

Algebraic Partial equations

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We generalized the concept of hyperbolic and trigonometric functions to the third and the fourth case which gives rise to the parametrization of the orbifold defined by $x^3 + y^3 + z^3 - 3xyz = 1$ in the third case.

Introduction

The trigonometric functions (cosinus and sinus) were historically discovered as real and imaginary parts of the complex exponential function which solves the first order partial equations. In addition, the hyperbolic functions are the results of the trigonometric function with imaginary arguments. Even if the trigonometric functions were find before the hyperbolic ones, try to imagine that this is the trigonometric which comes from the trigonometric ones.

Because the exponential function is defined as

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad (0.1)$$

It seems to be natural to generalized the exponential function as

$$e_p(z) = \sum_{k=0}^{\infty} \frac{z^{pk}}{(pk)!} \quad (0.2)$$

that we have call the p-exponential function in [1]. Here we recognize the hyperbolic cosinus if we take $p = 2$. In comparasion with the hyperbolic functions, we remarks that the p-th exponential (0.2) can be written as

$$e_p(z) = \frac{1}{p} \left(\sum_{k=0}^{p-1} e^{\omega_p^k z} \right) \quad (0.3)$$

where ω_p is the p-th root of unity.

Hence the hyperbolic and the trigonometric functions each satisfie a partial equation

$$y^2 - \left(\frac{\partial y}{\partial z} \right)^2 = 1 \quad (0.4)$$

for the hyperbolic functions and the sames with a minus for the trigonometric ones.

By the way those functions are viewed as the unique solutions of two manifolds whom defining equations are

$$x^2 - y^2 = 1 \quad (0.5)$$

which defines a hyperbol for the hyperbolic functions and the sames with a minus which defines a circle for the trigonometric ones.

As a consequence, we searched for the partial equation solved by the third exponential.

To this end, we showed the equivalent addition relation for the case $p = 3$. In the same way we find (0.4) when we put $b = -a$ in the trigonometric addition formulae ($\cosh(a + b)$), we have put $b = ja, c = j^2a$ in this equivalent trigonometric addition formulae ($\epsilon_3(a + b + c)$) and we have find the partial equation

$$\left(y(z)\right)^3 + \left(\frac{\partial y}{\partial z}\right)^3 + \left(\frac{\partial^2 y}{\partial z^2}\right)^3 - 3\left(y(z)\right)\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial^2 y}{\partial z^2}\right) = 1 \quad (0.6)$$

Thereby, the third exponential is the unique parametric solution of the orbifold defined by the implicit equation

$$x^3 + y^3 + z^3 - 3xyz = 1 \quad (0.7)$$

In a first time, we recall definitions of the third exponential and we give some expressions of this one. In a second time we give generalized hyperbolic parity and addition formulae of this third exponential. In a third time, we arbitrarily put 0 in the argument of the generalized hyperbolic addition relations to obtain the defining equation of the emerging orbifold and the partial equation solved by the third exponential. In a fourth, we give a definition of the hypercomplex number which are a generalization of the complex ones. Finally, we completely solve the case of the fourth exponential and give the orbifold and the partial equation solved by this fourth exponential.

1 The p-exponential

We recall the definition of the p-exponential given in [1] by :

$$e_p(z) = \sum_{k=0}^{\infty} \frac{z^{pk}}{(pk)!} = \frac{1}{p} \left(\sum_{k=0}^{p-1} e^{\omega_p^k z} \right) \quad (1.1)$$

where ω_p is the p-th root of unity. We have shown in [1] that the p-exponential is solution of the partial equation

$$\frac{\partial^p e_p}{\partial z^p} = e_p(z) \quad (1.2)$$

and

$$\sum_{k=0}^{p-1} \frac{\partial^k e_p}{\partial z^k} = e^z \quad (1.3)$$

Example : For p=3, the set of 3-exponential are

$$e_3(z) = \sum_{k=0}^{\infty} \frac{z^{3k}}{(3k)!} \quad (1.4)$$

$$e'_3(z) = \sum_{k=1}^{\infty} \frac{z^{3k-1}}{(3k-1)!} \quad (1.5)$$

$$e''_3(z) = \sum_{k=1}^{\infty} \frac{z^{3k-2}}{(3k-2)!} \quad (1.6)$$

which can also be written

$$e_3(z) = \frac{e^z + e^{jz} + e^{j^2z}}{3} \quad (1.7)$$

$$e'_3(z) = \frac{e^z + je^{jz} + j^2e^{j^2z}}{3} \quad (1.8)$$

$$e''_3(z) = \frac{e^z + j^2e^{jz} + je^{j^2z}}{3} \quad (1.9)$$

or

since $j = e^{\frac{2i\pi}{3}} = \frac{-1+i\sqrt{3}}{2}$ is the 3th root of unity

