Proof of the existence of Transfinite Cardinals strictly smaller than $\aleph_0$ with an ensuing solution to the Twin Prime conjecture

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Abstract

In this paper the author submits a proof using the Power Set relation for the existence of a transfinite cardinal strictly smaller than $\aleph_0$, the cardinality of the Naturals. Further, it can be established taking these arguments to their logical conclusion that even smaller transfinite cardinals exist. In addition, as a lemma using these new found and revolutionary concepts, the author conjectures that some outstanding unresolved problems in number theory can be brought to heel. Specifically, a proof of the twin prime conjecture is given.

1 Introduction

In this paper, the author attempts to establish a controversial and, if true, a revolutionary advancement in the 'theory of the infinite' as pioneered by Georg Cantor in the 1870's. What Cantor stated, which is now almost universally accepted as true, is that there are different sizes of infinity with the cardinality of the Naturals i.e. the counting numbers representing the smallest possible infinity designated by $\aleph_0$ (aleph zero / aleph naught / aleph null). Cantor established this by showing that the Power Set relation is universally applicable and that the number of elements of the Power Set of any Set is strictly greater than the number of elements of the original Set under consideration. This is trivially true for Sets with finite cardinality but vide his remarkable proof Cantor was able to show that the same is true even for Sets with infinite elements. It can be shown that the Reals have cardinality $2^{\aleph_0}$ which is an 'infinity' strictly larger than the 'infinity' of the Naturals having cardinality $\aleph_0$. Further, one can keep repeating this procedure to get still higher cardinalities. What got the author wondering is that though we are allowed to 'travel up' from $\aleph_0$, can we somehow 'travel down', meaning; are they any cardinalities which are transfinite but strictly smaller than $\aleph_0$? The rest of this paper attempts to answer this question...

The Set of Naturals is enumerated as $\mathbb{N} = \{1, 2, 3, 4, 5, ...\}$
It is realized that the Set of Naturals continues indefinitely, there being no largest Natural number and the cardinality of this Set is represented by $\aleph_0$. Any number strictly smaller than $\aleph_0$ has to be a finite Natural number say n. It is this particular statement that is objected to. Although many would claim this to be obvious and the author too for the longest time was of this same opinion, the author has till date not come across a rigorous proof for the same. However, here is a simple argument as to why this could indeed be true…

As a side note, the author attempts his own reasoning, though later proven untenable, as to why $\aleph_0$ is the smallest infinity (transfinite cardinal) possible. What is being claimed is well accepted in the annals of Set theory, though as stated earlier, the author has not come across a rigorous proof of the same. The following equations can been arrived at by establishing a bijection between the Set of Naturals and a proper subset of the Naturals;
$\aleph_0 - 1 = \aleph_0$
$\aleph_0 - 2 = \aleph_0$
$\aleph_0 - 3 = \aleph_0$
and continuing in the same manner we can derive
$\aleph_0 - n = \aleph_0$ where n is any finite Natural number. So subtracting any finite Natural number (however arbitrarily large) from the countable infinity $\aleph_0$ will still return the same infinity $\aleph_0$. The only way to get a number that is strictly smaller than $\aleph_0$ is to consider a finite Natural number. From this we conclude that there is no infinity lesser than $\aleph_0$ or conversely any number strictly less than $\aleph_0$ will be finite.

Now, where can this simple argument go wrong? Let us assume that there is indeed a transfinite cardinal strictly in between the finite Naturals n and the infinite $\aleph_0$. Based on established notation of representing the increasing orders of aleph numbers by $\aleph_0$, $\aleph_1$, $\aleph_2$ and so on, any transfinite cardinal strictly smaller than $\aleph_0$, I will represent by $\aleph_{-1}$. Thus we have,
$n < \aleph_{-1} < \aleph_0 \ldots$ (vide assumption)
(Some may object to use of the Aleph symbol to represent $\aleph_{-1}$ and would insist on using the Beth symbol arguing that the author hasn't proved that $\aleph_{-1}$ is the next smaller transfinite cardinal post $\aleph_0$. In principle, it is agreed, but the reader is asked for some leeway in notation, as for now the main aim is to prove that a smaller infinity than $\aleph_0$ exists and the author is not too concerned whether it is the immediate smaller transfinite cardinal to $\aleph_0$.)
The following equations will now also hold
$\aleph_{-1} - n = \aleph_{-1}$
$\aleph_0 - n = \aleph_0$
(Note that the above is the same equation used to clinch the argument of $\aleph_0$ being the smallest transfinite cardinal)

