted Modus Tollens rule:

$$\boldsymbol{P} \rightarrow \boldsymbol{Q}, \neg \boldsymbol{Q} \vdash \neg \boldsymbol{P}. \tag{1.2}$$

A paradox generator: we need only a very weak paradox generator. We take the $\boldsymbol{T}$ scheme in the following enthymematic form: $\boldsymbol{T}[\boldsymbol{A}] \wedge \boldsymbol{C} \vdash \boldsymbol{A}$; $\boldsymbol{A} \wedge \boldsymbol{C} \vdash \boldsymbol{T}[\boldsymbol{A}]$ for some true statement $\boldsymbol{C}$. The idea is simple: $\boldsymbol{T}[\boldsymbol{A}]$ need not entail $\boldsymbol{A}$. Take $\boldsymbol{C}$ to be the conjunction of all required background constraints.

Diagonalisation. To generate the paradox we use a technique of diagonalisation to construct a statement $\boldsymbol{\Psi}$ such that $\boldsymbol{\Psi}$ is equivalent to $\boldsymbol{T}[\boldsymbol{\Psi}] \rightarrow \boldsymbol{A}$, where $\boldsymbol{A}$ is any statement you please.

Curry's paradox, is a paradox within the family of so-called paradoxes of self-reference (or paradoxes of circularity). Like the liar paradox (e.g., 'this sentence is false') and Russell's paradox, Curry's paradox challenges familiar naive theories, including naïve truth theory (unrestricted $\boldsymbol{T}$-schema) and naive set theory (unrestricted axiom of abstraction), respectively. If one accepts naive truth theory (or naive set theory), then Curry's paradox becomes a direct challenge to one's theory of logical implication or entailment. Unlike the liar and Russell paradoxes Curry's paradox is negation-free; it may be generated irrespective of one's theory of negation.

There are basically two different versions of Curry's paradox, a truth-theoretic (or proof-theoretic) and a set-theoretic version; these versions will be presented below.

Truth-theoretic version.

Assume that our truth predicate satisfies the following $\boldsymbol{T}$-schema:

$$\boldsymbol{T}[\boldsymbol{A}] \leftrightarrow \boldsymbol{A},$$

Assume, too, that we have the principle called Assertion (also known as pseudo modus ponens):

Assertion: $(\boldsymbol{A} \wedge (\boldsymbol{A} \rightarrow \boldsymbol{B})) \rightarrow \boldsymbol{B}$

By diagonalization, self-reference we can get a sentence $\boldsymbol{C}$ such that $\boldsymbol{C} \leftrightarrow (\boldsymbol{T}[\boldsymbol{C}] \rightarrow \boldsymbol{F})$, where $\boldsymbol{F}$ is anything you like. (For effect, though, make $\boldsymbol{F}$ something obviously false, e.g. $\boldsymbol{F} \equiv \mathbf{0} = \mathbf{1}$) By an instance of the $\boldsymbol{T}$-schema ("$\boldsymbol{T}[\boldsymbol{C}] \leftrightarrow \boldsymbol{C}$") we immediately get: $\boldsymbol{T}[\boldsymbol{C}] \leftrightarrow (\boldsymbol{T}[\boldsymbol{C}] \rightarrow \boldsymbol{F})$,

Again, using the same instance of the $\boldsymbol{T}$-Schema, we can substitute $\boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}]$ for $\boldsymbol{T}[\boldsymbol{C}]$ in the above to get (1).

(1) $\vdash \boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}] \leftrightarrow (\boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}] \rightarrow \boldsymbol{F})$ [by $\boldsymbol{T}$-schema and Substitution]

(2) $\vdash (\boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}] \wedge (\boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}] \rightarrow \boldsymbol{F})) \rightarrow \boldsymbol{F}$ [by Assertion]

(3) $\vdash (\boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}] \wedge \boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}]) \rightarrow \boldsymbol{F}$ [by Substitution, from 2]

(4) $\vdash$ C[T,F] $\rightarrow$ F [by Equivalence of $\boldsymbol{C}$ and $\boldsymbol{C} \wedge \boldsymbol{C}$, from 3]

(5) $\vdash \boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}]$ [by Modus Ponens, from 1 and 4]

(6) $\vdash \boldsymbol{F}$ [by Modus Ponens, from 4 and 5]

Letting $\boldsymbol{F}$ be anything entailing triviality Curry's paradox quickly 'shows' that the world is trivial.

Set-Theoretic Version

The same result ensues within naive set theory. Assume, in particular, the

(unrestricted) axiom of abstraction (or naive comprehension (NC)):

Unrestricted Abstraction: $\boldsymbol{x} \in \{\boldsymbol{x} | \boldsymbol{A}(\boldsymbol{x})\} \leftrightarrow \boldsymbol{A}(\boldsymbol{x})$.

Moreover, assume that our conditional, $\rightarrow$, satisfies Contraction (as above),

which permits the deduction of $(\boldsymbol{s} \in \boldsymbol{s} \rightarrow \boldsymbol{A})$ from

$$\boldsymbol{s} \in \boldsymbol{s} \rightarrow (\boldsymbol{s} \in \boldsymbol{s} \rightarrow \boldsymbol{A}).$$

In the set-theoretic case, let $\boldsymbol{C}[\boldsymbol{F}], \{\boldsymbol{x} | \boldsymbol{x} \in \boldsymbol{x} \rightarrow \boldsymbol{F}\}$, where $\boldsymbol{F}$ remains as you please (but something obviously false, e.g. $\boldsymbol{F} \equiv \mathbf{0} = \mathbf{1}$). From here we reason thus:

(1) $\vdash \boldsymbol{x} \in \boldsymbol{C}[\boldsymbol{F}] \leftrightarrow (\boldsymbol{x} \in \boldsymbol{x} \rightarrow \boldsymbol{F})$ [by Unrestricted Abstraction]

(2) $\vdash \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \leftrightarrow (\boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow \boldsymbol{F})$ [by Universal Specification, from 1]

(3) $\vdash \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow (\boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow \boldsymbol{F})$ [by Simplification, from 2]

(4) $\vdash \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow \boldsymbol{F}$ [by Contraction, from 3]

(5) $\vdash \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}]$ [by Unrestricted Modus Ponens, from 2 and 4]

(6) $\vdash \boldsymbol{F}$ [by Unrestricted Modus Ponens, from 4 and 5]

So, coupling Contraction with the naive abstraction schema yields, via Curry's paradox, triviality.

This is a problem. Our true $\boldsymbol{C}[\boldsymbol{F}]$ entails an arbitrary $\boldsymbol{F}$.This inference arises independently of any treatment of negation. The form of the inference is reasonably well known. It is Curry's paradox, and it causes a great deal of trouble to any non-classical approach to the paradoxes. In the next sections we show how the tools for Curry's paradox are closer to hand than you might think.

II. Relevant First-Order Logics in General

Relevance logics are non-classical logics [2]-[15]. Called "relevant logics" in Britain and Australasia, these systems developed as attempts to avoid the paradoxes of material and strict implication. It is well known that relevant logic does not accept an axiom scheme $\boldsymbol{A} \rightarrow (\neg \boldsymbol{A} \rightarrow \boldsymbol{B})$ and the rule $\boldsymbol{A}, \neg \boldsymbol{A} \vdash \boldsymbol{B}$. Hence, in a natural way it might be used as basis for contradictory but non-trivial theories, i.e. paraconsistent ones. Among the paradoxes of material implication are: $\boldsymbol{p} \rightarrow (\boldsymbol{q} \rightarrow \boldsymbol{p}), \neg \boldsymbol{p} \rightarrow (\boldsymbol{p} \rightarrow \boldsymbol{q}), (\boldsymbol{p} \rightarrow \boldsymbol{q}) \vee (\boldsymbol{q} \rightarrow \boldsymbol{r})$. Among the paradoxes of strict implication are the following: $(\boldsymbol{p} \wedge \neg \boldsymbol{p}) \rightarrow \boldsymbol{q}, \boldsymbol{p} \rightarrow (\boldsymbol{q} \rightarrow \boldsymbol{q}), \boldsymbol{p} \rightarrow (\boldsymbol{q} \wedge \neg \boldsymbol{q})$. Relevant logicians point out that what is wrong with some of the paradoxes (and fallacies) is that is that the antecedents and consequents (or premises and conclusions) are on completely different topics. The notion of a topic, however, would seem not to be something that a logician should be interested in — it has to do with the content, not the form, of a sentence or inference. But there is a formal principle that relevant logicians apply to force theorems and inferences to "stay on topic". This is the variable sharing principle. The variable sharing principle says that no formula of the form $\boldsymbol{A} \rightarrow \boldsymbol{B}$ can be proven in a relevance logic if $\boldsymbol{A}$ and $\boldsymbol{B}$ do not have at least one propositional variable (sometimes called a proposition letter) in common and that no inference can be shown valid if the premises and conclusion do not share at least one propositional variable.

III. Curry's Paradox Resolution Using Canonical Systems of Relevant Logic

In the work of Anderson and Belnap [3] the central systems of relevance logic were the logic **E** of relevant entailment and the system **R** of relevant implication. The relationship between the two systems is that the entailment connective of **E** was supposed to be a strict (i.e. necessitated) relevant implication. To compare the two, Meyer added a necessity operator to **R** (to produce the logic **NR**).

It well known in set theories based on strong relevant logics, like **E** and **R,** as well as in classical set theory, if we add the naive comprehension axiom, we are able to derive any formula at all. Thus, naive set theories based on systems such as **E** and **R** are said to be "trivial" by Curry Paradox.

The existence of this paradox has led Grishen, Brady, Restall, Priest, and others to abandon the axiom of contraction which we have dubbed

$$\boldsymbol{K}: \quad ((\boldsymbol{A} \rightarrow (\boldsymbol{A} \rightarrow \boldsymbol{B})) \rightarrow (\boldsymbol{A} \rightarrow \boldsymbol{B})).$$

Brady has shown that by removing contraction, plus some other key theses, from R we obtain a logic that can accept naive comprehension without becoming trivial [4],[16],[17].

However, it is not just W that we must avoid. Shaw-Kwei [21] shows that a variant of Curry's paradox can trivialise a chain of weaker naive truth theories. Let us use the notations

$$\boldsymbol{\varphi} \rightarrow_{(\mathbf{0})} \boldsymbol{\psi} \text{ and } \boldsymbol{\varphi} \rightarrow_{(\boldsymbol{n}+\mathbf{1})} \boldsymbol{\psi}$$

to mean $\boldsymbol{\psi}$ and $\left(\boldsymbol{\varphi} \rightarrow \left(\boldsymbol{\varphi} \rightarrow_{(\boldsymbol{n})} \boldsymbol{\psi}\right)\right)$ correspondingly.

Then the following axioms also lead to triviality

$$\boldsymbol{K_n}: \quad \left(\boldsymbol{\varphi} \rightarrow \left(\boldsymbol{\varphi} \rightarrow_{(\boldsymbol{n})} \boldsymbol{\psi}\right)\right) \rightarrow \left(\boldsymbol{\varphi} \rightarrow_{(\boldsymbol{n})} \boldsymbol{\psi}\right).$$

We choose now a sentence $\boldsymbol{\gamma_n}$ via the diagonal lemma, that satisfies [22]:

$$\boldsymbol{\gamma_n} \leftrightarrow \left(\boldsymbol{Tr}(\widehat{\boldsymbol{\gamma_n}}) \rightarrow_{(\boldsymbol{n})} \boldsymbol{\varphi}\right),$$

where the notations $\widehat{\circ}$ to mean an fixed Godel numbering.

Then by full intersubstitutivity one obtain the equivalence

$$\boldsymbol{E_n}: \quad \boldsymbol{\gamma_n} \leftrightarrow \left(\boldsymbol{\gamma_n} \rightarrow_{(\boldsymbol{n})} \boldsymbol{\varphi}\right),$$

which by postulate $\boldsymbol{K_n}$ reduces to $\left(\boldsymbol{\gamma_n} \rightarrow_{(\boldsymbol{n})} \boldsymbol{\varphi}\right)$ and by $\boldsymbol{E_n}$ to $\boldsymbol{\gamma_n}$. But from $\boldsymbol{\gamma_n}$ and $\boldsymbol{\gamma_n} \rightarrow_{(\boldsymbol{n})} \boldsymbol{\varphi}$ one can deduce $\boldsymbol{\varphi}$ by n applications of unrestricted modus ponens (1.1). For example, a natural implicational logic without contraction is Łukasiewicz's 3-valued logic: $Ł_{\mathbf{3}}$. Although logic $Ł_{\mathbf{3}}$ does not contain $\boldsymbol{K}$, it does contain $\boldsymbol{K_2}$. In general the n+1-valued version of Łukasiewicz logic, $Ł_{\boldsymbol{n}+\mathbf{1}}$, validates $\boldsymbol{K_n}$ and is thus unsuitable for the same reason [22],[23].

