A CONDITION BY PAUL OF VENICE (1369-1429) SOLVES RUSSELL'S PARADOX, BLOCKS CANTOR'S DIAGONAL ARGUMENT, AND PROVIDES A CHALLENGE TO ZFC

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Abstract

Paul of Venice (1369-1429) provides a consistency condition that resolves Russell's Paradox in naive set theory without using a theory of types. It allows a set of all sets. It also blocks the (diagonal) general proof of Cantor's Theorem (in Russell's form, for the power set). It is not unlikely that the Zermelo-Fraenkel (ZFC) axioms for set theory are still too lax on the notion of a 'well-defined set'. The transfinites of ZFC may be a mirage, and a consequence of still imperfect axiomatics in ZFC w.r.t. the proper foundations for set theory. For amendment of ZFC two alternatives are mentioned: ZFC-PV (amendment of de Axiom of Separation) or BST (Basic Set Theory).

Keywords: Paul of Venice • Russell's Paradox • Cantor's Theorem • ZFC • naive set theory • well-defined set • set of all sets • diagonal argument • transfinites

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1. Introduction

Aristotle gave the first formalisation of the notions of *none*, *some* and *all*, of which an origin can be found in the Greek language. This developed into modern set theory, in which the notion of a *set* provides for the *all*. There is a parallel between constants in propositional logic and set theory: *and* giving *intersection*, *or* giving *union, implication* giving *subset*. Still, different axioms give different systems. A common contrast is between the formal ZFC system (from Zermelo, Fraenkel and the Axiom of Choice) and *naive set theory* (not quite defined, but perhaps Frege's system, and not to be confused with Halmos's verbal description of ZFC). There is a plethora - perhaps an infinity - of models for properties of sets.

In naive set theory, Russell's set is $R = \{x|\ x \notin x\}$. Subsequently $R \in R \Leftrightarrow R \notin R$ and naive set theory collapses. Russell's problem was a blow to Frege's system, and researchers spoke about a crisis in the foundations of logic and mathematics. The idea of a crisis was eventually put to rest by the ZFC system. A consequence of ZFC is a 'theory of types', so that a set cannot be member of itself, and with the impossibility of a 'set of all sets'.

Define however $S = \{x|\ (x \notin x)\ \&\ (\text{If}\,(x = S)\,\text{then}\,(x \in S))\}$ i.e. with the small consistency condition inspired by the discussion by Bochenski (1956, 1970:250) of Paulus Venetus or Paul of Venice (1368-1428). The consistency condition with the exception switch was presented in Colignatus *"A Logic of Exceptions"* (1981, 2007, 2011:129) (ALOE).

The *If*-switch gives a dynamic process of going through the steps, and it is not a mere static implication. We find $S \in S \Leftrightarrow (S \notin S\ \&\ S \in S)$, which reduces to $S \notin S$ without contradiction. One might hold that there would be infinite regress, if a test on *S* on the left causes a test on *S* on the right, which causes a test on the left again, and so on; but the truthtable of $A \Leftrightarrow (\neg A\ \&\ A)$ allows a formal decision.

It is not clear what Russell's set would be, since it is inconsistent; but who wants to work sensibly with a related notion can use *S* without problem. There is no reason for a crisis in the foundations of logic and mathematics and there is no need for a theory of types - though you can use them if needed.

PM 1. The dynamic *If*-switch may be replaced by static $S = \{x|\ (x \notin x)\ \&\ ((x = S) \Rightarrow (x \in S))\}$ but then the truthtable is a bit more involved. **PM 2**. Obviously $S = \{x \neq S|\ x \notin x\}$ has the same effect, but this has the suggestion of choice, while the point is that one must show that the property $x \neq S$ is necessary. **PM 3**. In some texts I have used the shorthand form $S = \{x|\ x \notin x\ \&\ x \in S\}$, as shorthand only. This allows students an introductory focus on *S*. Experts however do not regard themselves as students who need education; they quickly recognise that this shorthand form causes infinite regress when $x \neq S$, and then they put this analysis aside, disappointed that it contains such an elementary confusion. However, the shorthand only indicates the intuition by Paul of Venice on the Liar paradox, that must be developed into modern consistency for sets. It is rather curious that this intuition doesn't inspire the experts on set theory.

The use of a shorthand form remains useful, and thus I propose the following notation.

Notation: $V = \{x|\ f[x]\ \&\&\ x \in V\}$, with non-symmetric '&&', stands for the longer $V = \{x|\ f[x]\ \text{unless}\,(f[x]\ \&\ x \in V)\,\text{is contradictory (also formally, preventing infinite regress)}\}$. Alternatively $V = \{x|\ \text{If}\,(f[x]\ \&\ x \in V) \Leftrightarrow \textit{falsum}\ \text{then}\ \textit{falsum}\ \text{else}\ f[x]\}$ in which the first test can be formal again without infinite regress. In static logic this reduces to $V = \{x|\ f[x]\ \&\ x \in V\}$ but the idea is the dynamic switch, in which it is tested first whether the

Unless-condition reduces to a falsehood, formally without infinite regress, and if not, then the unprotected original rule $f[x]$ is applied.

