A CONDITION BY PAUL OF VENICE (1369-1429) SOLVES RUSSELL'S PARADOX, BLOCKS CANTOR'S DIAGONAL ARGUMENT, AND PROVIDES A CHALLENGE TO ZFC

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Abstract

Paul of Venice (1369-1429) provides a consistency enhancer that resolves Russell's Paradox in naive set theory without using a theory of types. It allows a set of all sets. It also blocks the (diagonal) general proof of Cantor's Theorem on the power set. It is not unlikely that the Zermelo-Fraenkel (ZFC) axioms for set theory are still too lax on the notion of a 'well-defined set'. The transfinites of ZFC may be a mirage, and a consequence of still imperfect axiomatics in ZFC w.r.t. the proper foundations for set theory. For amendment of ZFC two alternatives are mentioned: ZFC-PV (amendment of de Axiom of Separation) or BST (Basic Set Theory).

Keywords: Paul of Venice • Russell's Paradox • Cantor's Theorem • ZFC • naive set theory • well-defined set • set of all sets • diagonal argument • transfinites

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1. Introduction

Aristotle gave the first formalisation of the notions of *none*, *some* and *all*, of which an origin can be found in the Greek language. This developed into modern set theory, in which the notion of a *set* provides for the *all*. There is a parallel between constants in propositional logic and set theory: *and* giving *intersection*, *or* giving *union, implication* giving *subset*. Still, different axioms give different systems. A common contrast is between the formal ZFC system (from Zermelo, Fraenkel and the Axiom of Choice) and *naive set theory* (not quite defined, but perhaps Frege's system, and not to be confused with Halmos's verbal description of ZFC). There is a plethora - perhaps an infinity - of models for properties of sets.

In naive set theory, Russell's set is $R = \{x \mid x \notin x\}$. Subsequently $R \in R \Leftrightarrow R \notin R$ and naive set theory collapses. Russell's problem was a blow to Frege's system, and researchers spoke about a crisis in the foundations of logic and mathematics. The idea of a crisis was eventually put to rest by the ZFC system. A consequence of ZFC is a 'theory of types', so that a set cannot be member of itself, and with the impossibility of a 'set of all sets'.

Define however $S = \{x \mid x \in S \;\&\; x \notin x\}$, i.e. with the small consistency enhancer inspired by the discussion by Bochenski (1956, 1970:250) of Paulus Venetus or Paul of Venice

(1368-1428). We find $S \in S \Leftrightarrow \{S \in S \,\&\, S \notin S\}$, which reduces to $S \notin S$ without contradiction. There is no reason for a crisis in the foundations of logic and mathematics and there is no need for a theory of types - though you can use them if needed. It is not clear what Russell's set would be, since it is inconsistent, but who wants to work sensibly with some related notion can use S without problem.

An objection to ZFC is that a theory of types forbids the set of all sets while it is a useful concept. For formalisation of an alternative to ZFC there are at least two approaches. One approach is to forbid the formation of R by always requiring the Paul of Venice consistency enhancer. Alternatively we can allow that R is formally acceptable: then we need a three-valued logic to determine that R is nonsense. (It has meaning, that allows us to see that it is nonsense.) Observe that a theory of types has R in the category 'may not be formed' and thus already implies a 'third category' next to truth and falsehood. It would be illogical to reject such a third category. It is logical instead to generalise that third category to the general notion of 'nonsense'. This gives a three-valued logic with values *true, false, nonsense*. It remains an issue that three-valued logic is not without its paradoxes, but Colignatus (1981, 2007, 2011) holds that these can be solved too.

A closely related issue is what *infinity* actually means. When set theory (with perhaps infinite models) is used to help to explain *infinity* then there might be an infinite number of possible meanings for *infinity*. The real question becomes what would be consistent systems, and what systems might be used for what practical purposes. A critical property of ZFC is that it also allows for *transfinites*, and without models in reality those might be a mere product of nonsense.

The notion of infinity brings us to Cantor's Theorem on the power set. This theorem would hold in ZFC (see below). It need not hold if we amend ZFC.