However, it well known that contraction is not the only route to triviality . There are logics which are contraction free that still trivialize naive comprehension schema (NC) [18]. Abelian logic with axiom of relativity which we have dubbed

$$\boldsymbol{R}: \left((\boldsymbol{p} \rightarrow \boldsymbol{q}) \rightarrow \boldsymbol{q}\right) \rightarrow \boldsymbol{p}.$$

Let $\boldsymbol{a} = \{\boldsymbol{x} | \boldsymbol{\varphi}(\boldsymbol{x})\}$ and $\boldsymbol{\varphi}(\boldsymbol{x}) = \boldsymbol{p} \rightarrow \boldsymbol{x} \in \boldsymbol{x}$. Then as instance of NC one obtain $(\boldsymbol{p} \rightarrow \boldsymbol{a} \in \boldsymbol{a}) \rightarrow \boldsymbol{a} \in \boldsymbol{a}$. Thus we obtain

(1) $\vdash (\boldsymbol{p} \rightarrow \boldsymbol{a} \in \boldsymbol{a}) \rightarrow \boldsymbol{a} \in \boldsymbol{a}$ [by NC]

(2) $\vdash \left((\boldsymbol{p} \rightarrow \boldsymbol{a} \in \boldsymbol{a}) \rightarrow (\boldsymbol{a} \in \boldsymbol{a})\right) \rightarrow \boldsymbol{p}$ [by instance of $\boldsymbol{R}$]

(3) $\vdash \boldsymbol{p}$ [by 1,2 and Unrestricted Modus Ponens (1.1)].

IV. Relevant First-Order Logic $LP^{\#}$

In order to avoid the results mentioned in II and III, one could think of restrictions in initial formulation of the rule Unrestricted Modus Ponens (1.1). The postulates (or their axioms schemata) of propositional logic $\mathbf{LP}^{\#}[\mathbf{V}]$ are the following [19]:

I. Logical postulates:

(1) $\boldsymbol{A} \rightarrow (\boldsymbol{B} \rightarrow \boldsymbol{A})$,

(2) $(\boldsymbol{A} \rightarrow \boldsymbol{B}) \rightarrow ((\boldsymbol{A} \rightarrow (\boldsymbol{B} \rightarrow \boldsymbol{C})) \rightarrow (\boldsymbol{A} \rightarrow \boldsymbol{C}))$,

(3) $\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{A} \wedge \mathbf{B})$,

(4) $\boldsymbol{A} \wedge \boldsymbol{B} \rightarrow \boldsymbol{A}$,

(5) $\boldsymbol{A} \wedge \boldsymbol{B} \rightarrow \boldsymbol{B}$,

(6) $\boldsymbol{A} \rightarrow (\boldsymbol{A} \vee \boldsymbol{B})$,

(7) $\boldsymbol{B} \rightarrow (\boldsymbol{A} \vee \boldsymbol{B})$,

(8) $(\mathbf{A} \rightarrow \mathbf{C}) \rightarrow ((\mathbf{B} \rightarrow \mathbf{C}) \rightarrow (\mathbf{A} \vee \mathbf{B} \rightarrow \mathbf{C}))$,

(9) $\boldsymbol{A} \vee \neg \boldsymbol{A}$,

(10) $\boldsymbol{B} \rightarrow (\neg \boldsymbol{B} \rightarrow \boldsymbol{A})$.

II. Restricted Modus Ponens rule:

$$\boldsymbol{A}, \boldsymbol{A} \rightarrow \boldsymbol{B} \vdash \boldsymbol{B} \text{ iff } \boldsymbol{A} \notin \boldsymbol{V}. \quad (1.3)$$

or

$$\boldsymbol{A}, \boldsymbol{A} \rightarrow \boldsymbol{B} \vdash \boldsymbol{B} \text{ iff } \boldsymbol{B} \notin \boldsymbol{V}. \quad (1.4)$$

which we have write for short

$$\boldsymbol{A}, \boldsymbol{A} \rightarrow \boldsymbol{B} \vdash_{r} \boldsymbol{B} \text{ or } \boldsymbol{A}, \boldsymbol{A} \rightarrow \boldsymbol{B} \vdash_{r,V} \boldsymbol{B}.$$

V. Curry's Paradox and Shaw-Kwei's paradox Resolution Using Relevant First-Order Logic $\mathbf{LP}^{\#}$

In my paper [19] was shown that by removing only Unrestricted Modus Ponens rule (1.1) (without removing contraction etc.), plus some other key theses, from classical logic we obtain a logic that can accept naive comprehension without becoming trivial.

Let us consider Curry's paradox in a set theoretic version using Relevant First-Order Logic $\mathbf{LP}^{\#}$ with Restricted Modus Ponens rule (1.3). Let $\boldsymbol{C}[\boldsymbol{F}] = \{\boldsymbol{x} | \boldsymbol{x} \in \boldsymbol{x} \rightarrow \boldsymbol{F}\}$ and $\boldsymbol{\alpha}[\boldsymbol{F}]$ is a closed a well formed formula of *ZFC* (cwff) such that: $\boldsymbol{\alpha}[\boldsymbol{F}] \leftrightarrow \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}]$. We assume now $\boldsymbol{Con}(\boldsymbol{ZFC})$ and denote by $\boldsymbol{\Delta}$ a set of all cwff such that $\boldsymbol{\beta} \in \boldsymbol{\Delta} \leftrightarrow \neg \boldsymbol{Con}(\boldsymbol{ZFC} + \boldsymbol{\beta})$. Let us denote by symbol $\boldsymbol{W}_{\boldsymbol{\Delta}}$ a set

$$\boldsymbol{W}_{\boldsymbol{\Delta}} = \{ \boldsymbol{C}[\boldsymbol{F}] \, | \boldsymbol{F} \in \boldsymbol{\Delta} \}.$$

We set now in (1.3). $\boldsymbol{V} = \boldsymbol{W}_{\boldsymbol{\Delta}}$ From definition above we obtain the Restricted Modus Ponens rule:

$$\boldsymbol{A}, \boldsymbol{A} \rightarrow \boldsymbol{B} \vdash \boldsymbol{B} \text{ iff } \boldsymbol{A} \notin \boldsymbol{W}_{\boldsymbol{\Delta}}. \quad (1.5)$$

Let $\boldsymbol{F} \in \boldsymbol{\Delta}$. From here we reason thus:

(1) $\vdash \boldsymbol{x} \in \boldsymbol{C}[\boldsymbol{F}] \leftrightarrow (\boldsymbol{x} \in \boldsymbol{x} \rightarrow \boldsymbol{F})$ [by Unrestricted Abstraction]

(2) $\vdash \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \leftrightarrow (\boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow \boldsymbol{F})$ [by Universal Specification, from 1]

(3) $\vdash \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow (\boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow \boldsymbol{F})$ [by Simplification, from 2]

(4) $\vdash \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow \boldsymbol{F}$ [by Contraction, from 3]

(5) $\nvdash_{\boldsymbol{r}} \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}]$ [by Restricted Modus Ponens (1.5), from 2 and 4]

Let us denote by symbol $\breve{\boldsymbol{W}}_{\boldsymbol{\Delta}}$ a set

$$\breve{\boldsymbol{W}}_{\boldsymbol{\Delta}} = \{ \boldsymbol{C}[\boldsymbol{F}] \, | \boldsymbol{F} \notin \boldsymbol{\Delta} \}.$$

Therefore

$$\boldsymbol{A}, \boldsymbol{A} \rightarrow \boldsymbol{B} \vdash \boldsymbol{B} \text{ iff } \boldsymbol{A} \in \breve{\boldsymbol{W}}_{\boldsymbol{\Delta}}. \quad (1.6)$$

Let $\boldsymbol{F} \notin \boldsymbol{\Delta}$. From here we reason thus:

(1) $\vdash \boldsymbol{x} \in \boldsymbol{C}[\boldsymbol{F}] \leftrightarrow (\boldsymbol{x} \in \boldsymbol{x} \rightarrow \boldsymbol{F})$ [by Unrestricted Abstraction]

(2) $\vdash \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \leftrightarrow (\boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow \boldsymbol{F})$ [by Universal Specification, from 1]

(3) $\vdash \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow (\boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow \boldsymbol{F})$ [by Simplification, from 2]

(4) $\vdash \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow \boldsymbol{F}$ [by Contraction, from 3]

(5) $\vdash \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}]$ [by Restricted Modus Ponens (1.6), from 2 and 4]

(6) $\vdash \boldsymbol{F}$ [by Restricted Modus Ponens (1.6), from 4 and 5]

Let us consider now Curry's paradox in a set theoretic version using Relevant First-Order Logic $\mathbf{LP}^{\#}$ with Restricted Modus Ponens rule (1.4). We set now in (1.4). $\boldsymbol{V} = \boldsymbol{W}_{\boldsymbol{\Delta}}$ From definition above we obtain the Restricted Modus Ponens rule:

$$\boldsymbol{A}, \boldsymbol{A} \rightarrow \boldsymbol{B} \vdash \boldsymbol{B} \text{ iff } \boldsymbol{B} \notin \boldsymbol{W}_{\boldsymbol{\Delta}}. \quad (1.7)$$

Let $\boldsymbol{F} \in \boldsymbol{\Delta}$. From here we reason thus:

(1) $\vdash \boldsymbol{x} \in \boldsymbol{C}[\boldsymbol{F}] \leftrightarrow (\boldsymbol{x} \in \boldsymbol{x} \rightarrow \boldsymbol{F})$ [by Unrestricted Abstraction]

(2) $\vdash \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \leftrightarrow (\boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow \boldsymbol{F})$ [by Universal Specification, from 1]

(3) $\vdash \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow (\boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow \boldsymbol{F})$ [by Simplification, from 2]

(4) $\vdash \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}] \rightarrow \boldsymbol{F}$ [by Contraction, from 3]

(5) $\nvdash_{\boldsymbol{r}} \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}]$ [by Restricted Modus Ponens (1.7), from 2 and 4]

Let us consider now Curry's paradox in a set theoretic version using Abelian logic with axiom of relativity and Restricted Modus Ponens (1.4). We set now in (1.4). $\boldsymbol{V} = \boldsymbol{\Delta}$ From definition above we obtain the Restricted Modus Ponens rule:

$$\boldsymbol{A}, \boldsymbol{A} \to \boldsymbol{B} \vdash \boldsymbol{B} \textbf{ iff } \boldsymbol{B} \notin \boldsymbol{\Delta}. \qquad (1.8)$$

Let $\boldsymbol{C}[\boldsymbol{F}] = \{\boldsymbol{x}|\boldsymbol{\varphi}(\boldsymbol{x})\}$ and $\boldsymbol{\varphi}(\boldsymbol{x}) = \boldsymbol{F} \to \boldsymbol{x} \in \boldsymbol{x}$ and let $\boldsymbol{F} \in \boldsymbol{\Delta}$. Then as instance of NC one obtain $(\boldsymbol{F} \to \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}]) \to \boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}]$. Thus we obtain

(1) $\vdash (\boldsymbol{F} \to (\boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}])) \to (\boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}])$ [by NC]

(2) $\vdash ((\boldsymbol{F} \to (\boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}])) \to (\boldsymbol{C}[\boldsymbol{F}] \in \boldsymbol{C}[\boldsymbol{F}])) \to \boldsymbol{F}$ [by instance of $\boldsymbol{R}$]

(3) $\nvdash_{\boldsymbol{r}} \boldsymbol{F}$ [by 1,2 and Restricted Modus Ponens (1.7)].

Let us consider now Curry's paradox in a truth theoretic version using Relevant First-Order Logic $\mathbf{LP}^{\#}$ with Restricted Modus Ponens rule (1.4). We set now in (1.4). $\boldsymbol{V} = \boldsymbol{\Delta}$ From definition above we obtain the Restricted Modus Ponens rule:

$$\boldsymbol{A}, \boldsymbol{A} \to \boldsymbol{B} \vdash \boldsymbol{B} \textbf{ iff } \boldsymbol{B} \notin \boldsymbol{\Delta}. \qquad (1.9)$$

By diagonalization, self-reference we can get a sentence $\boldsymbol{C}$ such that $\boldsymbol{C} \leftrightarrow (\boldsymbol{T}[\boldsymbol{C}] \to \boldsymbol{F})$, where $\boldsymbol{F} \in \boldsymbol{\Delta}$.

By an instance of the $\boldsymbol{T}$-schema ("$\boldsymbol{T}[\boldsymbol{C}] \leftrightarrow \boldsymbol{C}$") we immediately get: $\boldsymbol{T}[\boldsymbol{C}] \leftrightarrow (\boldsymbol{T}[\boldsymbol{C}] \to \boldsymbol{F})$,

Again, using the same instance of the $\boldsymbol{T}$-Schema, we can substitute $\boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}]$ for $\boldsymbol{T}[\boldsymbol{C}]$ in the above to get (1).

(1) $\vdash \boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}] \leftrightarrow (\boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}] \to \boldsymbol{F})$ [by $\boldsymbol{T}$-schema and Substitution]

(2) $\vdash (\boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}] \wedge (\boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}] \to \boldsymbol{F})) \to \boldsymbol{F}$ [by Assertion]

(3) $\vdash (\boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}] \wedge \boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}]) \to \boldsymbol{F}$ [by Substitution, from 2]

(4) $\vdash \boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}] \to \boldsymbol{F}$ [by Equivalence of $\boldsymbol{C}$ and $\boldsymbol{C} \wedge \boldsymbol{C}$, from 3]

(5) $\vdash \boldsymbol{C}[\boldsymbol{T}, \boldsymbol{F}]$ [by Restricted Modus Ponens (1.9), from 1 and 4]

(6) $\nvdash_{\boldsymbol{r}} \boldsymbol{F}$ [by Restricted Modus Ponens (1.9), from 4 and 5].