Also: $V = \{\backslash x | f[x] /\}$ means $V = \{x | f[x] \,\&\&\, x \in V\}$.

Example: In the above we could write $S = \{\backslash x | (x \notin x) /\}$ - and compare this with R.

An objection to ZFC is that a theory of types forbids the set of all sets while it is a useful concept. For formalisation of an alternative to ZFC there are at least two approaches. One approach is to forbid the formation of *R* by always requiring the Paul of Venice consistency condition. Alternatively we can allow that *R* is formally acceptable: then we need a three-valued logic to determine that *R* is nonsense. (It has meaning, that allows us to see that it is nonsense.) Observe that a theory of types has *R* in the category 'may not be formed' and thus already implies a 'third category' next to truth and falsehood. It would be illogical to reject such a third category. It is logical instead to generalise that third category to the general notion of 'nonsense'. This gives a three-valued logic with values *true, false, nonsense*. It remains an issue that three-valued logic is not without its paradoxes, but Colignatus (1981, 2007, 2011) holds that these can be solved too.

A closely related issue is what *infinity* actually means. When set theory (with perhaps infinite models) is used to help to explain *infinity* then there might be an infinite number of possible meanings for *infinity*. The real question becomes what would be consistent systems, and what systems might be used for what practical purposes. A critical property of ZFC is that it also allows for *transfinites*, and without models in reality those might be a mere product of nonsense.

The notion of infinity brings us to Cantor's Theorem, in this paper in the form which Bertrand Russell created for the power set (Hart (2015:42 first column)). This theorem would hold in ZFC (see below). It need not hold if we amend ZFC.

Colignatus (1981, 2007, 2011:239) (ALOE) already (re-) presented in 1981 the Paul of Venice consistency condition for the Russell set, and applied it in 2007 also to Cantor's (diagonal) argument (in Russell's version for the power set). ALOE does not develop ZFC however. Thus ALOE's discussion might be seen as intermediate between naive set theory and this present paper. **Appendix A** discusses the versions of ALOE, for proper reference.

The new issue in this paper is the challenge to the ZFC axioms. The ZFC system may still be too lax on the notion of a 'well-defined set'. The transfinites of ZFC may be a mirage, and a consequence of still imperfect axiomatics of ZFC w.r.t. the foundations for set theory.

The following sections will make the argument formal. Section 2 reviews that diagonal argument, section 3 gives the challenge to ZFC. Section 4 concludes. **Appendix B** discusses recent Hart (2015) who covers the same topic in traditional manner. **Appendix C** refutes a related theorem and proof, communicated by Hart in 2012. **Appendix D** is an 'initial review' by an editor of a peer-reviewed journal for the December 31 2014 version of this paper.

2. Review of the standard proof of Cantor's Theorem

It is with some apology that this article now presents some material from a matricola course in mathematics. When a paper challenges a widely accepted theorem then the reader may require a substantial argument and a detailed reconstruction of the proof. Conventionally it would be necessary to go to the source too. In this case Cantor presented his theorem before ZFC existed, and our focus is rather on the challenge to ZFC. It suffices to restate the matricola material to show how we arrive at that challenge for ZFC. We take the course that is in use at the universities of Leiden and Delft for students majoring in mathematics. The online syllabus is by Coplakova et al. (2011), and the issue concerns theorem I.4.9, pages 18-19. We translate Dutch into English, also using the proof addendum by Edixhoven in Colignatus (2014).

Since this paper will refer to various forms of "Cantor's Theorem" it will be useful to collect them in a table, and see also Hart (2015) and Appendices B & C for a discussion.

Author & date	*Theorem*	*Refutation*
Cantor 1874	Reals are nondenumerable, via intervals	CCPO-PCWA 2012
Cantor 1890/91	Diagonal argument, binary, bijection	CCPO version 2007j
Russell 1907	Power set theorem, using bijection ("common")	ALOE 2007
Coplakova et al. 2011	Power set theorem, using surjection ("standard")	Here, Section 3
Hart 2012	Weakest theorem underlying Cantor's Theorem	Here, Appendix C

2.1. Cantor's Theorem and its standard proof

Definition (Coplakova et al. (2011:144-145)): ZFC.

Definition (Coplakova et al. (2011:18), I.4.7): Let *A* be a set. The power set of *A* is the set of all subsets of *A*. Notation: *P*[*A*]. (Another notation is 2^A.)

Cantor's Theorem (for the power set, Hart (2015:42)) (Coplakova et al. (2011:18), I.4.9): Let *A* be a set. There is no surjective function $f : A \rightarrow P[A]$.