Further, we also have
$\aleph_0 - \aleph_{-1} = \aleph_0$
It can be seen therefore that just by establishing $\aleph_0 - n = \aleph_0$ in itself may not be enough to prove that $\aleph_0$ is the smallest transfinite cardinal. In essence, the claim that the size of the Naturals i.e. $\aleph_0$ is the first transfinite cardinal as hypothesized by Cantor and generally accepted in the mathematical community without rigorous debate needs further scrutiny.

2 Tantalizing Hints

Some jugglery with Cardinal mathematics to reinforce this point...

2.1 Hint #1

We know $\aleph_0 < 2^{\aleph_0}$
Taking Logarithm's to base 2
$\log_2 \aleph_0 < \log_2 2^{\aleph_0}$
$\log_2 \aleph_0 < \aleph_0$

Possibility 1: $\log_2 \aleph_0$ is finite say n
$\log_2 \aleph_0 = n$
$\Rightarrow \aleph_0 = 2^n = finite$
As 2^n will be a finite Natural number for any Natural number n
A Contradiction...

Possibility 2: $\log_2 \aleph_0 = \aleph_0$ then
$\aleph_0 < \aleph_0$
A Contradiction...

The reader may well argue that taking Logarithms which applies to finite numbers does not necessarily translate to the infinite case and the inverse of the Power Set relation for $\aleph_0$ has not been shown to exist. The author agrees... At this stage, the reader is not being asked to accept these to be rigorous proofs but take them as hints of the possibility that $\aleph_{-1}$ may exist.

Before we proceed, let's postulate that indeed

$$n < \aleph_{-1} < 2^{\aleph_{-1}} = \aleph_0 \tag{1}$$

i.e. there exists a transfinite cardinal strictly smaller than $\aleph_0$ whose Power Set can be put in a bijection with $\aleph_0$ to which we assign a symbol $\aleph_{-1}$

2.2 Hint #2

In the same vein, consider that the number of primes '$\pi(n)$' up to any given number say n can be approximately given by the formula
$\pi(n) = \dfrac{n}{\ln(n)}$

If we extend this formula to cover the Naturals, we get RHS
$= \frac{\aleph_0}{\ln \aleph_0}$
Now
$\ln \aleph_0 = \ln 2 \cdot \log_2 \aleph_0$
$= \ln 2 \cdot \aleph_{-1}$
$= \aleph_{-1}$
Giving RHS as
$= \frac{\aleph_0}{\aleph_{-1}}$
$= \aleph_0$ as $\aleph_{-1} < \aleph_0$
proves quite easily that there are infinite primes as is already known by Euclid's theorem. However, if we take $\ln(\aleph_0) = \aleph_0$, the result will be either undefined or equal 1 or if we take $\ln(\aleph_0)$ as undefined, we cannot solve for the above equation. A note to the reader that the author has given this particular example not only because it has merit in its own right but as similar arguments will later be used, once we have proof that $\aleph_{-1}$ exists, to establish the twin prime conjecture... Till this stage, all we have are some interesting ideas. The author now proceeds to provide a proof of (1) i.e. there exists a transfinite cardinal strictly smaller than $\aleph_0$.
We now attempt a proof of (1) repeated below:

$$n < \aleph_{-1} < 2^{\aleph_{-1}} = \aleph_0$$

3 Proof of the existence of a Transfinite Cardinal strictly smaller than $\aleph_0$

We begin by first describing what will henceforth be referred to as the construction of a 'Power Set table'. Said construction is a systematic and procedural method of generating Power Sets of any given Set. The method involves constructing a table in which the first row represents the elements of the Set (whose Power Set is to be constructed) and the last column represents the elements of the Power Set so constructed. In this context, the word construction is to be taken not just as in ordinary English but also as in a rigorous method of assemblage of individual elements to from a Set.