It easy to see that by using logic with appropriate restricted modus ponens rule (1.4) Shaw-Kwei's paradox disappears by the same reason.

VI. The Resolution of ω-Inconsistency Problem for the Infnite Valued Łukasiewicz Logic Ł$_\infty$. Logic $\mathbf{LP}^{\#}_{\{\omega\}}$.

It well known that in the infinite valued Łukasiewicz logic, Ł$_\infty$, every instance of $\boldsymbol{K_n}$ is invalid, and in fact Ł$_\infty$ can consistently support a naive truth predicate [23]-[24]. However, Ł$_\infty$ is plagued with an apparently distinct problem – it is $\boldsymbol{\omega}$-inconsistent. This fact was first shown model theoretically by Restall in [25] and demonstrated a proof theoretically by Bacon in [24].

An classical extension of Peano Arithmetic is said to be $\boldsymbol{\omega}$ –inconsistent iff $\vdash \boldsymbol{\varphi}[\boldsymbol{n}/\boldsymbol{x}]$ for each n, but $\vdash \exists \boldsymbol{x} \neg \boldsymbol{\varphi}[\boldsymbol{x}]$. (1.10)

Note that while an $\boldsymbol{\omega}$ -inconsistent theory is not formally inconsistent. However $\boldsymbol{\omega}$ -inconsistency is generally considered to be an undesirable property, generally considered to be an undesirable property. It is generally considered undesirable if the theory becomes inconsistent in $\boldsymbol{\omega}$ –logic. In other words, if it cannot be consistently maintained in the presence of the infinitary $\boldsymbol{\omega}$ -rule:

$$\{\boldsymbol{\varphi}[\boldsymbol{n}/\boldsymbol{x}] | \boldsymbol{n} \in \boldsymbol{\omega}\} \vdash \forall \boldsymbol{x}\, \boldsymbol{\varphi}[\boldsymbol{x}]. \qquad (1.11)$$

Clearly $\boldsymbol{\omega}$ -inconsistency entails inconsistency with the $\boldsymbol{\omega}$ –rule (1.11), but the converse does not hold in general. We have dubbed any Logic $\mathbf{LP}^{\#}$ with the $\boldsymbol{\omega}$ –rule (1.11) by $\mathbf{LP}^{\#}_{\{\omega\}}$.

Definition 6.1. [23]. Weak $\boldsymbol{\omega}$ –inconsistency means:

$$\boldsymbol{\varphi}[\boldsymbol{n}/\boldsymbol{x}] \vdash \text{for each } \boldsymbol{n}, \text{ but } \vdash \exists \boldsymbol{x} \boldsymbol{\varphi}[\boldsymbol{x}]. \qquad (1.12)$$

Definition 6.2. [23]. Strong $\boldsymbol{\omega}$ –inconsistency means:

$$\vdash \boldsymbol{\varphi}[\boldsymbol{n}/\boldsymbol{x}] \text{ for each } \boldsymbol{n}, \text{ but } \vdash \exists \boldsymbol{x}(\boldsymbol{\varphi}[\boldsymbol{x}] \to \bot). \qquad (1.13)$$

Note that without the rule of reduction one cannot derive strong $\boldsymbol{\omega}$ -inconsistency from weak $\boldsymbol{\omega}$ -inconsistency [23].

Definition 6.3. [23]. By a classical "naive truth theory" (CNTT) we shall mean any set of first order sentences in the language of arithmetic with a truth predicate which, in addition to being closed under modus ponens, has the following properties:

(1) **Standard syntax:** it contains all the arithmetical consequences of classical Peano arithmetic.

(2) **Intersubstitutivity:** it contains φif and only if it contains $\varphi\left[Tr\left(\widetilde{\psi}\right)/\psi\right]$ for any sentence φ.

(3) **Compositionality:** it contains $Tr(x) \rightarrow Tr(y)$ if and only if it contains $Tr(x \dot{\rightarrow} y)$.

(4) **Unrestricted Modus Ponens rule:** it closed under unrestricted modus ponens rule (1.1).

(i) If $\varphi \vdash \psi$ then $\exists x\varphi \vdash \exists x\psi$,

(ii) $(\varphi \rightarrow \exists x\psi) \vdash \exists x(\varphi \rightarrow \psi)$.

Note that by using the diagonal lemma we can construct a sentence γ satisfying

$$\gamma \leftrightarrow \exists nTr\left(f\left(n, \widetilde{\gamma}\right)\right), \quad (1.14)$$

where the notations $\widetilde{\circ}$ to mean an fixed Gödel numbering and a function f is defined arithmetically by recursion [23]:

$$f(0, x) = x \dot{\rightarrow} \bot \text{ and } f(n+1, x) = x \dot{\rightarrow} f(n, x).$$

Theorem 6.1. [23]. Any classical naive truth theory closed under (1),(2),(3), (i) and (ii) can prove γ.

Theorem 6.2. [23]. Any naive truth theory closed under (i) and (ii) is weakly ω-inconsistent.

Proof. By theorem 6.1 one obtain

$$\text{CNTT} \vdash \exists nTr\left(f\left(n, \widetilde{\gamma}\right)\right).$$

By arithmetic and full intersubstitutivity we obtain that

$$Tr\left(f\left(n, \widetilde{\gamma}\right)\right) \vdash \gamma \rightarrow_{(n)} \bot.$$

Since we have $\vdash \gamma$ by theorem 2.1, by n applications of unrestricted modus ponens we obtain

$$\gamma \rightarrow_{(n)} \bot \vdash \bot. \quad (1.15)$$

So we have in general $Tr\left(f\left(n, \widetilde{\gamma}\right)\right) \vdash$ for any n, and

$$\vdash \exists nTr\left(f\left(n, \widetilde{\gamma}\right)\right).$$

Theorem 6.3. [23]. Any naive truth theory closed under (i) and (ii) is strongly ω-inconsistent.

Theorem 6.4. [25]. Infinitely valued Łukasiewicz logic, $Ł_\infty$, is strongly ω-inconsistent.

Definition 6.4. By a non-classical or generalized "naive truth theory" (GCNTT) we shall mean any set of first order sentences in the language of arithmetic with a truth predicate which, in addition to being closed under modus ponens, has the following properties:

(1) **Standard syntax:** it contains all the arithmetical consequences of classical Peano arithmetic.

(2) **Intersubstitutivity:** it contains φ if and only if it contains $\varphi\left[Tr\left(\widetilde{\psi}\right)/\psi\right]$ for any sentence φ.

(3) **Compositionality:** it contains $Tr(x) \rightarrow Tr(y)$ if and only if it contains $Tr(x \dot{\rightarrow} y)$.

(4) **Infinitary ω -rule:** $\{\varphi[n/x] | n \in \omega\} \vdash \forall x\, \varphi[x]$.

(5) **Restricted Modus Ponens rules**: it closed under restricted modus ponens rule (1.3) or (1.4)

Definition 6.5. Weak ω –consistency means:

$$\vdash \varphi[n/x] \text{ for each } n, \text{ but } \nvdash \exists x(\varphi[x] \rightarrow \bot). \quad (1.16)$$

Definition 6.6. [23]. Strong ω –consistency means:

$$\vdash \varphi[n/x] \text{ for each } n, \text{ but } \nvdash \exists x \neg(\varphi[x]). \quad (1.17)$$

Theorem 6.5. Any consistent GCNTT closed under restricted modus ponens rule (1.3) is strongly ω –consistent.

Theorem 6.6.

V.II. Applications to da Costa's Paraconsistent Set Theories.

da Costa [27] introduced a Family of paraconsistent logics C_n, $1 \leq n \leq \omega$, with unrestricted modus ponens rule (3) [28], designed to be able to support set theories NF_n^C, respectively, $1 \leq n \leq \omega$,incorporating unrestricted **Comprehension Schema:**

$\vdash \exists x\forall y[x \in y \leftrightarrow F(x)],$ (18)

where $F(x)$ is any formula in which *y* is not free but *x* may be, and $F(x)$ does not contain any sub formula of the form $A \rightarrow B$

Axiom of Extensionality:

$\vdash \forall x\forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y]$ (19)

Since Russell's paradox could be reproduced in these set theories, their underlying logics in the absence classical rule $A, \neg A \nvdash B$ had to be capable of tolerating such theorems as $\vdash \mathcal{R} \in \mathcal{R} \leftrightarrow \neg\mathcal{R} \in \mathcal{R}$ without collapse into triviality [29] but which is hardly less disastrous $\vdash \forall x\forall y[(x \in y) \wedge (x = y)]$.

Definition 7.1.[29].$\sim A$ iff $A \rightarrow \forall x\forall y[(x \in y) \wedge (x = y)]$.

Theorem 7.1. [29] In NF_n^C, **negation** $\sim$ is a minimal intuitionistic negation.

Theorem 7.2. [29]. (Cantor's Theorem) $\vdash \forall \alpha[\sim(\alpha = P(\alpha))]$.

Definition 7.2. [29]. The universal set V is defined as: $\forall x[x \in V \leftrightarrow (x = x)]$

Theorem 7.3. [29]. (Cantor's Paradox) $\vdash (V = P(V)) \wedge \sim(V = P(V))$.

Theorem 7.4.[29].

(i) $\forall x\forall y[(x = y) \wedge \sim(x = y)]$,

(ii) $\forall x\forall y[(x \in y) \wedge \sim(x \in y)]$,

(iii) $\forall x\forall y[(x \in x) \wedge \sim(x \in x)]$.

Proof. (i). By theorem 7.3 one obtain

$$\vdash (V = V) \wedge \sim(V = V). \quad (20)$$

From (20) and definition 7.1 one obtain

$$V = V \rightarrow \forall x\forall y[(x \in y) \wedge (x = y)]. \quad (21)$$

Therefore, as $V = V$, then $\vdash \forall x\forall y(x = y)$ and $\vdash \forall x\forall y(x \in y)$.

Note that statement (i) of the theorem 7.4 is called *paradox of identity.*

Definition7.3. Let us define paraconsistent da Costa type logics $C_n^\#, 1 \leq n < \omega$, with restricted modus ponens rule such that

$$A, A \rightarrow B \vdash B \text{ iff } B \notin V, \quad (22)$$

$$[\forall x\forall y(x = y)] \wedge [\forall x\forall y(x \in y)] \in V \quad (23)$$

for support set theories $NF_n^\#$, respectively, $1 \leq n < \omega$, incorporating unrestricted Comprehension Schema (18).

From the proof of the theorem 7.3 it follows directly that logics $C_n^\#, 1 \leq n < \omega$, in fact, provide an effective way of circumventing paradox of identity.

Arruda in [29] introduced a Family of set theories ZF_n, $1 \leq n \leq \omega$, in which any canonical axiom of ZFC: the axiom of pairing, axiom of union etc., are postulated in general and in which also postulated the existence of the Russell's set $\mathcal{R}$.

Definition7.4. [29]. $\neg_n^* A$ iff $\neg A \wedge A^{(n)}$.

Note that $\neg_n^*$ is a classical negation.

Theorem 7.5. [29].Any set theories ZF_n, $1 \leq n \leq \omega$ are trivial.

Proof. By axiom of separation there exist subset $\mathcal{R}_n$ of $\mathcal{R}$ such that (1) $\forall x[x \in \mathcal{R}_n \leftrightarrow (x \in \mathcal{R}) \wedge (x \in x)^{(n)}]$. From (1) we obtain (2) $\mathcal{R}_n \in \mathcal{R}_n \leftrightarrow \neg(\mathcal{R}_n \in \mathcal{R}_n) \wedge (\mathcal{R}_n \in \mathcal{R}_n)^{(n)}$. From (2) we obtain

$$(\mathcal{R}_n \in \mathcal{R}_n) \wedge \neg_n^*(\mathcal{R}_n \in \mathcal{R}_n). \quad (24)$$

But formula (24) trivializes the system ZF_n.

Definition7.5. Let us define paraconsistent da Costa type logics $\breve{C}_n^\#, 1 \leq n < \omega$, with restricted modus ponens rule such that

$$A, A \rightarrow B \vdash B \text{ iff } A \neg_n^* A \notin V_n. \quad (25)$$

$$(\mathcal{R}_n \in \mathcal{R}_n) \wedge \neg_n^*(\mathcal{R}_n \in \mathcal{R}_n) \in V_n \quad (26)$$

for support set theories $\widetilde{ZF}_n^\#$, respectively, $1 \leq n < \omega$, incorporating unrestricted Comprehension Schema (18).

From the proof of the theorem 7.5 it follows directly that logic $\breve{C}_n^\#, 1 \leq n < \omega$,in fact, provide an effective way of circumventing Russell's paradox.

Arruda and da Costa [30] introduced a Family of sentential logics, J_1 to J_5, designed to be able to support set theories, respectively ZF_1 to ZF_5 ,incorporating an unrestricted Comprehension Schema (18). These J logics are interesting in that they do not have modus ponens, but still seem to contain a lot of theorems that might be expected if modus ponens was included.

Theorem 7.6. [32]. $\vdash A$ is a theorem of positive intuitionistic logic if and only if $\rightarrow A$ is a theorem of J_1.