Proof (Coplakova et al. (2011:19), replacing their *B* by Φ, and inserting a [*NB*]): Assume that there is a surjective function $f : A \rightarrow P[A]$. Now consider the set $\Phi = \{x \in A \mid x \notin f[x]\}$. [*NB* (nota bene): Prove (iii) and (iv) below.] Since $\Phi \subseteq A$ we also have $\Phi \in P[A]$. Because of the assumption that *f* is surjective, there is a $\varphi \in A$ with $f[\varphi] = \Phi$. There are two possibilities: (i) $\varphi \in \Phi$ or (ii) $\varphi \notin \Phi$. If (i) then $\varphi \in \Phi$. Thus also $\varphi \in f[\varphi]$. From the definition of Φ it follows $\varphi \notin f[\varphi]$ or $\varphi \notin \Phi$. Thus (i) gives a contradiction. If (ii) then we know $\varphi \notin \Phi$ and thus also $\varphi \notin f[\varphi]$. With the definition of Φ it follows that $\varphi \in \Phi$. Thus (ii) gives a contradiction too. Both cases (i) and (ii) cannot apply, and hence we find a contraction. Q.E.D.

The insertion of [*NB*] is relevant here. Colignatus (2014) records this addendum by professor Edixhoven of Leiden: it holds that Φ belongs to ZFC because of the Axiom of Separation. Given this addendum, it now should be clearer that above standard proof actually provides a challenge to ZFC. If ZFC allows a paradoxical construct then one may feel that ZFC needs amendment.

[*NB*] Addendum for above Proof (writing out [*NB*]): (iii) Φ is in ZFC, (iv) ZFC provides for well-defined sets.

Proof for (iii) (Edixhoven in Colignatus (2014), appendix D): (a) *P*[*A*] exists because of the Axiom of the Powerset. (b) Note that *f* can be regarded as a subset of *A* x *P*[*A*]. Then *f* exists because of Axiom of Pairing. (c) Φ exists because of the Axiom of Separation. Q.E.D.

Proof for (iv): Not available. This is not proven but remains an assumption. (Finding a model in reality would be sufficient but might not be necessary.)

Comments:

(1) Colignatus (1981, 2007, 2011:239) used the version with the bijection rather than the surjection, and the following shorter proof, apparently created by Russell in 1907 (Hart (2015:42)). Regard an arbitrary set A. Let f: $A \to 2^A$ be the hypothetical bijection. Let $\Phi = \{x \in A \mid x \notin f[x]\}$. Clearly Φ is a subset of A and thus there is a $\varphi = f^{-1}[\Phi]$ so that $f[\varphi] = \Phi$. The question now arises whether $\varphi \in \Phi$ itself. We find that $\varphi \in \Phi \Leftrightarrow \varphi \notin f[\varphi] \Leftrightarrow \varphi \notin \Phi$ which is a contradiction. Ergo, there is no such f. This concludes the common short proof of Cantor's theorem. (The bijection is sufficient and the surjection is necessary, see Appendix B point 1)

(2) From the contradiction derived above, the proper conclusion is not that Cantor's Theorem is proven, but only that it is proven in ZFC. Either Cantor's Theorem is true *or* ZFC doesn't yet provide for well-defined sets.

(3) Sets A and B have 'the same size' when there is a bijection or one-to-one function between them. Cantor's Theorem holds that a set is always 'smaller' than its power set. For finite sets this can be proven by mathematical induction too. The standard proof, and in particular for infinite sets, uses a construction that strongly reminds of Russell's paradox (deconstructed in section 1 above).

(4) The Axiom of Separation blocks Russell's paradoxical set, but doesn't block Cantor's paradoxical Φ yet.

(5) Colignatus (1981, 2007, 2011) (ALOE) deals with logic and inference and thus keeps some distance from number theory and issues of the infinite. Historically, logic developed parallel to geometry and theories of the infinite (Zeno's paradoxes). Aristotle's syllogisms with *none, some* and *all* helped to discuss the infinite. Yet, to develop logic and inference proper, it appeared that ALOE could skip the tricky bits of number theory, non-Euclidean geometry, the development of limits, and Cantor's development of the transinfinite. Though it is close to impossible to discuss logic without mentioning the subject matter that logic is applied to, ALOE originally kept and keeps some distance from those subjects themselves. But, if logic uses the notion of *all,* it seems fair to ask whether there are limits to the use of this *all*. Thus it is explained why this present paper came about.

(6) It must also be observed that this author is no expert on Cantor's Theorem. We may reject the standard proof but perhaps there are other proofs. A marginal check shows that this proof is the only one given at various locations that seem to matter but this may only mean that it is a popular proof. For now, we have reproduced that standard proof and will now reproduce the refutation using the Paul of Venice consistency criterion, following Colignatus (1981, 2007, 2011:239). (PM. Hart (2015:41) gives Cantor's form of 1890/91, but then see Appendix B and the rejection in Colignatus (2012, 2013) (CCPO-PCWA).)