To understand the steps involved, we first consider a finite Set example, say with four elements a, b, c, d. There is a systematic way [3] of constructing the Power Set table in which the elements of the Set are listed in the first row and below each element is written the number 1 or 0 to indicate whether it is included or not in the corresponding subset, for e.g.,

Bin2Dec	a	b	c	d	*subsets*
$0 = 0 \times 2^0 + 0 \times 2^1 + 0 \times 2^2 + 0 \times 2^3$	0	0	0	0	$\{\}$
$1 = 1 \times 2^0 + 0 \times 2^1 + 0 \times 2^2 + 0 \times 2^3$	1	0	0	0	$\{a\}$
$2 = 0 \times 2^0 + 1 \times 2^1 + 0 \times 2^2 + 0 \times 2^3$	0	1	0	0	$\{b\}$
$3 = 1 \times 2^0 + 1 \times 2^1 + 0 \times 2^2 + 0 \times 2^3$	1	1	0	0	$\{a, b\}$
$4 = 0 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 + 0 \times 2^3$	0	0	1	0	$\{c\}$
$5 = 1 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 + 0 \times 2^3$	1	0	1	0	$\{a, c\}$
$6 = 0 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 0 \times 2^3$	0	1	1	0	$\{b, c\}$
$7 = 1 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 0 \times 2^3$	1	1	1	0	$\{a, b, c\}$
$8 = 0 \times 2^0 + 0 \times 2^1 + 0 \times 2^2 + 1 \times 2^3$	0	0	0	1	$\{d\}$
$9 = 1 \times 2^0 + 0 \times 2^1 + 0 \times 2^2 + 1 \times 2^3$	1	0	0	1	$\{a, d\}$
$10 = 0 \times 2^0 + 1 \times 2^1 + 0 \times 2^2 + 1 \times 2^3$	0	1	0	1	$\{b, d\}$
$11 = 1 \times 2^0 + 1 \times 2^1 + 0 \times 2^2 + 1 \times 2^3$	1	1	0	1	$\{a, b, d\}$
$12 = 0 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 + 1 \times 2^3$	0	0	1	1	$\{c, d\}$
$13 = 1 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 + 1 \times 2^3$	1	0	1	1	$\{a, c, d\}$
$14 = 0 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 1 \times 2^3$	0	1	1	1	$\{b, c, d\}$
$15 = 1 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 + 1 \times 2^3$	1	1	1	1	$\{a, b, c, d\}$

The Set of elements a, b, c, d is given in the second to fifth columns. For the time being, the author requests the reader to ignore the very first column '*Bin2Dec*'; it will be explained later. Below each element a, b, c, d, a digit 0 or 1 is placed. The corresponding subset in the sixth column will contain the element if a 1 is indicated and will not contain the element if a 0 is indicated. The sixth column lists the subsets based on this exclusion / inclusion 0/1 rule. It is to be realized that by all possible combinations of 0 and 1's, one can generate all possible subsets of a given Set. A rigorous way to construct this table is to consider the 0's and 1's as binary digits and keep incrementing them each succeeding row. This way all combinations of 0's and 1's are systematically exhausted. A point of note: I am incrementing the binary digits in reverse i.e. binary addition is done from left to right rather than the traditional right to left, a variation which is useful when dealing with infinite Sets. The rules for incrementing are simple; $0 + 0 = 0$, $0 + 1 = 1$, $1 + 1 = 0$ with 1 carried over to the element to the immediate right. One key realization is that the first element is the empty Set corresponding to all 0's (in this case 0000) and the last element of our table is generated when all the elements of the Set are included giving all 1's (in this case 1111).
No. of columns of the table = No. of elements of the Set = 4
No. of rows of the table = No. of all possible subsets (i.e. no. of elements of the Power Set)$= 2^4 = 16$
To the author, this represents an elegant and rigorous method of generating all subsets of any given Set.