The basic version of Curry's paradox shows that any such set theory is trivial if its underlying logic contains the rules of Unrestricted Modus Ponens (1) and Contraction, in addition to the usual Instantiation rules for the quantifiers and Simplification. Arruda and da Costa instead constructed their J-systems without *modus ponens.* Arruda and da Costa [27] announced that $A \equiv \neg A \vdash B \supset C$ is derivable in J_2 *to* J_5 for all formulas A, B and C. Consequently, by Russell's paradox, the set theories: ZF_1 to ZF_5 contain $\vdash B \supset C$ for all B and C. In the absence of modus ponens, this does not quite amount to triviality. It is rather a variant which can be called $\supset$-triviality, but which is hardly less disastrous: $\forall\mathbf{x}\forall\mathbf{y}(\mathbf{x}=\mathbf{y})$ directly follows by Axiom of Extensionality (19). Noting only that $A \equiv \neg A \vdash B \supset C$ is not similarly derivable in J_1. Arruda and da Costa [31] left open the question whether the sole remaining set theory ZF_1 is acceptably non-trivial, and thus whether the strategy of restricting *modus ponens* in the manner of the J -systems does in fact provide an effective way of circumventing Curry's paradox. These questions answered in the negative by the following variant of the Russell's paradox [33]:

$$\mathcal{R} \in \mathcal{R} \equiv (\mathcal{R} \in \mathcal{R} \supset C). \tag{27}$$

Theorem 7.7. [33]. ZF_1 **is** $\supset$-trivial.

In addition to Contraction, Simplification and Instantiation rules, J_1 contains the rules of Weakening, $B \vdash A \supset B$, and Transitivity, $A \supset B, B \supset C \vdash A \supset C$.

Definition7.6. Let us define paraconsistent logic $J_1^{\#}$, with restricted Weakening rule such that

$$B \vdash A \supset B \text{ iff } B \notin V, \tag{28}$$

$$(\mathcal{R} \in \mathcal{R} \supset C) \in V. \tag{29}$$

For support set theory $ZF_1^{\#}$, incorporating unrestricted Comprehension Schema (18).

From the proof of the theorem 7.7 it follows directly that logic $J_1^{\#}$in fact, provide an effective way of circumventing Curry's paradox.

V.III. Paraconsistent Nonstandard Analysis

Definition 8.1. Let us define paraconsistent da Costa type logics $\widetilde{C}_n^{\#}, 1 \leq n < \omega$, with restricted modus ponens rule such that

$$A, A \rightarrow B \vdash B \text{ iff } A\neg_n^* A \notin V_n \text{ and } B \notin V_n, \tag{30}$$

$$[\forall x \forall y(x=y)] \wedge [\forall x \forall y(x \in y)] \in V_n, \tag{31}$$

$$(\mathcal{R}_n \in \mathcal{R}_n) \wedge \neg_n^*(\mathcal{R}_n \in \mathcal{R}_n) \in V_n. \tag{32}$$

Definition 8.2. Let us define now paraconsistent logic $\widetilde{C}_\infty^{\#}$ with infinite hierarchy levels of contradiction [20]:

$$\widetilde{C}_\infty^{\#} = \bigcup_{n<\omega} \widetilde{C}_n^{\#}.$$

for support set theory $ZFC_\infty^{\#}$, in which any canonical axiom of ZFC: the axiom of pairing, axiom of union etc., are postulated in general and in which also postulated the existence of the Universal set $\mathbf{V}$.

In this subsection, we will to distinguish: (1) the relations (i) strong (consistent) equality denoted by $(\cdot =_s \cdot)$,(ii) weak equality denoted by $(\cdot =_w \cdot)$, (iii) weak paraconsistent equality denoted by $(\cdot =_{w_1} \cdot)$, and

(2) the relations (i) strong (consistent) membership relation denoted by $\in_s$ and (ii) weak membership relation denoted by $\in_w$,(iii) weak paraconsistent membership relation denoted by $\in_{w_1}$[20].

Remark8.1. We note, that in $ZFC_\infty^{\#}$ valid:

(i) $\forall x, y: (x =_s y) \wedge \neg(x =_s y) \vdash B, n = 1, 2, \ldots$
(ii) $\exists x, y: (x =_w y) \wedge \neg_n^*(x =_w y) \nvdash B, B \in V, n = 1, 2, \ldots$
(iii) $\forall x, y: (x \in_s y) \wedge \neg(x \in_s y) \vdash B, n = 1, 2, \ldots$
(iv) $\exists x, y: (x \in_w y) \wedge \neg_n^*(x \in_w y) \nvdash B, B \in V, n = 1, 2, \ldots$

Designations 8.1. We will write:

(i) $x =_{w_1} y$ for $(x =_w y) \wedge \neg(x =_w y)$,

(ii) $x = y$ for $(x =_s y) \vee (x =_w y) \vee (x =_{w_1} y)$,

(iii) $x \in_{w_1} y$ for $(x \in_w y) \wedge \neg(x \in_w y)$,

(iv) $x \in y$ for $(x \in_s y) \vee (x \in_w y) \vee (x \in_{w_1} y)$,

Definition 8.3. (1) By $\in$- consistent set we shall mean a set X such that there does not exist any weak element z of X , e.g., $\neg\exists z(z \in_w X) \vee (z \in_{w_1} X)$ and

(2) there exist at least one strong element of X, e.g., $\exists x(x \in_s X)$.

Designations 8.2. We will write: $\mathbf{con}(\boldsymbol{X})$ if $\boldsymbol{X}$ is $\in$-consistent set.

Assumption. (Postulate of the existence the Universe of the all $\in$- consistent sets) We assume now that, there exist a set $\mathbf{V}^{\mathbf{con}}$ such that $\forall X[con(X) \rightarrow X \in_s V^{con}]$.

Definition 8.4. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be a set.

(i) $X =_s Y$ iff $\forall z(z \in_s X \leftrightarrow z \in_s Y)$,

(ii) $X =_w Y$ iff $\forall z(z \in_w X \leftrightarrow z \in_w Y)$,

(iii) $X =_{w_1} Y$ iff $\forall z\left(z \in_{w_1} X \leftrightarrow z \in_{w_1} Y\right)$,

(iv)$X \subseteq_s Y$ iff $\forall x[x \in_s X \rightarrow x \in_s Y]$,

Definition 8.5.

(i) By consistent empty set we shall mean a set $\emptyset_s$ such that there does not exist any strong element z of $\emptyset_s$ e.g., $\neg\exists z(z \in_s \emptyset_s)$.

(ii) By a weakly consistent empty set we shall mean a set $\emptyset_w$ such that there does not exist any weak element z of $\emptyset_w$ e.g., $\neg\exists z(z \in_w \emptyset_w) \vee \left(z \in_{w_1} \emptyset_w\right)$.

(iii) By inconsistent empty set we shall mean a set $\emptyset_{w_1}$ such that $\forall z\left(z \in_{w_1} X\right)$.

Definition 8.6. The superstructure over Universal set $\mathbf{V}$, denoted by $\mathbf{W}(\mathbf{V})$, is defined by the following canonical recursion:

$$\mathbf{W}_0(\mathbf{V}) = \mathbf{V}, \mathbf{W}_{n+1}(\mathbf{V}) = \mathbf{W}_n(\mathbf{V}) \cup \boldsymbol{P}\left(\mathbf{W}_n(\mathbf{V})\right),$$

$$\mathbf{W}(\mathbf{V}) = \cup_{n<\omega} \boldsymbol{W}_n(\mathbf{V}). \qquad (33)$$

Remark8.2. Note that $\mathbf{W}(\mathbf{V}) = \mathbf{V}$.
The language $\boldsymbol{L}$ which describes $\mathbf{W}(\mathbf{V})$ consists of logical connectives $\neg, \wedge, \vee, \rightarrow$, quantifiers $\forall, \exists$, individual variables $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots$, individual constants $\boldsymbol{C_u}$ for all $\boldsymbol{u} \in \mathbf{W}(\mathbf{V})$ and two binary predicate constants $=, \in$. A formula of language $\boldsymbol{L}$ is constructed from the above constituents in the usual way. We will use the following abbreviations, called bounded quantifiers: $(\forall \boldsymbol{x} \in \boldsymbol{y})\boldsymbol{\varphi}$ means $\forall \boldsymbol{x}(\boldsymbol{x} \in \boldsymbol{y} \rightarrow \boldsymbol{\varphi})$, $(\exists \boldsymbol{x} \in \boldsymbol{y})\boldsymbol{\varphi}$ means$\exists \boldsymbol{x}\left((\boldsymbol{x} \in \boldsymbol{y}) \wedge \boldsymbol{\varphi}\right)$. A bounded formula is a formula in which every quantifier occurs as a bounded quantifier. We will write $\boldsymbol{\varphi}[\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}]$ for $\boldsymbol{\varphi}\left[\boldsymbol{C_{u_1}}, \boldsymbol{C_2}, \ldots, \boldsymbol{C_{u_n}}\right]$.

For any formula $\boldsymbol{\varphi}$ in $\boldsymbol{L}$, the relation $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{\varphi}$ is defined by the following rules:

(1) $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{C_{u_1}} =_s \boldsymbol{C_{u_2}}$ if and only if $\boldsymbol{u_1} =_s \boldsymbol{u_2}$ hold.

(2) $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{C_{u_1}} \in_s \boldsymbol{C_{u_2}}$ if and only if $\boldsymbol{u_1} \in_s \boldsymbol{u_2}$ hold.

(3) $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{C_{u_1}} =_w \boldsymbol{C_{u_2}}$ if and only if $\boldsymbol{u_1} =_w \boldsymbol{u_2}$ hold.

(4) $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{C_{u_1}} \in_w \boldsymbol{C_{u_2}}$ if and only if $\boldsymbol{u_1} \in_w \boldsymbol{u_2}$ hold.

(5) $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{C_{u_1}} = \boldsymbol{C_{u_2}}$ if and only if $\boldsymbol{u_1} =_s \boldsymbol{u_2}$ or $\boldsymbol{u_1} =_w \boldsymbol{u_2}$ hold.

(6) $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{C_{u_1}} \in \boldsymbol{C_{u_2}}$ if and only if $\boldsymbol{u_1} \in_s \boldsymbol{u_2}$ or $\boldsymbol{u_1} \in_w \boldsymbol{u_2}$ hold.

(7) $\mathbf{W}(\mathbf{V}) \vDash \neg\left(\boldsymbol{C_{u_1}} =_s \boldsymbol{C_{u_2}}\right)$ if and only if $\boldsymbol{u_1} =_s \boldsymbol{u_2}$ does not hold.

(8) $\mathbf{W}(\mathbf{V}) \vDash \neg\left(\boldsymbol{C_{u_1}} \in_s \boldsymbol{C_{u_2}}\right)$ if and only if $\boldsymbol{u_1} \in_s \boldsymbol{u_2}$ does not hold.

(9) $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{\varphi_1} \wedge \boldsymbol{\varphi_2}$ if and only if $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{\varphi_1}$ and $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{\varphi_2}$

(10) $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{\varphi_1} \vee \boldsymbol{\varphi_2}$ if and only if $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{\varphi_1}$ or $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{\varphi_2}$

(11) $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{\varphi_1} \rightarrow \boldsymbol{\varphi_2}$ if and only if $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{\varphi_1} \rightarrow \mathbf{W}(\mathbf{V}) \vDash \boldsymbol{\varphi_2}$ hold.

(12) $\mathbf{W}(\mathbf{V}) \vDash \forall \boldsymbol{x}\boldsymbol{\varphi}(\boldsymbol{x})$ if and only if $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{\varphi}[\boldsymbol{u}]$ for all $\boldsymbol{u}$ in $\mathbf{W}(\mathbf{V})$.

(13) $\mathbf{W}(\mathbf{V}) \vDash \exists \boldsymbol{x}\boldsymbol{\varphi}(\boldsymbol{x})$ if and only if $\mathbf{W}(\mathbf{V}) \vDash \boldsymbol{\varphi}[\boldsymbol{u}]$ for an $\boldsymbol{u}$ in $\mathbf{W}(\mathbf{V})$.

Definition 8.3. Let $\boldsymbol{f}$ be a function defined on a set $\boldsymbol{X} \subseteq \mathbf{V}$ and taking values in a set $\boldsymbol{Y} \subseteq \mathbf{V}$. Then $\boldsymbol{f}$ is said to be an paraconsistent injection (or injective map, or embedding) if, whenever (1) $\boldsymbol{f}(\boldsymbol{x}) =_s \boldsymbol{f}(\boldsymbol{y})$, it must be the case that $\boldsymbol{x} =_s \boldsymbol{y}$ and (2) $\boldsymbol{f}(\boldsymbol{x}) =_w \boldsymbol{f}(\boldsymbol{y})$, it must be the case that $(\boldsymbol{x} =_s \boldsymbol{y}) \vee (\boldsymbol{x} =_w \boldsymbol{y})$.