Colignatus (1981, 2007, 2011:239) (ALOE) already (re-) presented the Paul of Venice consistency enhancer in 1981 for the Russell set, and applied it in 2007 also to Cantor's (diagonal) argument for the power set. ALOE does not develop ZFC however. Thus ALOE's discussion might be seen as intermediate between naive set theory and this present paper. The Appendix discusses the versions of ALOE, for proper reference.

The new issue in this paper is the challenge to the ZFC axioms. The ZFC system may be still too lax on the notion of a 'well-defined set'. The transfinites of ZFC may be a mirage, and a consequence of still imperfect axiomatics of ZFC w.r.t. the foundations for set theory.

The following sections will make the argument formal. Section 2 reviews that diagonal argument, section 3 gives the challenge to ZFC.

2. Review of the standard proof of Cantor's Theorem

It is with some apology that this article now presents some material from a matricola course in mathematics. When a paper challenges a widely accepted theorem then the reader may require a substantial argument and a detailed reconstruction of the proof. Conventionally it would be necessary to go to the source too. In this case Cantor presented his theorem before ZFC existed, and our focus is rather on the challenge to ZFC. It suffices to restate the matricola material to show how we arrive at that challenge for ZFC. We take the course that is in use at the universities of Leiden and Delft for students majoring in mathematics. The online syllabus is by Coplakova et al. (2011), and the issue concerns theorem I.4.9, pages 18-19. We translate Dutch into English, also using the proof supplement by Edixhoven in Colignatus (2014).

2.1. Cantor's Theorem and its standard proof

Definition (Coplakova et al. (2011:144-145)): ZFC.

Definition (Coplakova et al. (2011:18), I.4.7): Let A be a set. The power set of A is the set of all subsets of A. Notation: $P[A]$. (Another notation is 2^A.)

Cantor's Theorem (Coplakova et al. (2011:18), I.4.9): Let A be a set. There is no surjective function $f : A \rightarrow P[A]$.

Proof (Coplakova et al. (2011:19), replacing their B by Φ, and inserting a [NB]): Assume that there is a surjective function $f : A \rightarrow P[A]$. Now consider the set $\Phi = \{x \in A \mid x \notin f[x]\}$. [NB (nota bene): Prove (iii) and (iv) below.] Since $\Phi \subseteq A$ we also have $\Phi \in P[A]$. Because of the assumption that f is surjective, there is a $\varphi \in A$ with $f[\varphi] = \Phi$. There are two possibilities: (i) $\varphi \in \Phi$ or (ii) $\varphi \notin \Phi$. If (i) then $\varphi \in \Phi$. Thus also $\varphi \in f[\varphi]$. From the definition of Φ it follows $\varphi \notin f[\varphi]$ or $\varphi \notin \Phi$. Thus (i) gives a contradiction. If (ii) then we know $\varphi \notin \Phi$ and thus also $\varphi \notin f[\varphi]$. With the definition of Φ it follows that $\varphi \in \Phi$. Thus (ii) gives a contradiction too. Both cases (i) and (ii) cannot apply, and hence we find a contraction. Q.E.D.

Colignatus (2014) records the following NB supplement to this proof, provided on request by professor Edixhoven of Leiden, holding that Φ belongs to ZFC because of the Axiom of Separation. Given this supplement, it now should be clearer that above standard proof actually provides a challenge to ZFC. If ZFC allows a paradoxical construct then one may feel that ZFC needs amendment.

Addendum for above Proof (writing out NB): (iii) Φ is in ZFC, (iv) ZFC provides for well-defined sets.

Proof for (iii) (Edixhoven in Colignatus (2014), appendix D): (a) $P[A]$ exists because of the Axiom of the Powerset. (b) Note that f can be regarded as a subset of $A \times P[A]$. Then f exists because of Axiom of Pairing. (c) Φ exists because of the Axiom of Separation. Q.E.D.

Proof for (iv): Not available. This is not proven but remains an assumption. (Finding a model in reality would be sufficient but might not be necessary.)