Now, let us consider the very first column '*Bin2Dec*', which till now we have kept in abeyance. This column is appended to the Power Set table as it will be required for our proof. If the aim is to just generate a Power Set, this first column is not required. This column '*Bin2Dec*' on the left treats the 1's and 0's as binary digits whose decimal conversions are calculated. Do note that as indicated earlier, I am reading the binary numbers from right to left rather than the traditional left to right. For e.g., the second row is to be read as being the binary number 0001 rather than the traditional way of 1000 and so on. The entire column, as expected, has 16 elements whose decimal conversions range from 0 to 15.

To get to our proof, we now extend this procedure to infinite Sets. Specifically, consider a Set as given in the first row in the Table below. Although it is tempting to think of Set X given as $\{1, 2, 3, 4, 5, 6, ...\}$ to be the Set of Naturals, the author requests that for the time being the reader not jump to this conclusion and keep an open mind. As previously indicated, the last column enumerates the subsets given by the exclusion / inclusion 0/1 rule. The first column calculates the decimal conversions of the corresponding binary numbers; one small modification being that we are adding one to the decimal value of the corresponding binary sequence so that the decimal number 1 is paired with the null Set $\{\}$ and so on.

$Bin2Dec + 1$	1	2	3	4	5	6	...	$subsets$
1	0	0	0	0	0	0	...	$\{\}$
2	1	0	0	0	0	0	...	$\{1\}$
3	0	1	0	0	0	0	...	$\{2\}$
4	1	1	0	0	0	0	...	$\{1, 2\}$
5	0	0	1	0	0	0	...	$\{3\}$
6	1	0	1	0	0	0	...	$\{1, 3\}$
7	0	1	1	0	0	0	...	$\{2, 3\}$
8	1	1	1	0	0	0	...	$\{1, 2, 3\}$
9	0	0	0	1	0	0	...	$\{4\}$
...	...	...	...	...	...	...	...	...

Now, we form a Set Y of the denary numbers so formed by enumerating the table. The Set Y is given as
$Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, ...\}$
Now comes the essential and, to the author, the beautiful part. We end table construction when the Set Y exhausts the entire Naturals i.e. when $Y = \mathbb{N}$. Enormous repercussions follow:
$|Y| = |\mathbb{N}| = \aleph_0$
$|\wp(X)| = |Y|$
$\therefore |\wp(X)| = \aleph_0$
i.e. $2^{|X|} = \aleph_0$
(Further, though not essential to our proof, do note $\wp(X) \sim Y$ i.e. $\wp(X)$ is similar to Y) which from (1) of our definition for $\aleph_{-1}$ becomes

$$|X| = \aleph_{-1} < 2^{\aleph_{-1}} = \aleph_0 = |\wp(X)|$$

Thus there exists a Set given by X of transfinite cardinality $\aleph_{-1}$ strictly smaller than $\aleph_0$. As earlier indicated $\aleph_{-1}$ cannot be finite say n as 2^n will be a finite Natural number for any Natural number n. This completes the simple and to the author elegant proof... In this proof, the author has of course used Cantor's famous theorem that the Power Set of any Set has strictly greater elements than the original Set.

One possible objection from the reader could be that the author has not ensured that while we exhaust the Naturals in all the rows, the very last row of the table contains all 1's to ensure proper and complete Power Set construction. The author accepts this criticism but will provide a simple solution to the problem. In this regard, we will need to take the help of the ordinal numbers. The smallest ordinal associated with $\aleph_0$ is ω. Let us take the smallest ordinal associated with $\aleph_{-1}$ as psi 'ψ' (Some ordinal will have to be assigned to $\aleph_{-1}$ which also implies that a revised theory of ordinals would now be required to take into account the cardinal $\aleph_{-1}$). The table is redrawn below assuming two extreme possibilities. The simplest option being that once we exhaust all the Naturals we indeed have all 1's in the last row. The worst case option being that once we exhaust all the Naturals, we are at the earliest stage with 1 being present only in the last column. Any other combination of 0's and 1's will represent some intermediate stage.