Definition 8.4. Paraconsistent nonstandard universe is a triple $\langle \mathbf{W}(\mathbf{V}), \mathbf{W}(\mathbf{V}'), \# \rangle$ consisting of superstructures $\mathbf{W}(\mathbf{V})$, $\mathbf{W}(\mathbf{V}')$ and a map $\#: \mathbf{W}(\mathbf{V}) \rightarrow \mathbf{W}(\mathbf{V}')$ satisfying the following conditions:

(1) (**Paraconsistent Transfer Principle**) The map $\#: \mathbf{W}(\mathbf{V}) \rightarrow \mathbf{W}(\mathbf{V}')$ is an paraconsistent injective mapping from $\mathbf{W}(\mathbf{V})$ into $\mathbf{W}(\mathbf{V}')$, and for any

bounded formula $\varphi(x_1, x_2, \dots, x_n)$ in L

$$\mathbf{W(V)} \vDash \varphi[u_1, u_2, \dots, u_n] \Longrightarrow$$

$$\mathbf{W(V')} \vDash \varphi[\,{}^{\#}u_1, {}^{\#}u_2, \dots, {}^{\#}u_n] \quad (33)$$

for any $u_1, u_2, \dots, u_n$ in $\mathbf{W(V)}$

(2) ${}^{\#}\mathbf{V} = \mathbf{V'} = \mathbf{V}$.

Theorem 8.1.[36]**.** There exist paraconsistent nonstandard universe $\langle \mathbf{W(V)}, \mathbf{W(V')}, \# \rangle$ as required above.

Definition 8.5. An inconsistent filter $\mathcal{F}^{\mathbf{inc}}$ over $\mathbf{V}$ is a family of subsets of $\mathbf{V}$ satisfying the following properties:

(1) (i) $\mathbf{V} \in_s \mathcal{F}^{\mathbf{inc}}$, $\emptyset_s \notin_s \mathcal{F}^{inc}$, (ii) $\emptyset \in_w \mathcal{F}^{inc} \wedge \emptyset \notin_w \mathcal{F}^{inc}$,

(2) $\forall n \in \mathbb{N}$: **if** $A_1, \dots, A_n \in \mathcal{F}^{\mathbf{inc}}$, then

$$A_1 \cap \dots \cap A_n \in \mathcal{F}^{\mathbf{inc}}.$$

(3) $\forall n \in \mathbb{N}: A_1, \dots, A_n \in_s \mathcal{F}^{\mathbf{inc}} \rightarrow A_1 \cap_s \dots \cap_s A_n \in_s \mathcal{F}^{\mathbf{inc}}$.

(4) $\forall n \in \mathbb{N}$: **if** $A_1, \dots, A_n \in_w \mathcal{F}^{\mathbf{inc}}$, then

$$A_1 \cap_w \dots \cap_w A_n \in_w \mathcal{F}^{\mathbf{inc}}.$$

Let us consider now paraconsistent nonstandard extension of $\mathbb{R}$. Let $\mathcal{F}^{\mathbf{inc}}$ be a free inconsistent ultrafilter on $\mathbf{V}$ and introduce an paraconsistent equivalence relation on functions in $\mathbb{R}^{\mathbf{V}}$ as

$$f \sim_{\mathbf{inc}} g \text{ iff } \{i \in \mathbf{V} | f(i) = g(i)\} \in \mathcal{F}^{inc} \quad (34)$$

We will to distinguish the relations:

(i) strong (consistent) equivalence relation denoted by $(\cdot \sim_s \cdot)$

$$f \sim_s g \text{ iff } \{i \in_s \mathbf{V} | f(i) = g(i)\} \in_s \mathcal{F}^{inc}, \quad (35)$$

(ii) weakly consistent equivalence relation denoted by $(\cdot \sim_w \cdot)$

$$f \sim_w g \text{ iff } \{i \in_w \mathbf{V} | f(i) = g(i)\} \in_w \mathcal{F}^{inc} \text{ and}$$

$$\{i \in_w \mathbf{V} | f(i) = g(i)\} \notin_s \mathcal{F}^{inc}, \quad (36)$$

(iii) inconsistent equivalence relation denoted by $(\cdot \sim_{w_1} \cdot)$

$$f \sim_{w_1} g \text{ iff } \{i \in_{w_1} \mathbf{V} | f(i) = g(i)\} \in_{w_1} \mathcal{F}^{inc}. \quad (37)$$

Definition 8.6. If $f \in \mathbb{R}^{\mathbf{V}}$ (i) we denoted by $[f]_s$ a set

$$[f]_s =_{def} \{g | f \sim_s g\}, \quad (38)$$

(ii) we denoted by $[f]_w$ a set

$$[f]_w =_{def} \{g | f \sim_w g\}, \quad (39)$$

(iii) we denoted by $[f]_{w_1}$ a set

$$[f]_{w_1} =_{def} \{g | f \sim_{w_1} g\}. \quad (40)$$

$\mathbb{R}^{\mathbf{V}}$ divided out by the paraconsistent equivalence relation $\sim_{\mathbf{inc}}$ gives us the paraconsistent nonstandard extension ${}^{\#}\mathbb{R}$, the #-hyperreals; in symbols, ${}^{\#}\mathbb{R} = \mathbb{R}^{\mathbf{V}} / \mathcal{F}^{inc}$.

Remark8.4. We note that every element in ${}^{\#}\mathbb{R}$ is only of the form (i) $[f]_s$, (ii) $[f]_w$ or (iii) $[f]_{w_1}$ for some $f \in \mathbb{R}^{\mathbf{V}}$.

Definition 8.7. For any real number $r \in \mathbb{R}$

(i) we denoted by r_s the constant function in $\mathbb{R}^{\mathbf{V}}$ with value r and $r_s(i) = r$, for all $i \in_s \mathbf{V}$. We then have a natural consistent embedding

$$\#_s : \mathbb{R} \rightarrow {}^{\#_s}\mathbb{R} \subset {}^{\#}\mathbb{R} \quad (41)$$

by setting ${}^{\#_s}r = [r_s]_s$, for all $r \in \mathbb{R}$,

(ii) we denoted by r_w the constant function in $\mathbb{R}^{\mathbf{V}}$ with value r and $r_w(i) = r$, for all $i \in_w \mathbf{V}$. We then have a natural weakly consistent embedding

$$\#_w : \mathbb{R} \rightarrow {}^{\#_w}\mathbb{R} \subset {}^{\#}\mathbb{R} \quad (42)$$

by setting ${}^{\#_w}r = [r_w]_w$, for all $r \in \mathbb{R}$,

(iii) we denoted by r_{w_1} the constant function in $\mathbb{R}^{\mathbf{V}}$ with value r and $r_{w_1}(i) = r$, for all $i \in_{w_1} \mathbf{V}$. We then have a natural inconsistent embedding

$$\#_{w_1} : \mathbb{R} \rightarrow {}^{\#_{w_1}}\mathbb{R} \subset {}^{\#}\mathbb{R} \quad (43)$$

by setting ${}^{\#_{w,1}}r = [r_{w,1}]_{w,1}$, for all $r \in \mathbb{R}$.

(iv) we denoted a set ${}^{\#s}\mathbb{R} \cup {}^{\#w}\mathbb{R}$ by ${}^{\#s,w}\mathbb{R}$

we then have embedding :

$$\#_{s,w}: \mathbb{R} \to {}^{\#s,w}\mathbb{R} \subset {}^{\#}\mathbb{R}, \quad (44)$$

(v) we denoted a set ${}^{\#s}\mathbb{R} \cup {}^{\#w}\mathbb{R} \cup {}^{\#w_1}\mathbb{R}$ by ${}^{\#}\mathbb{R}$

we then have embedding :

$$\#: \mathbb{R} \to {}^{\#}\mathbb{R}. \quad (45)$$

As an algebraic structure, $\mathbb{R}$ is a complete ordered field, i.e., a structure of the form

$$\langle \mathbb{R}, +, \times, =, <, \mathbf{0}, \mathbf{1} \rangle, \quad (46)$$

where $\mathbb{R}$ is the set of elements of the structure, $+$ and $\times$ are the binary operations of addition and multiplication, $<$ is the ordering relation, and $\mathbf{0}$ and $\mathbf{1}$ are two distinguished elements of the domain. And it is complete in the sense that every nonempty set bounded from above has a least upper bound.

Remark8.5. We note that:

(i) the $\#_s$-embedding of (41) sends $\mathbf{0}$ to $[\mathbf{0}]_s =_{def} \mathbf{0}_s$ and 1 to $[\mathbf{1}]_s =_{def} \mathbf{1}_s$, we denoted $\mathbf{0}_s$ by $\mathbf{0}$ and $\mathbf{1}_s$ by 1;

(ii) the $\#_w$-embedding of (42) sends $\mathbf{0}$ to $[\mathbf{0}]_w =_{def} \mathbf{0}_w$ and 1 to $[\mathbf{1}]_w =_{def} \mathbf{1}_w$;

(iii) the $\#_{w_1}$-embedding of (43) sends $\mathbf{0}$ to

$[\mathbf{0}]_{w_1} =_{def} \mathbf{0}_{w_1}$ and 1 to $[\mathbf{1}]_{w_1} =_{def} \mathbf{1}_{w_1}$;

We must lift now the operations and relations of $\mathbb{R}$ to ${}^{\#}\mathbb{R}$.

Definition 8.8. We get the clue from (35)-(37), which tells us when any two elements of ${}^{\#}\mathbb{R}$ are "equal":

(i) two elements $[\boldsymbol{f}]_s$ and $[\boldsymbol{g}]_s$ of ${}^{\#}\mathbb{R}$ are $\boldsymbol{s}$-equal:

$$[\boldsymbol{f}]_s =_s [\boldsymbol{g}]_s \text{ iff } \{\boldsymbol{i} \in_s \mathbf{V} | \boldsymbol{f}(\boldsymbol{i}) = \boldsymbol{g}(\boldsymbol{i})\} \in_s \boldsymbol{\mathcal{F}}^{inc}, \quad (47)$$

(ii) two elements $[\boldsymbol{f}]_w$ and $[\boldsymbol{g}]_w$ of ${}^{\#}\mathbb{R}$ are $\boldsymbol{w}$-equal:

$$[\boldsymbol{f}]_w =_w [\boldsymbol{g}]_w \text{ iff } \{\boldsymbol{i} \in_s \mathbf{V} | \boldsymbol{f}(\boldsymbol{i}) = \boldsymbol{g}(\boldsymbol{i})\} \in_w \boldsymbol{\mathcal{F}}^{inc}, \quad (48)$$

(iii) two elements $[\boldsymbol{f}]_{w_1}$ and $[\boldsymbol{g}]_{w_1}$ of ${}^{\#}\mathbb{R}$ are

$\boldsymbol{w_1}$-equal:

$$[\boldsymbol{f}]_{w_1} =_{w_1} [\boldsymbol{g}]_{w_1} \text{ iff } \{\boldsymbol{i} \in_s \mathbf{V} | \boldsymbol{f}(\boldsymbol{i}) = \boldsymbol{g}(\boldsymbol{i})\} \in_{w_1} \boldsymbol{\mathcal{F}}^{inc}, \quad (49)$$

(iv) two elements $[\boldsymbol{f}]_s$ and $[\boldsymbol{g}]_s$ of ${}^{\#}\mathbb{R}$ are

$\boldsymbol{w}$ -equal:

$$[\boldsymbol{f}]_s =_w [\boldsymbol{g}]_s \text{ iff } \{\boldsymbol{i} \in_w \mathbf{V} | \boldsymbol{f}(\boldsymbol{i}) = \boldsymbol{g}(\boldsymbol{i})\} \in_w \boldsymbol{\mathcal{F}}^{inc}, \quad (50)$$

(v) two elements $[\boldsymbol{f}]_s$ and $[\boldsymbol{g}]_s$ of ${}^{\#}\mathbb{R}$ are

$\boldsymbol{w_1}$ -equal:

$$[\boldsymbol{f}]_s =_{w_1} [\boldsymbol{g}]_s \text{ iff } \{\boldsymbol{i} \in_{w_1} \mathbf{V} | \boldsymbol{f}(\boldsymbol{i}) = \boldsymbol{g}(\boldsymbol{i})\} \in_{w_1} \boldsymbol{\mathcal{F}}^{inc}, \quad (51)$$

(vi) two elements $[\boldsymbol{f}]_w$ and $[\boldsymbol{g}]_w$ of ${}^{\#}\mathbb{R}$ are $\boldsymbol{w_1}$-equal:

$$[\boldsymbol{f}]_w =_{w_1} [\boldsymbol{g}]_w \text{ iff } \{\boldsymbol{i} \in_{w_1} \mathbf{V} | \boldsymbol{f}(\boldsymbol{i}) = \boldsymbol{g}(\boldsymbol{i})\} \in_{w_1} \boldsymbol{\mathcal{F}}^{inc}. \quad (52)$$