Comments:

(1) Colignatus (1981, 2007, 2011:239) used the version with the bijection and the following shorter proof. Regard an arbitrary set A. Let f: $A \rightarrow 2^A$ be the hypothetical bijection. Let $\Phi = \{x \in A \mid x \notin f[x]\}$. Clearly Φ is a subset of A and thus there is a $\varphi = f^{-1}[\Phi]$ so that $f[\varphi] = \Phi$. The question now arises whether $\varphi \in \Phi$ itself. We find that $\varphi \in \Phi \Leftrightarrow \varphi \notin f[\varphi] \Leftrightarrow \varphi \notin \Phi$ which is a contradiction. Ergo, there is no such f. This concludes the standard proof of Cantor's theorem.

(2) From the contradiction derived above, the proper conclusion is not that Cantor's Theorem is proven, but only that it is proven in ZFC. Either Cantor's Theorem is true *or* ZFC doesn't yet provide for well-defined sets.

(3) Sets A and B have 'the same size' when there is a bijection or one-to-one function between them. Cantor's Theorem holds that a set is always 'smaller' than its power set. For finite sets this can be proven by mathematical induction too. The standard proof, and in particular for infinite sets, uses a construction that strongly reminds of Russell's paradox (deconstructed in section 1 above).

(4) The Axiom of Separation blocks Russell's paradoxical set, but doesn't block Cantor's paradoxical Φ yet.

(5) Colignatus (1981, 2007, 2011) (ALOE) is on logic and inference and thus keeps some distance from number theory and issues of the infinite. Historically, logic developed parallel to geometry and theories of the infinite (Zeno's paradoxes). Aristotle's syllogisms with *none, some* and *all* helped to discuss the infinite. Yet, to develop logic and inference proper, it appeared that ALOE could skip the tricky bits of number theory, non-Euclidean geometry, the development of limits, and Cantor's development of the transinfinite. Though it is close to impossible to discuss logic without mentioning the subject matter that logic is applied to, ALOE originally kept and keeps some distance from those subjects themselves. But, if logic uses the notion of *all,* it seems fair to ask whether there are limits to the use of this *all*. Thus it is explained why this present paper came about.

(6) It must also be observed that this author is no expert on Cantor's Theorem. We may reject the standard proof but perhaps there are other proofs. A marginal check shows that this proof is the only one given at various locations that seem to matter but this may only mean that it is a popular proof. For now, we have reproduced that standard proof and will now reproduce the refutation using the Paul of Venice consistency criterion, following Colignatus (1981, 2007, 2011:239).

The subsequent discussion intends to show that the standard proof cannot be accepted. For the discussion below, relabel Φ in this subsection 2.1 into Φ'.

2.2. Rejection of this proof (in ALOE)

We might hold that above Φ' is badly defined since it is self-contradictory under the hypothesis. A badly defined 'something' may just be a weird expression and need not represent a true set. A test on this line of reasoning is to insert a small consistency condition, giving us $\Phi = \{x \in A \mid x \notin f[x] \;\&\; x \in \Phi\}$. Now we conclude that $\varphi \notin \Phi$ since it cannot satisfy the condition for membership, i.e. we get $\varphi \in \Phi \Leftrightarrow (\varphi \notin f[\varphi] \;\&\; \varphi \in \Phi) \Leftrightarrow (\varphi \notin \Phi \;\&\; \varphi \in \Phi) \Leftrightarrow$ *falsum*. Puristically speaking, the Φ defined in 2.1 differs lexically from the Φ defined here, with the first expression being nonsensical and the present one consistent. It will be useful to reserve the term Φ for the proper definition in 2.2, and use Φ' for the expression in 2.1. The latter symbol is part of the lexical description but does not meaningfully refer to a set. Using this, we can also use $\Phi^* = \Phi \cup \{\varphi\}$ and we can express consistently that $\varphi \in \Phi^*$. So the 'proof' in 2.1 can be seen as using a confused mixture of Φ and Φ^*.