The subsequent discussion intends to show that the standard proof cannot be accepted. For the discussion below, relabel Φ in this subsection 2.1 into Φ'.

2.2. Rejection of this proof (in ALOE)

We might hold that above Φ' is badly defined since it is self-contradictory under the hypothesis. A badly defined 'something' may just be a weird expression and need not represent a true set. A test on this line of reasoning is to insert a small consistency condition, giving us $\Phi = \{x \in A \mid x \notin f[x] \mathbin{\&\&} x \in \Phi\}$ (see above Notation on '&&'). Now we conclude that $\varphi \notin \Phi$ since it cannot satisfy the condition for membership, i.e. we get $\varphi \in \Phi \Leftrightarrow (\varphi \notin f[\varphi] \;\&\; \varphi \in \Phi)) \Leftrightarrow (\varphi \notin \Phi \;\&\; \varphi \in \Phi) \Leftrightarrow$ *falsum*. Alternatively said, the *Unless*-condition $(\varphi \notin \Phi \;\&\; \varphi \in \Phi)$ finds falsehood on formal grounds, whence $\varphi \notin \Phi$. This closes the argument against the proof.

Puristically speaking, the Φ defined in 2.1 differs lexically from the Φ defined here, with the first expression being nonsensical and the present one consistent. It will be useful to reserve the term Φ for the proper definition in 2.2, and use Φ' for the expression in 2.1. The latter symbol is part of the lexical description but does not meaningfully refer to a set. Using this, we can use $\Phi^* = \Phi \cup \{\varphi\}$ and we can express consistently that $\varphi \in \Phi^*$. So the 'proof' in 2.1 can be seen as using a confused mixture of Φ and Φ^*.

PM. In writing CCPO-PCWA in 2012 I already considered using this more general &&-construction, but back then I preferred $\Phi = \{(x \in A)\ \&\ (x \neq f^{-1}[\Phi]) \mid x \notin f[x]\ \}$ to avoid the infinite regress. Now, looking at the challenge to ZFC, it seems better not to linger in ad hoc solutions but to emphasize the general idea. If one feels uncomfortable with the &&-construction then it is useful to know that there is still this ad hoc definition for Φ.

3. The challenge to ZFC

3.1. What is the difference between Φ' in 2.1 and Φ in 2.2 ?

Above deduction in section 2 poses a challenge to ZFC. Sets R and S above were in naive set theory, so it has relatively little meaning - for now - to ask about the difference between R and S. However, Φ' in 2.1 and Φ in 2.2 are in ZFC, and thus the question is (more) meaningful. Users of ZFC will have a hard time trying to clarify (a) that the consistency condition should have no effect but (b) actually does have an effect. To try to answer the question we might use the axiom of extensionality, see Coplakova et al. (2011:145):

$(A = B) \Leftrightarrow ((\forall x)\ (x \in A \Leftrightarrow x \in B))$

I have not pursued this question further since I have no vested interest in ZFC. I have requested Edixhoven who agrees with (a) to explain (b), and to describe the relation between Φ' in 2.1 and Φ in 2.2. I leave it to him or other users of ZFC to clarify this.

My solution of this issue is that Φ' in 2.1 is badly defined and that Φ in 2.2 is well-defined. Accepting that Φ' is ill-defined (rejecting (iv) above) has the effect of the collapse of the standard proof to Cantor's theorem (in the version of Russell for the power set). I am interested in an argument to the contrary but haven't seen it yet.

3.2. Amendments to the Axiom of Separation in ZFC

The proof in 2.1 relies on the separation axiom in ZFC.

Definition of the Axiom of Separation (Coplakova et al. (2011:145), adding a by-line on freedom): If A is a set and $\varphi[x]$ is a formula with variable x, then there exists a set B that consists of the elements of A that satisfy $\varphi[x]$, while B is not free in $\varphi[x]$:

$(\forall A)\ (\exists B)\ (\forall x)\ (x \in B \Leftrightarrow ((x \in A)\ \&\ \varphi[x]))$

Note the condition "B is not free in $\varphi[x]$". The consistency condition by Paul of Venice in the definition of Φ in 2.2 uses $\varphi'[x] = (\varphi[x]\ \&\&\ (x \in \Phi))$, in which $B = \Phi$ is not free since it is bound by the existential quantifier $(\exists B)$. Thus the formation of Φ in 2.2 is allowed in ZFC.

To meet the challenge in 3.1 we would require the PV-condition in general.