Case 1: The ideal case in which last row corresponds to all 1's and the Naturals are exhausted simultaneously.

$Bin2Dec+1$	1	2	3	4	...	ψ	$subsets$
1	0	0	0	0	...	0	$\{\}$
2	1	0	0	0	...	0	$\{1\}$
3	0	1	0	0	...	0	$\{2\}$
4	1	1	0	0	...	0	$\{1,2\}$
5	0	0	1	0	...	0	$\{3\}$
6	1	0	1	0	...	0	$\{1,3\}$
7	0	1	1	0	...	0	$\{2,3\}$
8	1	1	1	0	...	0	$\{1,2,3\}$
9	0	0	0	1	...	0	$\{4\}$
...	...	...	...	...	...	...	...
ω	1	1	1	1	...	1	$\{1,2,3,4,...,\psi\}$

Now
$ord\{1,2,3,...,\omega\} = \omega + 1$
$card\{1,2,3,...,\omega\} = \aleph_0$
$ord\{1,2,3,...,\psi\} = \psi + 1$
$card\{1,2,3,...,\psi\} = \aleph_{-1}$
Thus the rows exhaust the Naturals and the Power Set table is completed simultaneously, resulting in the Set in the first row being transfinite and having strictly lesser cardinality than that of the Naturals as earlier demonstrated.

Case 2: The non-ideal case in which the last row does not contain all 1's but a mixture of 1's and 0's indicating that the Power Set table is still incomplete. Let us take the extreme case of this in which we travel all the way up to the previous series of all 1's and then take one step down to the row in which we have all 0's except the last column having a single 1. The completed table looks like below:

$Bin2Dec + 1$	1	2	3	4	...	ψ	$subsets$
1	0	0	0	0	...	0	$\{\}$
2	1	0	0	0	...	0	$\{1\}$
3	0	1	0	0	...	0	$\{2\}$
4	1	1	0	0	...	0	$\{1,2\}$
5	0	0	1	0	...	0	$\{3\}$
6	1	0	1	0	...	0	$\{1,3\}$
7	0	1	1	0	...	0	$\{2,3\}$
8	1	1	1	0	...	0	$\{1,2,3\}$
9	0	0	0	1	...	0	$\{4\}$
...	...	...	...	...	...	...	...
ω	0	0	0	0	...	1	$\{\psi\}$
$\omega + 1$	1	0	0	0	...	1	$\{1,\psi\}$
$\omega + 2$	0	1	0	0	...	1	$\{2,\psi\}$
$\omega + 3$	1	1	0	0	...	1	$\{1,2,\psi\}$
...	...	...	...	...	...	...	...
$(\omega - 1) \cdot 2 = \omega 2 - 2$	1	1	1	1	...	1	$\{1,2,3,4,...,\psi\}$

It is to be noted that if the ordinal number corresponding to the row having $0,0,0,0,...,1$ is x, then the ordinal number corresponding to the row having $1,1,1,1,...,1$ is $(x-1) \cdot 2$ which is the formula used to determine the last row; given row ω having $0,0,0,0,...,1$. Now,
$ord\{1,2,3,...,\omega\} = \omega + 1$
$card\{1,2,3,...,\omega\} = \aleph_0$
$ord\{1,2,3,...,\omega,\omega+1,\omega+2,\omega+3,...,\omega 2-2\} = \omega 2 - 1$
$card\{1,2,3,...,\omega,\omega+1,\omega+2,\omega+3,...,\omega 2-2\} = \aleph_0$
Similarly
$ord\{1,2,3,...,\psi\} = \psi + 1$
$card\{1,2,3,...,\psi\} = \aleph_{-1}$
thus the cardinality of the Set given by the rows does not change being the same for both cases. Thus the Proof of the Power Set table holds...