3. The challenge to ZFC

3.1. What is the difference between Φ' in 2.1 and Φ in 2.2 ?

Above deduction in section 2 poses a challenge to ZFC. Sets *R* and *S* above were in naive set theory, so it has relatively little meaning - for now - to ask about the difference between *R* and *S*. However, Φ' in 2.1 and Φ in 2.2 are in ZFC, and thus the question is (more) meaningful. Users of ZFC will have a hard time trying to clarify (a) that the consistency enhancer should have no effect but (b) actually does have an effect. To try to answer the question we might use the axiom of extensionality, see Coplakova et al. (2011:145):

$(A = B) \Leftrightarrow ((\forall x)\,(x \in A \;\Leftrightarrow x \in B))$

I have not pursued this question further since I have no vested interest in ZFC. I have requested Edixhoven who agrees with (a) to explain (b), and to describe the relation between Φ' in 2.1 and Φ in 2.2. I leave it to him or other users of ZFC to clarify this.

My solution of this issue is that Φ' in 2.1 is badly defined and that Φ in 2.2 is well-defined. Accepting that Φ' is ill-defined (rejecting (iv) above) has the effect of the collapse of the standard proof to Cantor's theorem. I am interested in an argument to the contrary but haven't seen it yet.

3.2. Amendments to the Axiom of Separation in ZFC

The proof in 2.1 relies on the separation axiom in ZFC.

Definition of the Axiom of Separation (Coplakova et al. (2011:145), adding a by-line on freedom): If A is a set and $\varphi[x]$ is a formula with variable x, then there exists a set B that consists of the elements of A that satisfy $\varphi[x]$, while B is not free in $\varphi[x]$:

$(\forall A)\ (\exists B)\ (\forall x)\ (x \in B \iff ((x \in A)\ \&\ \varphi[x]))$

Note the condition "B is not free in $\varphi[x]$". The consistency enhancer by Paul of Venice in the definition of Φ in 2.2 uses $\varphi'[x] = (\varphi[x]\ \&\ (x \in \Phi))$, in which $B = \Phi$ is not free since it is bound by the existential quantifier $(\exists B)$. Thus the formation of Φ in 2.2 is allowed in ZFC.

To meet the challenge in 3.1 we would require the PV-enhancer in general.

Possibility 3.2.1: Amendment by Paul of Venice to the Axiom of Separation:

$(\forall A)\ (\exists B)\ (\forall x)\ (x \in B \iff ((x \in A)\ \&\ \varphi[x]\ \&\ (x \in B)))$

In this case, 2.1 is no longer possible, the proof for Cantor's theorem collapses, and question 3.1 disappears since Φ' becomes ill-formed and nonsensical. My suggestion is to call this the 'neat' solution, and use the abbreviation ZFC-PV.

Another possibility is to move from ZFC closer to naive set theory, discard the axiom of separation, and adopt an axiom that allows greater freedom to create sets from formulas.

Possibility 3.2.2: Discard the separation axiom and have extensionality of formula's:

$(\forall \varphi)\ (\exists B)\ (\forall x)\ (x \in B \iff ((x \in B)\ \&\ \varphi[x]))$

This axiom protects against Russell's paradox and destroys the standard proof of Cantor's theorem. This resulting system might be called ZFC-S+PV.

The Axiom of Regularity forbids that sets are member of themselves. Instead, it is useful to be able to speak about the set of all sets. Though it is another discussion my suggestion is to drop this axiom too, then to call this the 'basic' solution, and use the abbreviation BST (basic set theory), thus BST = ZFC-S+PV-R. I would also propose a rule that the enhancer could be dropped in particular applications if it could be shown to be superfluous. However, for paradoxical $\varphi[x]$ it would not be superfluous.

I am not aware of a contradiction yet. I have not looked intensively for such a contradiction, since my presumption is that others are better versed in set theory and that the problem only is that those authors aren't aware of the potential relevance of the consistency enhancer by Paul of Venice. A question for historians is: Zermelo (1871-1953) and Fraenkel (1891-1965) might have embraced the Paul of Venice's enhancer if they had been aware of it.

4. Conclusion

Colignatus (1981, 2007, 2011) concludes, and we now supplement with the questions on ZFC:

1. The standard proof for Cantor's Theorem (given above) is based upon a badly defined and inherently paradoxical construct. This proof evaporates once a sound construct is used.

2. The theorem is proven for finite sets by means of induction but is still unproven for (vaguely defined) infinite sets: that is, this author is not aware of other proofs. We would better speak about "Cantor's Impression" or "Cantor's Supposed Theorem". It is not quite a

conjecture since Cantor might not have done such a conjecture (without proof) if he would have known about above refutation.