Possibility 3.2.1: Amendment by Paul of Venice to the Axiom of Separation:

$(\forall A)\ (\exists B)\ (\forall x)\ (x \in B \Leftrightarrow ((x \in A)\ \&\ \varphi[x]\ \&\&\ (x \in B)))$

In this case, 2.1 is no longer possible, the proof for Cantor's theorem collapses, and question 3.1 disappears since Φ' becomes ill-formed and nonsensical. My suggestion is to call this the 'neat' solution, and use the abbreviation ZFC-PV.

Another possibility is to move from ZFC closer to naive set theory, discard the axiom of separation, and adopt an axiom that allows greater freedom to create sets from formulas.

Possibility 3.2.2: Discard the separation axiom and have extensionality of formula's:

$(\forall \varphi)\ (\exists B)\ (\forall x)\ (x \in B \iff (\varphi[x]\ \&\&\ (x \in B)))$

This axiom protects against Russell's paradox and destroys the standard proof of Cantor's theorem. This resulting system might be called ZFC-S+PV.

The Axiom of Regularity forbids that sets are member of themselves. Instead, it is useful to be able to speak about the set of all sets. Though it is another discussion, my suggestion is to drop this axiom too, then to call this the 'basic' solution, and use the abbreviation BST (basic set theory), thus BST = ZFC-S+PV-R. I would also propose a rule that the condition could be dropped in particular applications if it could be shown to be superfluous. However, for paradoxical $\varphi[x]$ it would not be superfluous.

I am not aware of a contradiction yet. I have not looked intensively for such a contradiction, since my presumption is that others are better versed in set theory and that the problem only is that those authors aren't aware of the potential relevance of the consistency condition by Paul of Venice. A question for historians is: Zermelo (1871-1953) and Fraenkel (1891-1965) might have embraced the Paul of Venice's condition if they had been aware of it.

4. Conclusion

Colignatus (1981, 2007, 2011) concludes, and we now supplement with the questions on ZFC:

1. The standard proof for Cantor's Theorem (given above) is based upon a badly defined and inherently paradoxical construct. This proof evaporates once a sound construct is used. The earlier proofs by Cantor himself were already rejected by Colignatus (2012, 2013) (CCPO-PCWA) (the one of 1874 directly and 1890/91 for the decimal form - see Appendix B below).

2. The theorem is proven for finite sets by means of induction but is still unproven for (vaguely defined) infinite sets: that is, this author is not aware of other proofs. We would better speak about 'Cantor's Impression' or 'Cantor's Supposed Theorem'. It is not quite a conjecture since Cantor might not have done such a conjecture (without proof) if he would have known about above refutation.

3. It becomes feasible to speak again about the 'set of all sets'. This has the advantage that we do not need to distinguish (i) sets versus classes, (ii) *all* versus *any*.

4. The transfinites that are defined by using 'Cantor's Theorem' evaporate with it.

5. The distinction between the natural and the real numbers now rests (only) upon the specific diagonal argument (that differs from the standard proof). See Colignatus (2012, 2013) (CCPO-PCWA) for the conclusion that Cantor's original proof for the natural and real numbers evaporates too, specifically for a convenient level of constructivity. (CCPO-PCWA indeed looks at Cantor's original argument (in German) - also given by Hart (2015), see Appendix B.)

6. Users of ZFC should give an answer to 3.1, and clarify why they accept 2.1 and not 2.2 that has a better definition of a well-defined set. ZFC might be consistent but allows the construction of a 'proof' for 'Cantor's Impression' that generates the transfinites, which makes one wonder what this system is a model for. We can agree with Cantor that the essence of mathematics lies in its freedom, but the freedom to create nonsense somehow would no longer be mathematics proper. Useful alternatives are in ZFC-PV or BST.

7. The prime importance of this discussion lies in education. Mathematics education should respect that education itself is an empirical issue. In teaching, there is the logic that students can grasp and the idea to challenge them with more; and there is the wish for good history and and still not burden students with the confusions of the past. My suggestion is that Cantor's transfinites can hardly be grasped, are not challenging, and are rather burdening than enlightening. Colignatus (2012, 2013) clarifies that highschool education and matricola for non-math majors could be served well with a theory of the infinite that consistently develops both the natural and real numbers, without requiring more than the denumerable infinite ($\aleph \sim \Re$), using the notion of *bijection by abstraction*. See Colignatus (2015) for a discussion on abstraction.

It was Cantor himself who emphasised the freedom in mathematics, but that freedom is limited if alternatives are not mentioned. Even a university course like Coplakova et al. (2011) currently presents students only with 'Cantor's Theorem' without mentioning the alternative analysis in Colignatus (1981, 2007, 2011).

Acknowledgements

I thank Richard Gill (Leiden) for various discussions, and Klaas Pieter Hart (Delft) in 2012 and Bas Edixhoven (Leiden) in 2014 for some comments and for causing me to look closer at ZFC. Hart and Edixhoven seem to have seen only the shorthand form (see the paper) and seem to have missed the full argument of this paper. All errors remain mine.

