A note on the harmonic series and the logarithm

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Abstract

An (assumed) new relationship between the harmonic series H_n and the natural logarithm $log(n)$ is presented.

The logarithm can be understood as part(s) of the harmonic series:

$$\underbrace{\frac{1}{1}+\cdots\frac{1}{N}}_{H_N} + \underbrace{\cdots\frac{1}{N^2}}_{\approx\, log(N)} + \underbrace{\cdots\frac{1}{N^3}}_{\approx\, log(N)} + \underbrace{\cdots\frac{1}{N^4}}_{\approx\, log(N)} + \quad \cdots$$

which implies the (arguable) most elementary formulation of the Euler-Mascheroni constant:

$$\gamma = \lim_{n\to\infty} \quad \frac{1}{1}+\cdots+\frac{1}{n}-\cdots-\frac{1}{n^2}$$

Numerical results:

Approximation of $log(n)$	n = 1,000	n = 10,000	n = 100,000
$log(n)-(H_{n^2+n}-H_n)$	-5.0008e-04	-5.0001e-05	-5.0000e-06
$log(n)-(H_{n^3+n}-H_{n^2+n})$	9.9900e-04	9.9990e-05	9.9999e-06
$log(n)-(H_{n^4+n}-H_{n^3+n})$	9.9950e-07	9.9995e-09	1.0000e-10

Approximation of γ	n = 1,000	n = 10,000	n = 100,000
$\gamma-(\frac{1}{1}+\cdots+\frac{1}{n}-\cdots-\frac{1}{n^2})$	-9.9933e-04	-9.9993e-05	9.9999e-06
$\gamma-(H_n-(H_{n^2+n}-H_n))$	1.6650e-07	1.6666e-09	1.6665e-11
$\gamma-(H_n-(H_{n^3+n}-H_{n^2+n}))$	-0.0014989	-1.4999e-04	-1.5000e-05
$\gamma-(H_n-(H_{n^4+n}-H_{n^3+n}))$	-5.0092e-04	-5.0009e-05	-5.0001e-06
$\gamma-(H_n-log(n)$	-4.9992e-04	-4.9999e-05	-5.0000e-06

Proof:

$$\begin{aligned} H_{n^{a+1}}-H_{n^a} &\approx (log(n^{a+1})+\gamma)-(log(n^a)+\gamma) \\ &\approx (log(n^a)+log(n)+\gamma)-(log(n^a)+\gamma) \approx log(n) \end{aligned}$$