3. It becomes feasible to speak again about the 'set of all sets'. This has the advantage that we do not need to distinguish (i) sets versus classes, (ii) *all* versus *any*.

4. The transfinites that are defined by using "Cantor's Theorem" evaporate with it.

5. The distinction between the natural and the real numbers now rests (only) upon the specific diagonal argument (that differs from the standard proof). See Colignatus (2012, 2013) for the conclusion that Cantor's original proof for the natural and real numbers evaporates too, specifically for a convenient level of constructivity. (This paper indeed looks at Cantor's original argument in German.)

6. Users of ZFC should give an answer to 3.1, and clarify why they accept 2.1 and not 2.2 that has a better definition of a well-defined set. ZFC might be consistent but allows the construction of a 'proof' for "Cantor's Impression" that generates the transfinites, which makes one wonder what this system is a model for. We can agree with Cantor that the essence of mathematics lies in its freedom, but the freedom to create nonsense somehow would no longer be mathematics proper. Useful alternatives are in ZFC-PV or BST.

7. The prime importance of this discussion lies in education. Mathematics education should respect that education itself is an empirical issue. Teachers should respect the logic that students can grasp and not burden mathematics education with the confusions of the past. Colignatus (2012, 2013) clarifies that highschool education could be served well with a theory of the infinite that consistently develops both the natural and real numbers, without requiring more than the denumerable infinite ($\aleph \sim \Re$), using the notion of *bijection by abstraction*. It was Cantor himself who emphasised the freedom in mathematics, but that freedom is limited if alternatives are not mentioned. Even a university course like Coplakova et al. (2011) currently presents students only with "Cantor's Theorem" without mentioning the alternative analysis in Colignatus (1981, 2007, 2011).

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I thank Richard Gill (Leiden) for various discussions, and Bas Edixhoven (Leiden) for reminding me of the link to the ZFC separation axiom.

Appendix

The following comments are relevant for accurate reference.

(1) Colignatus (1981, 2007, 2011) existed first unpublished in 1981 as *In memoriam Philetas of Cos,* then in 2007 rebaptised and self-published. It was both retyped and programmed in the computer-algebra environment of *Mathematica* to allow ease of use of three-valued logic. In 2011 it was marginally adapted with a new version of *Mathematica*. At that moment it could also refer to a new rejection of Cantor's particular argument for the natural and real numbers, using the notion of *bijection by abstraction* - in 2011 still called *bijection in the limit* but now developed in Colignatus (2012, 2013).

(2) Gill (2008) reviewed the 1st edition of ALOE of 2007. That edition refers to Cantor's standard set-theoretic argument and rejects it, as in the above. ALOE refers to Wallace (2003) as the book that caused me to look into the issue again. Wallace's book is critically reviewed by Harris (2004). It will be useful to mention that ALOE does not rely on Wallace's book but indeed only mentions it as a source of inspiration to look into the issue again.

(3) Gill (2008) did not review the 2nd edition of ALOE of 2011. That edition also refers to Cantor's original argument on the natural and real numbers in particular. That edition of ALOE mentions the suggestion that $\aleph \sim \Re$. The discussion itself is not in ALOE but is now in Colignatus (2012, 2013), using the notion of *bijection by abstraction*.

(4) A visit to a restaurant and subsequent e-mail exchange led to the memo Colignatus (2014), and the inspiration to write this present article on the challenge to ZFC. Edixhoven also refers to Coplakova et al. (2011), theorem I.4.9, pp. 18-19, that gives the standard theorem and proof, also reproduced and challenged in above section 2.

(5) Colignatus (1981, 2007, 2011) is a book on logic and not a book on set theory. It presents the standard notions of naive set theory (membership, intersection, union) and the standard axioms for first order predicate logic that of course are relevant for set theory. But I have always felt that discussing *axiomatic* set theory (with ZFC) was beyond the scope of the book and my actual interest and developed expertise. This present paper is in my sentiment rather exploratory, by discussing axiomatic set theory in section 3 and actually presenting two possible alternatives.

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