Definition 8.9. In a similar way we extend $<$ to ${}^{\#}\mathbb{R}$ by setting:

$$[\boldsymbol{f}]_s <_s [\boldsymbol{g}]_s \text{ iff } \{\boldsymbol{i} \in_s \mathbf{V} | \boldsymbol{f}(\boldsymbol{i}) < \boldsymbol{g}(\boldsymbol{i})\} \in_s \boldsymbol{\mathcal{F}}^{inc}, \quad (53)$$

$$[\boldsymbol{f}]_w <_w [\boldsymbol{g}]_w \text{ iff } \{\boldsymbol{i} \in_w \mathbf{V} | \boldsymbol{f}(\boldsymbol{i}) < \boldsymbol{g}(\boldsymbol{i})\} \in_w \boldsymbol{\mathcal{F}}^{inc}, \quad (54)$$

$$[\boldsymbol{f}]_{w_1} <_{w_1} [\boldsymbol{g}]_{w_1} \text{ iff } \{\boldsymbol{i} \in_s \mathbf{V} | \boldsymbol{f}(\boldsymbol{i}) < \boldsymbol{g}(\boldsymbol{i})\} \in_{w_1} \boldsymbol{\mathcal{F}}^{inc}, \quad (55)$$

$$[\boldsymbol{f}]_s <_w [\boldsymbol{g}]_s \text{ iff } \{\boldsymbol{i} \in_w \mathbf{V} | \boldsymbol{f}(\boldsymbol{i}) < \boldsymbol{g}(\boldsymbol{i})\} \in_w \boldsymbol{\mathcal{F}}^{inc}, \quad (56)$$

$$[\boldsymbol{f}]_s <_{w_1} [\boldsymbol{g}]_s \text{ iff } \{\boldsymbol{i} \in_{w_1} \mathbf{V} | \boldsymbol{f}(\boldsymbol{i}) < \boldsymbol{g}(\boldsymbol{i})\} \in_{w_1} \boldsymbol{\mathcal{F}}^{inc}, \quad (57)$$

$$[\boldsymbol{f}]_w <_{w_1} [\boldsymbol{g}]_w \text{ iff } \{\boldsymbol{i} \in_{w_1} \mathbf{V} | \boldsymbol{f}(\boldsymbol{i}) < \boldsymbol{g}(\boldsymbol{i})\} \in_{w_1} \boldsymbol{\mathcal{F}}^{inc}. \quad (58)$$

With this definition of $<$ in ${}^{\#}\mathbb{R}$ we easily show that the extended domain *R is linearly ordered. As an example we verify transitivity of $<$ in ${}^{\#}\mathbb{R}$: let $[\boldsymbol{f}]_s <_s [\boldsymbol{g}]_s$ and $[\boldsymbol{g}]_s <_s [\boldsymbol{h}]_s$, **i. e.,**

$$\wp_1 = \{\boldsymbol{i} \in_s \mathbf{V} | \boldsymbol{f}(\boldsymbol{i}) < \boldsymbol{g}(\boldsymbol{i})\} \in_s \boldsymbol{\mathcal{F}}^{inc} \text{ and}$$

$$\wp_2 = \{\boldsymbol{i} \in_s \mathbf{V} | \boldsymbol{g}(\boldsymbol{i}) < \boldsymbol{h}(\boldsymbol{i})\} \in_s \boldsymbol{\mathcal{F}}^{inc}.$$

By the finite intersection property from definition 8.5.(3), $\wp_1 \cap \wp_2 \in_s \boldsymbol{\mathcal{F}}^{inc}$. If $\boldsymbol{i} \in_s \wp_1 \cap \wp_2$, then $\boldsymbol{f}(\boldsymbol{i}) < \boldsymbol{g}(\boldsymbol{i})$ and $\boldsymbol{g}(\boldsymbol{i}) < \boldsymbol{h}(\boldsymbol{i})$. Therefore by transitivity of $<$ in $\mathbb{R}$, $\boldsymbol{f}(\boldsymbol{i}) < \boldsymbol{h}(\boldsymbol{i})$. And thus $\wp_1 \cap \wp_2 \subseteq_s \{\boldsymbol{i} \in_s \boldsymbol{V} | \boldsymbol{g}(\boldsymbol{i}) < \boldsymbol{h}(\boldsymbol{i})\} \in_s \boldsymbol{\mathcal{F}}^{inc}$, i.e., $[\boldsymbol{f}]_s <_s [\boldsymbol{g}]_s$.

Theorem 8.2.

(i) Let $[\boldsymbol{f}]_s <_s [\boldsymbol{g}]_s$ and $[\boldsymbol{g}]_s <_s [\boldsymbol{h}]_s$ then $[\boldsymbol{g}]_s <_s [\boldsymbol{h}]_s$

(ii)

Definition 8.10.

(i) A (positive) consistent infinitesimal $\boldsymbol{\varepsilon}$ in ${}^{\#}\mathbb{R}$ is an element $\boldsymbol{\varepsilon} \in {}^{\#s}\mathbb{R}$ such that ${}^{\#s}\mathbf{0} <_s \boldsymbol{\varepsilon} <_s {}^{\#s}\boldsymbol{r}$ for all $\boldsymbol{r} > \mathbf{0}, \boldsymbol{r} \in \mathbb{R}$.

(ii) A (positive) consistent infinite number $\boldsymbol{\omega}$ in ${}^{\#}\mathbb{R}$ is an element $\boldsymbol{\omega} \in {}^{\#s}\mathbb{R}$ such that ${}^{\#s}\boldsymbol{r} <_s \boldsymbol{\omega}$ for all $\boldsymbol{r} > \mathbf{0}, \boldsymbol{r} \in \mathbb{R}$.

(iii) A (positive) weakly consistent infinitesimal $\boldsymbol{\varepsilon}$ in ${}^{\#}\mathbb{R}$ is an element $\boldsymbol{\varepsilon} \in {}^{\#w}\mathbb{R}$ such that ${}^{\#w}\mathbf{0} <_w \boldsymbol{\varepsilon} <_w {}^{\#w}\boldsymbol{r}$ for all $\boldsymbol{r} > \mathbf{0}, \boldsymbol{r} \in \mathbb{R}$.

(iv) A (positive) weakly consistent infinite number $\boldsymbol{\omega}$ in ${}^{\#}\mathbb{R}$ is an element $\boldsymbol{\omega} \in {}^{\#w}\mathbb{R}$ such that ${}^{\#w}\boldsymbol{r} <_w \boldsymbol{\omega}$ for all $\boldsymbol{r} > \mathbf{0}, \boldsymbol{r} \in \mathbb{R}$.

(v) A (positive) inconsistent infinitesimal $\boldsymbol{\varepsilon}$ in ${}^{\#}\mathbb{R}$ is an element $\boldsymbol{\varepsilon} \in {}^{\#w_1}\mathbb{R}$ such that ${}^{\#w_1}\mathbf{0} <_{\#w_1} \boldsymbol{\varepsilon} <_{\#w_1} {}^{\#w_1}\boldsymbol{r}$ for all $\boldsymbol{r} > \mathbf{0}, \boldsymbol{r} \in \mathbb{R}$.

(vi) A (positive) inconsistent infinite number $\boldsymbol{\omega}$ in ${}^{\#}\mathbb{R}$ is an element $\boldsymbol{\omega} \in {}^{\#w_1}\mathbb{R}$ such that ${}^{\#w_1}\boldsymbol{r} <_{\#w_1} \boldsymbol{\omega}$ for all $\boldsymbol{r} > \mathbf{0}, \boldsymbol{r} \in \mathbb{R}$.

It remains to extend the operations $+$ and $\times$ to ${}^{\#}\mathbb{R}$.

Definition 8.11.

(i) $[\boldsymbol{f}_1]_s + [\boldsymbol{f}_2]_s =_s [\boldsymbol{g}]_s$ iff

$$\{\boldsymbol{i} \in_s \mathbf{V} | \boldsymbol{f}_1(\boldsymbol{i}) + \boldsymbol{f}_2(\boldsymbol{i}) = \boldsymbol{g}(\boldsymbol{i})\} \in_s \boldsymbol{\mathcal{F}}^{inc},$$

(ii) $[\boldsymbol{f}_1]_s \times [\boldsymbol{f}_2]_s =_s [\boldsymbol{g}]_s$ iff

$$\{\boldsymbol{i} \in_s \mathbf{V} | \boldsymbol{f}_1(\boldsymbol{i}) \times \boldsymbol{f}_2(\boldsymbol{i}) = \boldsymbol{g}(\boldsymbol{i})\} \in_s \boldsymbol{\mathcal{F}}^{inc},$$

(iii) $[\boldsymbol{f}_1]_w + [\boldsymbol{f}_2]_w =_w [\boldsymbol{g}]_w$ iff

$$\{\boldsymbol{i} \in_w \mathbf{V} | \boldsymbol{f}_1(\boldsymbol{i}) + \boldsymbol{f}_2(\boldsymbol{i}) = \boldsymbol{g}(\boldsymbol{i})\} \in_w \boldsymbol{\mathcal{F}}^{inc},$$

(iv) $[\boldsymbol{f}_1]_w \times [\boldsymbol{f}_2]_w =_w [\boldsymbol{g}]_w$ iff

$$\{\boldsymbol{i} \in_w \mathbf{V} | \boldsymbol{f}_1(\boldsymbol{i}) \times \boldsymbol{f}_2(\boldsymbol{i}) = \boldsymbol{g}(\boldsymbol{i})\} \in_w \boldsymbol{\mathcal{F}}^{inc},$$

(v) $[\boldsymbol{f}_1]_{w_1} + [\boldsymbol{f}_2]_{w_1} =_{w_1} [\boldsymbol{g}]_{w_1}$ iff

$$\{\boldsymbol{i} \in_{w_1} \mathbf{V} | \boldsymbol{f}_1(\boldsymbol{i}) + \boldsymbol{f}_2(\boldsymbol{i}) = \boldsymbol{g}(\boldsymbol{i})\} \in_{w_1} \boldsymbol{\mathcal{F}}^{inc},$$

(vi) $[\boldsymbol{f}_1]_{w_1} \times [\boldsymbol{f}_2]_{w_1} =_{w_1} [\boldsymbol{g}]_{w_1}$ iff

$$\{\boldsymbol{i} \in_{w_1} \mathbf{V} | \boldsymbol{f}_1(\boldsymbol{i}) \times \boldsymbol{f}_2(\boldsymbol{i}) = \boldsymbol{g}(\boldsymbol{i})\} \in_{w_1} \boldsymbol{\mathcal{F}}^{inc}.$$

With these definitions one can prove easily that ${}^{\#}\mathbb{R}$ is an paraconsistent ordered field extension of $\mathbb{R}$.

Definition 8.12. Let $\boldsymbol{F}$ be an $\boldsymbol{n}$-ary function on $\mathbb{R}$, i.e., $\boldsymbol{F}: \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}$. We introduce now:

(i) the extended consistent function ${}^{\#s}\boldsymbol{F}$ by the strong equivalence

$${}^{\#s}\boldsymbol{F}([\boldsymbol{f}_1]_s, \ldots, [\boldsymbol{f}_n]_s) =_s [\boldsymbol{g}]_s \text{ iff} \quad (59)$$

$$\{\boldsymbol{i} \in_s \mathbf{V} | \boldsymbol{F}(\boldsymbol{f}_1(\boldsymbol{i}), \ldots, \boldsymbol{f}_n(\boldsymbol{i})) = \boldsymbol{g}(\boldsymbol{i})\} \in_s \boldsymbol{\mathcal{F}}^{inc},$$

(ii) the extended weakly consistent function ${}^{\#w}\boldsymbol{F}$ by the weakly consistent equivalence

$${}^{\#w}\boldsymbol{F}([\boldsymbol{f}_1]_w, \ldots, [\boldsymbol{f}_n]_w) =_w [\boldsymbol{g}]_w \text{ iff}$$

(60)

$$\{\boldsymbol{i} \in_w \mathbf{V} | \boldsymbol{F}(\boldsymbol{f}_1(\boldsymbol{i}), \ldots, \boldsymbol{f}_n(\boldsymbol{i})) = \boldsymbol{g}(\boldsymbol{i})\} \in_w \boldsymbol{\mathcal{F}}^{inc},$$

(iii) the extended inconsistent function ${}^{\#w_1}\boldsymbol{F}$ by the inconsistent equivalence

$${}^{\#w_1}\boldsymbol{F}([\boldsymbol{f}_1]_{w_1}, \ldots, [\boldsymbol{f}_n]_{w_1}) =_{w_1} [\boldsymbol{g}]_{w_1} \text{ iff}$$

(61)

$$\{\boldsymbol{i} \in_{w_1} \mathbf{V} | \boldsymbol{F}(\boldsymbol{f}_1(\boldsymbol{i}), \ldots, \boldsymbol{f}_n(\boldsymbol{i})) = \boldsymbol{g}(\boldsymbol{i})\} \in_{w_1} \boldsymbol{\mathcal{F}}^{inc}.$$

Definition 8.13. Let $\boldsymbol{R} \subseteq \mathbb{R}^{\boldsymbol{n}}$ be an $\boldsymbol{n}$-ary relation on $\mathbb{R}$. We introduce now:

(i) the extended consistent relation ${}^{\#s}\boldsymbol{R}$ by the consistent condition

$$\langle [\boldsymbol{f}_1]_s, \ldots, [\boldsymbol{f}_n]_s \rangle \in_s {}^{\#s}\boldsymbol{R} \quad \text{iff}$$

$$\{\boldsymbol{i} \in_s \mathbf{V} | \langle \boldsymbol{f}_1(\boldsymbol{i}), \ldots, \boldsymbol{f}_n(\boldsymbol{i}) \rangle \in \boldsymbol{R}\} \in_s \boldsymbol{\mathcal{F}}^{inc},$$

(ii) the extended weakly consistent relation ${}^{\#w}\boldsymbol{R}$ by the weakly consistent condition

$$\langle [\boldsymbol{f}_1]_w, \ldots, [\boldsymbol{f}_n]_w \rangle \in_w {}^{\#w}\boldsymbol{R} \quad \text{iff}$$

$$\{\boldsymbol{i} \in_w \mathbf{V} | \langle \boldsymbol{f}_1(\boldsymbol{i}), \ldots, \boldsymbol{f}_n(\boldsymbol{i}) \rangle \in \boldsymbol{R}\} \in_w \boldsymbol{\mathcal{F}}^{inc},$$

