

A note on the harmonic series and the logarithm

Martin Schlueter

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Abstract

Numerical evidence for an (assumed) new relationship between the harmonic series H_n and the natural logarithm $\log(n)$ is presented. The formula $H_n - \log(n)$ for the Euler-Mascheroni constant γ is adopted accordingly and reveals its (arguable) most elementary form.

The logarithm can be understood as part(s) of the harmonic series:

$$\underbrace{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{N}}_{H_N} + \underbrace{\frac{1}{N+1} \cdots \frac{1}{N+N^2}}_{\approx \log(N)} + \underbrace{\frac{1}{N+N^2+1} \cdots \frac{1}{N+N^3}}_{\approx \log(N)} + \cdots$$

This relationship can be simplified as follows:

$$\underbrace{\frac{1}{1} + \cdots \frac{1}{N}}_{H_N} + \underbrace{\cdots \frac{1}{N^2}}_{\approx \log(N)} + \underbrace{\cdots \frac{1}{N^3}}_{\approx \log(N)} + \underbrace{\cdots \frac{1}{N^4}}_{\approx \log(N)} + \cdots$$

which implies the (arguable) most elementary formulation of the Euler-Mascheroni constant:

$$\gamma = \lim_{n \rightarrow \infty} \frac{1}{1} + \cdots + \frac{1}{n} - \cdots - \frac{1}{n^2}$$

Numerical results:

Approximation of $\log(n)$	n = 1,000	n = 10,000	n = 100,000
$\log(n) - (H_{n^2+n} - H_n)$	-5.0008e-04	-5.0001e-05	-5.0000e-06
$\log(n) - (H_{n^3+n} - H_{n^2+n})$	9.9900e-04	9.9990e-05	9.9999e-06
$\log(n) - (H_{n^4+n} - H_{n^3+n})$	9.9950e-07	9.9995e-09	1.0000e-10

Approximation of γ	n = 1,000	n = 10,000	n = 100,000
$\gamma - (H_n - (H_{n^2+n} - H_n))$	1.6650e-07	1.6666e-09	1.6665e-11
$\gamma - (H_n - (H_{n^3+n} - H_{n^2+n}))$	-0.0014989	-1.4999e-04	-1.5000e-05
$\gamma - (H_n - (H_{n^4+n} - H_{n^3+n}))$	-5.0092e-04	-5.0009e-05	-5.0001e-06
$\gamma - (H_n - \log(n))$	-4.9992e-04	-4.9999e-05	-5.0000e-06