Appendix A: Versions of ALOE

The following comments are relevant for accurate reference.

(1) Colignatus (1981, 2007, 2011) existed first unpublished in 1981 as *In memoriam Philetas of Cos,* then in 2007 rebaptised and self-published. It was both retyped and programmed in the computer-algebra environment of *Mathematica* to allow ease of use of three-valued logic. In 2011 it was marginally adapted with a new version of *Mathematica*. At that moment it could also refer to a new rejection of Cantor's particular argument for the natural and real numbers, using the notion of *bijection by abstraction* - in 2011 still called *bijection in the limit* but now developed in Colignatus (2012, 2013).

(2) Gill (2008) reviewed the 1st edition of ALOE of 2007. That edition refers to Cantor's standard set-theoretic argument and rejects it, as in the above. ALOE refers to Wallace (2003) as the book that caused me to look into the issue again. Wallace's book is critically reviewed by Harris (2004). It will be useful to mention that ALOE does not rely on Wallace's book but indeed only mentions it as a source of inspiration to look into the issue again.

(3) Gill (2008) did not review the 2nd edition of ALOE of 2011. That edition also refers to Cantor's original argument on the natural and real numbers in particular. That edition of ALOE mentions the suggestion that $\aleph \sim \Re$. The discussion itself is not in ALOE but is now in Colignatus (2012, 2013) (CCPO-PCWA), using the notion of *bijection by abstraction*.

(4) A visit to a restaurant and subsequent e-mail exchange led to the memo Colignatus (2014), and the inspiration to write this present article on the challenge to ZFC. Edixhoven also refers to Coplakova et al. (2011), theorem I.4.9, pp. 18-19, that gives the standard theorem and proof, also reproduced and challenged in above section 2.

(5) Colignatus (1981, 2007, 2011) is a book on logic and not a book on set theory. It presents the standard notions of naive set theory (membership, intersection, union) and the

standard axioms for first order predicate logic that of course are relevant for set theory. But I have always felt that discussing *axiomatic* set theory (with ZFC) was beyond the scope of the book and my actual interest and developed expertise. This present paper is in my sentiment rather exploratory, by discussing axiomatic set theory in section 3 and actually presenting two possible alternatives.

Appendix B: The traditional view, in Hart (2015)

This article was basically written in November 2014 and has been slightly updated with some clarifications following some comments from others.

It so happens that Hart (2015) recently reviews the issue too, giving the now traditional view. He starts with Cantor's orginal arguments on non-denumerability and the diagonal. For foreign readers a discussion of this Dutch article will be difficult to follow, but let me still give my comments now that I am dealing with the subject.

(1) Hart (2015:43) holds correctly that a bijection doesn't have to be used, but only the surjection. He however holds incorrectly that the common short proof with the bijection would rely on a 'spurious contradiction' - referring here to Gillman 1987. This would be incorrect if we rely on the common meaning of 'spurious': (a) there is a real contradiction: the assumption of the bijection implies the assumption of the surjection, which causes the contradiction, (b) the context of discussion is infinity, for which we use isomorphisms, and thus injections, and in that case the properties of surjection and bijection are equivalent: and then the shortness of the proof must be appreciated. Indeed Hart (2015:41) explains that Cantor himself also used 'eindeutig' (column 1) and injection (column 3 - below the photograph of 'Georde Cantor'). PM. Hart (2015:42 first column) suggests that the power set version of Cantor's Theorem was given by Bertrand Russell 1907, using a bijection.

(2) On page 42, third column, Hart agrees that Cantor's distinction between proper sets and improper sets ('classes'), or the distinction between *all* and *any*, still is used informally. Thus mathematics uses both a formal ZFC and an informal naive set system. It is useful to see this confirmed. It remains curious that Hart as a mathematician is happy to live with this incongruity. Hart then discusses the axiom of separation, but it gives a wrong impression that not its main weaknesses and alternatives are discussed.

(3) On page 43 Hart mentions the argument concerning $\aleph \sim \Re$ that uses decimal expressions. He states that this particular form does not occur in Cantor's work. This is not quite true. Cantor's proof of 1890/91 uses a binary representation - see Hart (2015:41) - which, for these purposes, is equivalent to using decimals. Hart traces the proof with decimals to Young & Young in 1906, who explicitly refer to Cantor 1890/91, and who explicitly call it his 'second proof'. Thus mathematicians were aware already in 1906 that binaries and decimals are equivalent here. It is curious that Hart in 2015 does not express that awareness. His review of what Cantor originally did thus is biased.

(4) We may wonder why Hart's paper might be biased. It is a good hypothesis that he wants to emphasize that some authors still have questions about Cantor's argument.