Using this Power Set table concept, we can construct any general Set S with say elements $S = \{x_1, x_2, x_3, x_4, x_5, ...\}$ whose cardinality will be $|S| = \aleph_{-1} < \aleph_0$ as shown below:

$Bin2Dec + 1$	x_1	x_2	x_3	x_4	x_5	...	x_ψ	$subsets$
1	0	0	0	0	0	...	0	$\{\}$
2	1	0	0	0	0	...	0	$\{x_1\}$
3	0	1	0	0	0	...	0	$\{x_2\}$
4	1	1	0	0	0	...	0	$\{x_1, x_2\}$
5	0	0	1	0	0	...	0	$\{x_3\}$
6	1	0	1	0	0	...	0	$\{x_1, x_3\}$
7	0	1	1	0	0	...	0	$\{x_2, x_3\}$
8	1	1	1	0	0	...	0	$\{x_1, x_2, x_3\}$
9	0	0	0	1	0	...	0	$\{x_4\}$
...	...	...	...	...	...	...	...	...
ω	1	1	1	1	1	...	1	$\{x_1, x_2, x_3, x_4, ..., x_\psi\}$

4 Helpful Hints to visualize this Set

How does one visualize this Set that seems to go on like the Naturals but stops well before and is still transfinite. The author makes the following thought experiment. Consider the following Set
$S = \{2^1, 2^2, 2^3, 2^4, 2^5, ...\} = \{2, 4, 8, 16, 32, ...\}$
It can be realized that the magnitude of each element represents the number of elements of the Power Set. For e.g. for a Set with three elements $\{2, 4, 8\}$, the third element 8 corresponds to the number of elements of the Power Set of $\{2, 4, 8\}$. In essence, the Power Set of $\{2, 4, 8\}$ can be put in a bijection with the Set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, with both the original Set and the Power Set sharing the same last element.

Now the Naturals play two roles

1. as counting the number of elements of a Set (cardinal number)
2. as ordering and identifying the last element of a Set (ordinal number)
   The author asks the reader to also consider an option 3
3. helps represent the magnitude of an element

Using option 3, the reader can visualize that as we exhaust the Naturals, the Set S reaches the same magnitude (not cardinality or ordinarily) of the last element i.e. ω simultaneously. Further, as shown by the Power Set relation, Set S reaches the said magnitude quicker, in fact, strictly so and thus has strictly lesser cardinality.

5 Even smaller transfinite cardinals

The same powerful concept of the Power Set can then be used to prove the existence of even smaller transfinite cardinals. The Set of cardinality $\aleph_{-1}$ so obtained in the first row by constructing the Power Set table is now flipped to the first column and a fresh Power Set table is constructed to give in the first row a Set having cardinality strictly smaller than $\aleph_{-1}$ which the author will designate as $\aleph_{-2}$

$\aleph_{-2} < 2^{\aleph_{-2}} = \aleph_{-1}$
Using similar reasoning as above we try to visualize this Set. We defined Set S as
$\mathbb{N} = \{1, 2, 3, 4, 5, ...\}$
$S = \{2^1, 2^2, 2^3, 2^4, 2^5, ...\} = \{2, 4, 8, 16, 32, ...\}$
Continuing in the same vein, we can define another Set T as
$T = \{2^{2^1}, 2^{2^2}, 2^{2^3}, 2^{2^4}, 2^{2^5}, ...\} = \{2^2, 2^4, 2^8, 2^{16}, 2^{32}, ...\}$
$= \{4, 16, 256, 65536, 4294967296, ...\}$
which using similar arguments as earlier, will reach a magnitude of infinity quicker than Set S.