(i) the extended inconsistent relation ${}^{\#w_1}\boldsymbol{R}$ by the inconsistent condition

$$\langle [f_1]_{w_1}, \dots, [f_n]_{w_1} \rangle \in_{w_1} {}^{\#_s}\boldsymbol{R} \quad \text{iff}$$

$$\{ i \in_{w_1} \mathbf{V} | \langle f_1(i), \dots, f_n(i) \rangle \in \boldsymbol{R} \} \in_{w_1} \mathcal{F}^{inc}.$$

Remark8.6. Note a few elementary observations on the #-extensions of subsets of $\mathbb{R}$:

(i) ${}^{\#_s}\emptyset$ is the consistent empty set in ${}^{\#}\mathbb{R}$, e.g., $\forall z \in_s {}^{\#}\mathbb{R} \neg (z \in_s {}^{\#_s}\emptyset)$,

(ii) ${}^{\#_w}\emptyset$ is the weakly consistent empty set in ${}^{\#}\mathbb{R}$, e.g., $\forall z \in_w {}^{\#}\mathbb{R} \neg (z \in_w {}^{\#_w}\emptyset)$,

(iii) ${}^{\#_{w_1}}\emptyset$ is the inconsistent empty set in ${}^{\#}\mathbb{R}$, e.g., $\forall z \in_{w_1} {}^{\#}\mathbb{R} \neg (z \in_{w_1} {}^{\#_w}\emptyset)$,

(iv) if $E \subseteq \mathbb{R}$ then ${}^{\#_s}r \in_s {}^{\#}\mathbb{R}$ for all $r \in \mathbb{R}$,

(v) if $E \subseteq \mathbb{R}$ then ${}^{\#_w}r \in_w {}^{\#}\mathbb{R}$ for all $r \in \mathbb{R}$,

(vi) if $E \subseteq \mathbb{R}$ then ${}^{\#_{w_1}}r \in_{w_1} {}^{\#}\mathbb{R}$ for all $r \in \mathbb{R}$,

Remark8.7. Note that # is a Boolean homomorphism in the sense that for any sets $E_1, E_2 \subseteq \mathbb{R}$:

(i) ${}^{\#_s}(E_1 \cup_s E_2) =_s {}^{\#_s}E_1 \cup_s {}^{\#_s}E_2$,

(ii) ${}^{\#_s}(E_1 \cap_s E_2) =_s {}^{\#_s}E_1 \cap_s {}^{\#_s}E_2$,

(iii) ${}^{\#_w}(E_1 \cup_w E_2) =_w {}^{\#_w}E_1 \cup_w {}^{\#_w}E_2$,

(iv) ${}^{\#_w}(E_1 \cap_w E_2) =_w {}^{\#_w}E_1 \cap_w {}^{\#_w}E_2$,

(v) ${}^{\#_{w_1}}(E_1 \cup_{w_1} E_2) =_{w_1} {}^{\#_{w_1}}E_1 \cup_{w_1} {}^{\#_{w_1}}E_2$,

(vi) ${}^{\#_{w_1}}(E_1 \cap_{w_1} E_2) =_{w_1} {}^{\#_{w_1}}E_1 \cap_{w_1} {}^{\#_{w_1}}E_2$.

Remark8.8. Note that for any sets $E_1, E_2 \subseteq \mathbb{R}$

(i) ${}^{\#_s}E_1 =_s {}^{\#_s}E_2$ iff $E_1 = E_2$,

(ii)

Remark8.9. By virtue of (59)-(61) the absolute-value function $|\circ|$ has an extension to ${}^{\#}\mathbb{R}$ that we will denote by the usual $|\circ|$ rather than the "correct" ${}^{\#}|\circ|$.

Definition8.14.

(i) An element $x \in_s {}^{\#_s}\mathbb{R}$ is called s-finite if

$|x| < {}^{\#_s}r$ for some $r \in \mathbb{R}, r > 0$,

Definition 8.15.

(1) An element $u \in_s \mathbf{W}(\mathbf{V}')$ is called an s-standard set if there is $x \in_s \mathbf{W}(\mathbf{V}) = \mathbf{V}$ such that $u =_s {}^{\#_s}x$.

(2)

(3) An element $u \in \mathbf{W}(\mathbf{V}')$ is called an internal set if there is $x \in \mathbf{W}(\mathbf{V}) = \mathbf{V}$ such that $u \in {}^{\#}x$.

Remark8.3. Note that: (i) any standard set is internal set and (ii) $\mathbf{W}(\mathbf{V}') \subseteq \mathbf{W}(\mathbf{V}) = \mathbf{V}$.

Theorem 8.2. Any element $u \in \mathbf{W}(\mathbf{V}')$ is internal set.

Proof. Assume that $u \in \mathbf{W}(\mathbf{V}')$. Then $u \in \mathbf{V} = \mathbf{V}' = {}^{\#}\mathbf{V}$ and therefore $u \in {}^{\#}\mathbf{V}$.

For a set S, let ${}^{\sigma}S = \{ {}^{\sigma}s | s \in S \}$. We identify any ${}^{\#}z$ with z for all z in $\mathbb{C}$. Hence, ${}^{\sigma}S = S$ if S is a subset of $\mathbb{C}$, e.g., ${}^{\sigma}\mathbb{C} = \mathbb{C}$, ${}^{\sigma}\mathbb{R} = \mathbb{R}$, etc.

Designations 8.4. Let $\mathbb{R}^+$, ${}^{\#}\mathbb{R}$, ${}^{\#}\mathbb{R}^+$, ${}^{\#}\mathbb{R}_0$, ${}^{\#}\mathbb{R}_0^+$, ${}^{\#}\mathbb{R}_\infty^+$, ${}^{\#}\mathbb{N}$, ${}^{\#}\mathbb{N}_\infty$, $\mathbf{fin}\ {}^{\#}\mathbb{R}^+ = {}^{\#}\mathbb{R}^+ / {}^{\#}\mathbb{R}_\infty^+$, denote the sets of positive real numbers, hyperreal numbers, positive hyperreal numbers, infinitesimal hyperreal numbers, positive infinitesimal hyperreal numbers, positive infinite hyperreal numbers, hypernatural numbers, infinite hypernatural numbers and finite positive hyperreal numbers respectively.

Designations 8.6. Let $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

Remark8.5. Note that: ${}^{\#}\mathbb{R}_\infty^+ = {}^{\#}\mathbb{R}^+/\mathbb{R}^+$, ${}^{\#}\mathbb{N}_\infty = {}^{\#}\mathbb{N}/\mathbb{N}$,

$${}^{\#}\widetilde{\mathbb{R}} = {}^{\#}\mathbb{R} \cup {}^{\#}\{\pm\infty\} =_{def} {}^{\#}\mathbb{R} \cup \{\pm {}^{\#}\infty\}.$$

Theorem 8.3. (i) Every nonempty internal subset of ${}^{\#}\mathbb{N}$ has a $<$ $-$least element at least in inconsistent sense. (ii) Every nonempty internal subset of ${}^{\#}\mathbb{R}$ with an upper bound has a $<$ $-$least upper bound at least in inconsistent sense. Proof. We prove (i), so let $A \subseteq {}^{\#}\mathbb{N}$ be internal. Then

$A \in {}^{\#}\mathbf{W_2(V)}$. One can express the fact that any nonempty internal subset X of ${}^{\#}\mathbb{N}$ has a least element by the condition $\varphi\left({}^{\#}\mathbb{N}, {}^{\#}\mathbf{W_2(V)}\right) =_{def} \forall X \in {}^{\#}\mathbf{W_2(V)}\left[(X \neq \emptyset) \wedge \left(X \subseteq {}^{\#}\mathbb{N}\right) \to X \textbf{ has } a < -\textbf{least element}\right]$, where we write $x < y$ for $(x <_s y) \vee (x <_w y)$. The condition that any $X \subseteq \mathbb{N}$ has a <-least element are: $\varphi(\mathbb{N}, \mathbf{W_2(V)}) =_{def} \forall X \in \mathbf{W_2(V)} \exists x \in X[\forall y \in X \neg(y < x)]$. We thus have a condition $\varphi(\mathbb{N}, \mathbf{W_2(V)})$ such that $\varphi(\mathbb{N}, \mathbf{W_2(V)})$ is true in $\mathbf{W_2(V)}$. By paraconsistent transfer $\varphi\left({}^{\#}\mathbb{N}, {}^{\#}\mathbf{W_2(V)}\right)$ is true in ${}^{\#}\mathbf{W_2(V)}$ at least in inconsistent sense.

Theorem 8.4. (i) The subset of infinite hypernatural numbers ${}^{\#}\mathbb{N}_\infty \subset {}^{\#}\mathbb{N}$ has a $<_{w_1}-$least element.

(ii) The subset of positive real numbers $\mathbb{R}^+ \subset {}^{\#}\mathbb{R}$ has a $<_{w_1}-$least upper bound.

(iii) Proof. Immediately follows by theorem 8.2 and theorem 8.3.

Remark8.5. Let $\breve{\mathbb{N}}_\infty$ be a $<_{w_1}-$least element of ${}^{\#}\mathbb{N}_\infty$.

It is clear that $\breve{\mathbb{N}}_\infty$ satisfy inconsistent properties:

(i) $\breve{\mathbb{N}}_\infty - n <_{w_1} \breve{\mathbb{N}}_\infty$ and

(ii) $\neg\left(\breve{\mathbb{N}}_\infty - n <_{w_1} \breve{\mathbb{N}}_\infty\right)$, where $n \in \mathbb{N}$.

However by restricted modus ponens

$$\left(\breve{\mathbb{N}}_\infty - n <_{w_1} \breve{\mathbb{N}}_\infty\right) \wedge \neg\left(\breve{\mathbb{N}}_\infty - n <_{w_1} \breve{\mathbb{N}}_\infty\right) \nvdash A.$$

Theorem 8.5. Peano induction postulate valid for ${}^{\#}\mathbb{N}$, e.g.,

$$\forall X \subseteq {}^{\#}\mathbb{N}\left[1 \in X \wedge \forall x[x \in X \to x+1 \in X] \to X = {}^{\#}\mathbb{N}\right] \quad (34)$$

at least in inconsistent sense.

Proof. The condition $\varphi\left({}^{\#}\mathbb{N}, {}^{\#}\mathbf{W_2(V)}\right)$ that for any $X \subseteq {}^{\#}\mathbb{N}$ Peano induction postulate valid for ${}^{\#}\mathbb{N}$ are:

$$\varphi\left({}^{\#}\mathbb{N}, {}^{\#}\mathbf{W_2(V)}\right) =_{def} \quad (35)$$

$$\forall X \in {}^{\#}\mathbf{W_2(V)}\left[1 \in X \wedge \forall x[x \in X \to x+1 \in X] \to X = {}^{\#}\mathbb{N}\right].$$

The condition $\varphi(\mathbb{N}, \mathbf{W_2(V)})$ that for any $X \subseteq \mathbb{N}$ Peano induction postulate valid for $\mathbb{N}$ are:

$$\varphi(\mathbb{N}, \mathbf{W_2(V)}) =_{def} \quad (36)$$

$$\forall X \in \mathbf{W_2(V)}[1 \in X \wedge \forall x[x \in X \to x+1 \in X] \to X = \mathbb{N}].$$

By paraconsistent transfer $\varphi\left({}^{\#}\mathbb{N}, {}^{\#}\mathbf{W_2(V)}\right)$ is true in ${}^{\#}\mathbf{W_2(V)}$ at least in inconsistent sense.

Remark8.5. Note that by other hand, theorem 8.5 follows directly from theorem 8.3. Proof. Assume that:

(i) there exist some $X \subseteq {}^{\#}\mathbb{N}$ such that $1 \in X \wedge \forall x[x \in X \to x+1 \in X]$ and

(ii) $X \neq {}^{\#}\mathbb{N}$ even in inconsistent sense, e.g., there exist some $n \in {}^{\#}\mathbb{N}$ such that: $\left(n \in {}^{\#}\mathbb{N}\right) \wedge (n \in X) \vdash A.$

Let $Y \subseteq {}^{\#}\mathbb{N}$ be a set:

$$Y = \left\{m \in {}^{\#}\mathbb{N} \middle| \left(m \in {}^{\#}\mathbb{N}\right) \wedge (m \in X) \vdash A\right\}. \quad (37)$$

By theorem 8.3 a set Y has a $<_{w_1}-$least element $r \in Y$ (at least in inconsistent sense) and therefore by statement (i) we obtain $r - 1 \in X$ at least in inconsistent sense. Then again by statement (i) we obtain $(r-1) + r = r \in X$. By definition (37) finally we obtain that there exist $r \in X$ and $r \in X \vdash A$. But this is a contradiction.

Definition 8.6. A hypersequence is a function whose domain is a hypernatural number or ${}^{\#}\mathbb{N}$. A shypersequence whose domain is some hypernatural number $n \in {}^{\#}\mathbb{N}$ is called a hyperfinite sequence of length n.

Definition 8.7. A function $t : (m + 1) \to A$ is called an m-step computation based on a and g if $t_0 = a$, and for all k such that $0 \le k < m, t_{k+1} = g(t_k, k)$.