(4a) On page 43 Hart refers to Wilfrid Hodges (1998) who discusses "hopeless papers". Hart does not mention Hodges's email to me that I cited in CCPO-PCWA that I informed him about.

(4b) Hart accuses those "hopeless papers" of that they don't check what Cantor did himself originally. This is an improper accusation since such authors discuss a particular argument, that so happens to go by the name of 'Cantor's diagonal argument', while it is not always at issue what Cantor himself did.

(4c) Just to be sure: My own first contact with Hart - in 2012 - was about Cantor 1874. CCPO-PCWA wanted to know whether there were more proofs, and thus also looked at Cantor 1874, and found it inadequate. Hart's page 40 with Cantor 1874 finds a refutation in the appendix of CCPO-PCWA - but he knows about the latter and does not refer to that refutation.

(4d) Hart suggests that the proof with decimals causes most "hopeless papers", but that this proof can be "thrown in the trash can", because Cantor's original proof from 1874 and his second and more general proof of 1890/91 would be more attractive.

(4d1) This is improper, since it evades the question whether the argument with the decimals is a good deduction or not. Mathematics should not ditch arguments because they cause questions but should answer the questions.

(4d2) It also is an inconsistent argument, see (3): the proofs are equivalent, differ only in binaries versus decimals. Thus Hart suggests to throw Cantor's own proof into the trash can - but doesn't do so.

(4e) Hart holds that such "hopeless papers" and/or internet discussions quickly replace mathematics by ad hominem fallacies. An ad hominem would be: "You have no mathematics degree and hence I will not listen to your arguments." Obviously Hart presents himself as not falling into that trap. My problem however is that he applies an 'ad gentem fallacy', by reducing critique on Cantor's Theorem into "hopeless papers" and/or internet ad hominem fallacies. This is a racket or ballyhoo to induce a sentiment amongst his readership to no longer look at critique on Cantor's Theorem, and to join in the slaughter binge of such critics. We thus may understand why Hart (2015) is a biased presentation, unworthy of mathematics that wants to claim to be scientific.

(5) Hart (2015:42, last column): "The best known impossibility theorems in mathematical logic all use a version of Cantor's idea to flip all elements on a diagonal" - and then he refers to Gödel's first incompleteness theorem. This is not quite true. Gödel's theorem uses self-reference. This property was already known in antiquity in the Liar Paradox. Gödel's use of number-coding has historical explanations, like the trust in arithmetic in a period of a foundations crisis in mathematical logic. Gödel's numerical listing is not crucial to the argument. The influence of Cantor should not be made greater than it is. Hart could have known about this, reading both ALOE and Gill (2008) in the same Dutch journal for mathematics, with my refutation of Gödels two theorems.

(6) Hart does not refer to ALOE or CCPO-PCWA that he knows about, thus misinforms his readership. He reproduces Cantor's 'proofs' of 1874 and 1890/91 without mentioning their refutations. He states the common misconceptions and adds some new ones.

Appendix C: Refutation of another theorem and proof

In a personal communication in 2012 K.P. Hart (TU Delft) presented me this theorem and proof. If no one else presented this theorem earlier it may be called the Cantor-Hart Theorem but for now I label it for what it does. In 2012 my reply was asking Hart whether he understood the refutation of Cantor's Theorem that uses the power set and bijection, but I did not receive a response on that. If he had understood, he could have given below refutation himself.

Weakest Theorem underlying Cantor's Theorem (for the power set, Hart 2012): Let A be a set. For every $f : A \rightarrow P[A]$ there is a subset Φ in A - thus Φ in $P[A]$ - such that for all a in A it holds that $\Phi \neq f[a]$.

Proof: Define Φ = $\{x \in A \mid x \notin f[x]\}$. Take a in A. Check the two possibilities.

Case 1: $a \in \Phi$. In that case $a \notin f[a]$. Thus $\Phi \neq f[a]$. (We have a in $\Phi \setminus f[a]$.)

Case 2: $a \notin \Phi$. In that case $a \in f[a]$. Thus $\Phi \neq f[a]$. (We have a in $f[a] \setminus \Phi$.)

Q.E.D.

Discussion:

Positive is: This would hold for any set and function. Obviously, once the theorem is accepted, it follows that there can be no surjection and hence no bijection. The strength of the theorem and proof is that (1) it avoids using concepts like surjection, injection and bijection, (2) it would be constructive and avoids the *reductio ad absurdum*.

Negative is: For finite sets the proper constructive method uses mathematical induction, and then the method is beyond doubt. The problem lies with infinity, for which I have proposed the notion of 'bijection by abstraction'. If such a bijection would exist for the natural and real numbers, then there is something wrong with above 'proof'. Indeed, our refutation of the *reductio ad absurdum* proof of Cantor's Theorem shows what is the problem with above 'proof' too. There is a 'spurious non-contradiction': the 'proof' looks without contradiction but in fact relies on a hidden assumption that causes a contradiction. It may be mentioned that also above 'proof' should deal with the [*NB*]-addendum, see above.