Further, we can extend the Power Set table construction to prove the existence of even smaller transfinite cardinals $\aleph_{-3}, \aleph_{-4}, \aleph_{-5}$ and so on and establish the general relation $2^{\aleph_{-a}} = \aleph_{-a+1}$ for $a = 1, 2, 3, ...$

In essence, what referred to as the 'ladder of infinity' [2] with $\aleph_0$ being the smallest infinity and the Power Set of $\aleph_0$ being strictly larger and so on can now be rephrased with the same 'ladder of infinity' not only ascending from $\aleph_0$ but also descending from $\aleph_0$ with say $\aleph_0$, the countable infinity being the 'central rung'. To me, this seems a more elegant formulation of Cantorian Set theory...

6 Solution to the Twin Prime conjecture

The above result opens up new avenues for mathematicians. The author can think of one unresolved problem in Number theory which can be bought to heel based on this new finding. I refer to the 'Twin Prime conjecture'. Twin primes are Sets of prime numbers differing by 2. For e.g. $(17, 19)$, $(41, 43)$ etc. are known as twin primes. The conjecture states that there are infinitely many such primes. Based on earlier work by Viggo Brun and more recently as established by Christopher Hooley [1], the Brun – Hooley estimate for the number of twin primes is of the Order

$$\pi_2(x) = O\left(\frac{n}{(\log n)^2}\right)$$

which can be reformulated into

$$\pi_2(x) = \frac{cn}{(\log n)^2} \text{ for some constant } c > 0.$$

To get to the solution, we extend this formula to when we exhaust the Naturals to get

$$\frac{c\aleph_0}{(\log \aleph_0)^2}$$

Note 1:
log from any base can be converted to log to base 2 by multiplication of a constant, for e.g. $(\log_a n)^2 = \left(\frac{\log_2 n}{\log_2 a}\right)^2$. This just leads to a modification of our constant from c to c'.

Note 2:
Traversing the Power Set table in one direction given us the exponentiation relation

$2^{\aleph_{-1}} = \aleph_0$
Similarly traversing the same table in the other direction lets us rigorously define the Logarithm relation to the base 2 which is inverse of the exponentiation given as
$\aleph_{-1} = \log_2 \aleph_0$

Note 3:
$\aleph_{-1}^2 = \aleph_{-1}$ can be proved just like $\aleph_0^2 = \aleph_0$ by enumerating a Set of a pair of elements; the elements taken from a Set with cardinality $\aleph_{-1}$.

Note 4:
Similar to $\frac{\aleph_0}{n} = \aleph_0$ and $\frac{2^{\aleph_0}}{\aleph_0} = 2^{\aleph_0}$, as $\aleph_{-1} < \aleph_0$, similarly $\frac{\aleph_0}{\aleph_{-1}} = \aleph_0$

Therefore
$\frac{c\aleph_0}{(\log \aleph_0)^2}$
$= \frac{c'\aleph_0}{(\log_2 \aleph_0)^2}$
$= c' \frac{\aleph_0}{(\aleph_{-1})^2}$
$= c' \frac{\aleph_0}{\aleph_{-1}}$
$= c'\aleph_0$
$= \aleph_0$
Thus there are infinitely many twin primes...

7 Conclusion

A proof using the construction of a Power Set table is presented for the existence of a transfinite cardinal $\aleph_{-1}$ strictly smaller than $\aleph_0$. Further, taking these arguments to their logical conclusion it has been shown that even smaller transfinite cardinals exist. In addition, as a lemma using these new-found and revolutionary concepts, a proof of the twin prime conjecture is given.

References

[1] Christopher Hooley. On an almost pure sieve. *Acta Arithmetica*, 66:359–368, 1994.

[2] Roger Penrose. *The Road to Reality: A Complete Guide to the Laws of the Universe*, page 357. Vintage, London, UK, 2005.

[3] Charles Petzold. *The Annotated Turing: A Guided Tour through Alan Turing's Historic Paper on Computability and the Turing Machine*, chapter 2, pages 30–31. Wiley Publishing Inc., Indianapolis, Indiana, USA, 2008.