Definition 8.8. For any set $A \subseteq {}^{\#}\mathbb{N}$ and any function $f : A \times {}^{\#}\mathbb{N} \to A$:

(i) Function f are called a weakly consistent, if whenever $f(x) =_w f(y)$, it must be the case that $x =_w y$ and both statements $\neg(f(x) = f(y))$ and $\neg(x =_w y)$ does not true in ${}^{\#}\mathbf{W_2(V)}$.

(ii) Function f are called inconsistent, if $f(x) =_w f(y)$, it must be the case that $x =_w y$ and both statement $\neg[(f(x) = f(y)) \wedge (x =_w y)]$ is true in ${}^{\#}\mathbf{W_2(V)}$.

Definition 8.9. For any set $A \subseteq {}^{\#}\mathbb{N}$ and any functions $f : A \times {}^{\#}\mathbb{N} \to A, g : A \times {}^{\#}\mathbb{N} \to A$:

(i) Functions f and g are called compatible if $f(x) = g(x)$ for all $x \in \textbf{dom } f \cap \textbf{dom } g$.

(ii) A set of functions F is called a compatible system of functions if any two functions f and

$\boldsymbol{g}$ from $\boldsymbol{F}$ are compatible.

(iii) Functions $\boldsymbol{f} \in {}^{\#}\mathbf{W}_2(\mathbf{V})$ and $\boldsymbol{g} \in {}^{\#}\mathbf{W}_2(\mathbf{V})$ are called a strongly compatible if $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbf{dom}\,\boldsymbol{f} \cap \mathbf{dom}\,\boldsymbol{g}$ and statement $\neg(\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}))$ does not true in ${}^{\#}\mathbf{W}_2(\mathbf{V})$.

(iv) A set of functions $\boldsymbol{F}$ is called a strongly compatible system of functions if any two functions $\boldsymbol{f}$ and $\boldsymbol{g}$ from $\boldsymbol{F}$ are strongly compatible.

Theorem 8.6.(1) If $\boldsymbol{F}$ is a compatible system of functions, then $\cup\,\boldsymbol{F}$ is a function with

$$\mathbf{dom}(\cup\,\boldsymbol{F}) = \bigcup\{\mathbf{dom}\boldsymbol{f}\,|\boldsymbol{f} \in \boldsymbol{F}\}.$$

The function $\mathbf{dom}(\cup\,\boldsymbol{F})$ extends all $\boldsymbol{f} \in \boldsymbol{F}$.

(2)If $\boldsymbol{F}$ is a strongly compatible system of functions, then $\cup\,\boldsymbol{F}$ is a function with $\mathbf{dom}(\cup\,\boldsymbol{F}) = \bigcup\{\mathbf{dom}\boldsymbol{f}\,|\boldsymbol{f} \in \boldsymbol{F}\}$.

Theorem 8.7. (**Recursion Theorem**) (**I**) For any set $\boldsymbol{A} \subseteq {}^{\#}\mathbb{N}$, any $\boldsymbol{a} \in \boldsymbol{A}$, and any function $\boldsymbol{g} : \boldsymbol{A} \times {}^{\#}\mathbb{N} \to \boldsymbol{A}$, there exists a hypersequence $\boldsymbol{f} : {}^{\#}\mathbb{N} \to \boldsymbol{A}$ such that

(1) $\boldsymbol{f}_{\mathbf{0}} = \boldsymbol{a}$

(2) $\boldsymbol{f}_{\boldsymbol{n}+\mathbf{1}} = \boldsymbol{g}(\boldsymbol{f}_{\boldsymbol{n}}, \boldsymbol{n})\ \forall \boldsymbol{n} \in {}^{\#}\mathbb{N}$

Proof. (The existence of $\boldsymbol{f}$)

Let $\boldsymbol{a} \in \boldsymbol{A}$, and $\boldsymbol{g} : \boldsymbol{A} \times {}^{\#}\mathbb{N} \to \boldsymbol{A}$. Let

$$\boldsymbol{F} = \{\boldsymbol{t} \in \boldsymbol{P}(\boldsymbol{N} \times \boldsymbol{A})\mid \boldsymbol{t}\ \mathbf{is\ an}\ \boldsymbol{m} - \mathbf{step\ computation}$$

$$\mathbf{on}\ \boldsymbol{a}\ \mathbf{and}\ \boldsymbol{g}\ \mathbf{for\ some}\ \boldsymbol{m} \in {}^{\#}\mathbb{N}\}.$$ Let $\boldsymbol{f} = \cup\,\boldsymbol{F}$.

Claim 1: $\boldsymbol{f}$ is a function.

IX.Carleson's theorem and generalizations in dimentions $N \geq 1$.

Carleson's celebrated theorem of 1965 [38] asserts the pointwise convergence of the partial Fourier sums of square integrable functions. The Fourier transform has a formulation on each of the Euclidean groups $\mathbb{R}, \mathbb{Z}$ and $\mathbf{T}$. Carleson's original proof worked on T. Fefferman's proof translates very easily to $\mathbb{R}$. Mate [39] extended Carleson's proof to $\mathbb{Z}$. Each of the statements of the theorem can be stated in terms of a maximal Fourier multiplier theorem [40]. Inequalities for such operators can be transferred between these three Euclidean groups, and was done P. Auscher and M.J. Carro [41]. But L. Carleson's original proof and another proofs very long and very complicated. A very short proof Carleson's theorem using Paraconsistent Transfer Principle is given in [36]. In contrast to Carleson's method, which is based on profound properties of trigonometric series, the proposed approach is quite general and allows to research a wide class of analogous problems for the general orthogonal series. Let us consider any general orthogonal series in Hilbert space $\boldsymbol{L}_2(\boldsymbol{\Omega}), \boldsymbol{\Omega} \subseteq \mathbb{R}$:

Conclusions

We pointed out that appropriate resolution of Curry's Paradox and Shaw-Kwei's paradox resolution can be given without rejection any contraction postulate.

References

[1] Curry, H. (1942): "The Inconsistency of certain formal logics", *Journal of Symbolic Logic* 7, 115–17.

[2] *M.Jago, Recent Work in Relevant Logic, Analysis (2013) 73 (3): 526-541. doi: 10.1093/analys/ant043*

[3] Anderson, A.R. and N.D. Belnap, Jr., 1975, *Entailment: The Logic of Relevance and Necessity*, Princeton, Princeton University Press, Volume I. Anderson, A.R. N.D. Belnap, Jr. and J.M. Dunn (1992)*Entailment*, Volume II. [These are both collections of slightly modified articles on relevance logic together with a lot of material unique to these volumes. Excellent work and still the standard books on the subject. But they are very technical and quite difficult.]C. Smorynski, Handbook of mathematical logic, Edited by J. Barwise.North-Holland Publishing Company,1977

[4] Brady, R.T., 2005, Universal Logic, Stanford: CSLI Publications, 2005. [A difficult, but extremely important book, which gives details of Brady's semantics and his proofs that naïve set theory and higher order logic based on his weak relevant logic are consistent.]D. Marker, Model theory: an introduction.(Graduate Texts in Mathematics, Vol. 217). Springer 2002.

[5] Dunn, J.M., 1986, "Relevance Logic and Entailment" in F. Guenthner and D. Gabbay (eds.), Handbook of Philosophical Logic, Volume 3, Dordrecht: Reidel, pp. 117–24. [Dunn has rewritten this piece together with Greg Restall and the new version has appeared in volume 6 of the new edition of the Handbook of Philosophical Logic, Dordrecht: Kluwer, 2002,

[6] Mares, E.D., Relevant Logic: A Philosophical Interpretation, Cambridge: Cambridge University Press. U. R. Schmerl, Iterated Reflection Principles and the ω-Rule. The Journal of Symbolic Logic, 2004,Vol. 47, No. 4, pp.721-733.

[7] Mares, E.D. and R.K. Meyer, 2001, "Relevant Logics" in L. Goble (ed.), The Blackwell Guide to Philosophical Logic, Oxford: Blackwell.S. Feferman,Systems of predicative analysis. Journal of Symbolic Logic 29:1-30.

[8] Paoli, F., 2002, Substructural Logics: A Primer, Dordrecht: Kluwer. [Excellent and clear introduction to a field of logic that includes relevance logic.]S. Feferman, C. Spector, Incompleteness along paths in progressions of theories. Journal of Symbolic Logic 27:383--390.

[9] Priest, G., 2008, An Introduction to Non-Classical Logic: From If to Is, Cambridge: University of Cambridge Press. P.Lindstrom, "First order predicate logic and generalized quantifiers", Theoria, Vol. 32, No.3. pp. 186-195, December 1966.

[10] Read, S., 1988, Relevant Logic, Oxford: Blackwell.

[11] Restall, G., 2000, An Introduction to Substructural Logics, London: Routledge.

[12] [10] Routley, R., R.K. Meyer, V. Plumwood and R. Brady, 1983,Relevant Logics and its Rivals. Volume I.

[13] R.Brady (ed.), Relevant Logics and their Rivals. Volume II.

[14] Anderson, A.R., 1967, "Some Nasty Problems in the Formal Logic of Ethics," *Noûs*, 1: 354–360.

[15] Belnap, N.D., 1982, "Display Logic," *Journal of Philosophical Logic*, 11: 375–417.

[16] [15] Robles, G., Méndez, J. M., Curry's Paradox, Generalized Modus Ponens Axiom and Depth Relevance, Studia Logica February 2014, Volume 102, Issue 1, pp. 185-217.

[17] [16] G. Robles, J. M. Méndez, Blocking the Routes to Triviality with Depth Relevance, Journal of Logic, Language and Information December 2014, Volume 23, Issue 4, pp. 493-526.

[18] Rogerson, S., Restall, G. Routes to Triviality, Journal of Philosophical Logic August 2004, Volume 33, Issue 4, pp. 421-436.

[19] Foukzon,J., Relevant First-Order Logic LP# and Curry's Paradox. April 2008 http://arxiv.org/abs/0804.4818

[20] Foukzon,J.,Paraconsistent First-Order Logic LP# with infinite hierarchy levels of contradiction. May 2008 http://arxiv.org/abs/0805.1481

[21] Shaw-Kwei, M., Logical paradoxes for many-valued systems. Journal of Symbolic Logic, pages 37-40, 1954.

[22] Restall. G., How to be really contraction free. Studia Logica, 52(3):381{391,1993.

[23] Bacon, A, Curry's Paradox and ω -Inconsistency, Studia Logica February 2013, Volume 101, Issue 1, pp. 1-9.

[24] Hajek, P., Paris, J., and Shepherdson, J., The liar paradox and fuzzy logic. Journal of Symbolic Logic, 65(1):339{346, 2000.

[25] Restall, G., Arithmetic and truth in Lukasiewicz's innitely valued logic. Logique et Analyse, 140:303{12, 1992.

[26] Restall, G., An introduction to substructural logics. Routledge, 2000.

[27] da Costa, N.C.A.: On the theory of inconsistent formal systems. Notre Dame Journal of Formal Logic 15 (1974) 497–510

[28] Urbas, I., Pαrαconsistency and the C-Systems of da Costa, Notre Dame Journal of Formal Logic Volume 30, Number 4, Fall 1989

[29] A.I. Arruda, Remarks In Da Costa's Paraconsistent Set Theories, Revista colombiana de matematicas. Volume 19 (1985) pp.9-24.

[30] Arruda, A.I. and da Costa, N.C.A. "Sur le schéma de la séparation", *Nagoya Mathcmatical Journal*, v. 38, (1970), pp. 71-84.

[31] Arruda, A.I. and da Costa, N.C.A., "Le schéma de la séparation et les calculs J_n", *Mathematica Japonicae*, v. 19, (1974), pp. 183-186.

[32] Bunder. M.V., On Arruda, A.I. and da Costa's logics J_1toJ_5, Journal of Non-Classical Logic 2,43-48.

[33] Urbas, I., CURRY'S PARADOX AND *MODUS PONENS,* Journal of Non-Classical Logic Volume 8, Number 1, May 1991.

[34] Urbas, I., "On subsystems of the system J_1 of Arruda and da Costa", to appear in *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik,* v. 36, (1990).

[35] Urbas, I., "Paraconsistency and subsystems of the system J_1 of Arruda and da Costa", Journal *of Non-Classical Logic* v. 5, (1988), pp. 49-73.

[36] Foukzon, J., Foundation of paralogical nonstandard analysis and its application to some famous problems of trigonometrical and orthogonal series. Part I,II Volume 2004, Issue 3, March 2004,Pages 343-356. Mathematics Preprint Archive, http://www.sciencedirect.com/preprintarchive

[37] Foukzon,J., Foundation of paralogical nonstandard analysis and its application to some famous problems of trigonometrical and orthogonal series 4ECM Stockholm 2004 Contributedpapers. http://www.math.kth.se/4ecm/poster.list.html

[38] Carleson,L. On convergence and growth of partial sumas of Fourier series, Acta Math.116 (1966), 135--157. 33 #7774

[39] Mate, A., Convergence of Fourier series of square integrable functions,Mat. Lapok 18 (1967), 195--242. MR 39 #701 (Hungarian, with English summary)

[40] Michael,T.L., Georgia Institute of Technology, Carleson's Theorem: Proof, Complements, Variations, arXiv:math.CA/0307008 v2 22 Oct 2003.

[41] Auscher, P.M., Carro,J., On relations between operators on $\mathbb{R}^N$,$\mathbf{T}^N$ and $\mathbb{Z}^N$,Studia Math.101 (1992),165--182. MR 94b:42007.