Refutation: While the 'proof' in cases 1 and 2 assumes any f, it ought to distinguish between kinds of functions, for simplicity the bijections versus the non-bijections.

The Φ above is relabelled into Φ', and we reuse the symbol for a proper Φ.

Proper proof structure - that however fails:

If f is not bijective, then like the above.

If f is bijective, then there is a $\varphi = f^{-1}[\Phi]$. From the discussion above we know that above definition of Φ' causes a contradiction, so it is no useful Φ'. We require the consistency condition especially when this φ is tested. Thus the definition of Φ requires the additional conditions to prevent reliance on hidden contradictions - and let us split the subcases on the risk of infinite regress:

Define Φ = $\{x \in A \;\&\; x \neq f^{-1}[\Phi] \mid x \notin f[x]\} \cup \{x \in A \;\&\; x = f^{-1}[\Phi] \mid x \notin f[x] \;\&\; x \in \Phi\}$.

We subsequently distinguish cases $a = \varphi$ and $a \neq \varphi$.

Case A. $a \neq \varphi$. All is like the above.

Case 1: $a \in \Phi$. In that case $a \notin f[a]$. Thus $\Phi \neq f[a]$.

Case 2: $a \notin \Phi$. In that case $a \in f[a]$. Thus $\Phi \neq f[a]$.

Case B. $a = \varphi$.

Case 3: $\varphi \in \Phi$. Then ($\varphi \notin \Phi \;\&\; \varphi \in \Phi$): contradiction. This case cannot occur.

Case 4: $\varphi \notin \Phi$. Then ($\varphi \notin \Phi$ or $\varphi \in \Phi$): no contradiction. It is false that $\Phi \neq f[a]$ however since we have $a = \varphi = f^{-1}[\Phi]$.

Ergo: The 'proof' fails. Q.E.D.

Discussion:

(1) The theorem cannot stand. For some *f*, namely bijections, the given example Φ has an element in *A,* namely a = $\varphi = f^{-1}[\Phi]$, such that $\Phi = f[a]$.

(2) This does not mean that there might be other sets such that the theorem still stands. This may be doubted however.

(3) The crux of our interest now lies in a construction of a bijection between infinite sets, like the natural numbers and the reals. For this I refer to CCPO-PCWA.

Appendix D: An inadequate 'initial review'

The December 31 2014 version got this response from a peer-reviewed journal:

"An initial review of "A condition by Paul of Venice (1369-1429) solves Russell's Paradox, blocks Cantor's Diagonal Argument, and provides a challenge to ZFC" has made it clear that this submission does not meet the minimal requirements for publication in [our journal]. It is not sufficiently clear what the goal of the paper is, and (most importantly) it is not at all shown that the two possibilities listed on p.5 [i.e. in Section 3] have the intended consequences."

This April 29 2015 version has only made small changes. The major difference is to reduce the confusion on the shorthand form, now also with the &&-construction and notation. New are appendices B, C and D.

Thus, you can check that it is curious that the editor holds that the goal of the paper would not be clear. Also, the two possibilities listed in Section 3 directly modify the application of ZFC in the [*NB*]-addendum in the proof for Cantor's theorem, and block that proof, as indeed has been shown in the discussion in Section 2. There is no indication that the changes would be inconsistent.

This 'initial review' by the editor is inadequate. There should have been a full review with decent reports.

PM. Another reader wrote: "You seem to be saying that ZFC makes Cantor's theorem true which you find paradoxical and therefore you feel that ZFC needs amendment. But I think Cantor's theorem is cool so I am happy with ZFC." This reader got the main idea, contrary to the editor who suggested that it was not clear. However, observe that the word 'paradox' means 'seeming contradiction' (which is no real contradiction), and that the paper identifies these problems:

(a) the inclusion of a *consistency*-criterion causes the proofs to collapse,

(b) the question about Φ' in 2.1 and Φ in 2.2,

(c) the rejection of the other proofs, see CCPO-PCWA, (d) the lack of a 'set of all sets' and the schizophrenia of the formal use of proper sets and the informal use of improper sets;

(d) It is not irrelevant that we lack a model for the transfinites, preferably an empirical application: but with this caveat: see Colignatus (2015) for abstraction versus empirics, and Wigner, while mathematical modeling indeed creates ideas that don't exist in 'reality'.

Also: "You say *"the proper conclusion is not that Cantor's Theorem is proven, but only that it is proven in ZFC"*. Of course, whether or not Cantor's theorem is true, depends on your axioms of set theory, especially axioms pertaining to the infinite." Indeed, well understood again. One presumes however that some fundamental notions must be selected for their foundational values, not for their cool results when you neglect problems like (a) - (d).

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