2 Generalized trigonometric relations

Theorem 1. *There is addition relation between the p-exponential and its derivatives*

$$e_p(a+b) = f(e_p(a), e_p(b), e'_p(a), e'_p(b), e''_p(a), e''_p(b)) \quad (2.10)$$

Proof : Admitted

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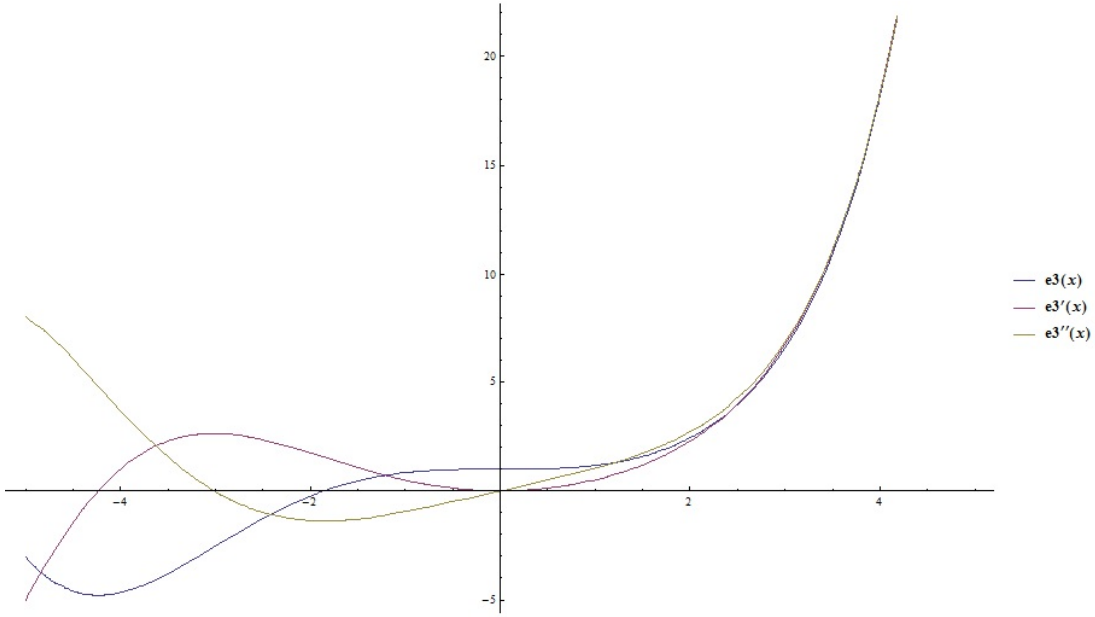


FIGURE 1 – The 3-exponential and its two derivatives

Example : For $p=3$, we generalized the parity relations for the third exponential

Proposition 1. *The parity relations are given by*

$$e_3(jz) = e_3(z) \qquad e_3(j^2z) = e_3(z) \qquad (2.11)$$

$$e_3'(jz) = j^2 e_3'(z) \qquad e_3'(j^2z) = j e_3'(z) \qquad (2.12)$$

$$e_3''(jz) = j e_3''(z) \qquad e_3''(j^2z) = j^2 e_3''(z) \qquad (2.13)$$

Proof :

From de definition (5.45), we obtain these relations.

For example, to show $e_3''(jz) = j e_3''(z)$

$$\begin{aligned} 3e_3''(jz) &= e^{jz} + j^2 e^{j(jz)} + j e^{j^2(jz)} \\ &= e^{jz} + j^2 e^{j^2z} + j e^z \\ &= j(j^2 e^{jz} + j e^{j^2z} + e^z) \\ 3e_3''(jz) &= 3j e_3''(z) \end{aligned} \qquad (2.14)$$

From (2.14), we obtain the other relation of (2.13) by successive derivations or by doing the same reasonment.

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Now, we generalized the addition relations for the third exponential

Proposition 2. *The addition relations are given by*

$$e_3(a+b) = e_3(a)e_3(b) + e'_3(a)e''_3(b) + e''_3(a)e'_3(b) \quad (2.15)$$

$$e'_3(a+b) = e''_3(a)e''_3(b) + e'_3(a)e_3(b) + e_3(a)e'_3(b) \quad (2.16)$$

$$e''_3(a+b) = e'_3(a)e'_3(b) + e_3(a)e''_3(b) + e''_3(a)e_3(b) \quad (2.17)$$

Proof :

Developping the expression of e_3, e'_3, e''_3 from their definition (5.45), we have the relations

$$\begin{aligned} 9e_3(a)e_3(b) &= e^{a+b} + e^{a+jb} + e^{a+j^2b} + e^{ja+b} + e^{ja+jb} + e^{ja+j^2b} + e^{j^2a+b} + e^{j^2a+jb} + e^{j^2a+j^2b} \\ 9e'_3(a)e'_3(b) &= e^{a+b} + je^{a+jb} + j^2e^{a+j^2b} + je^{ja+b} + j^2e^{ja+jb} + e^{ja+j^2b} + j^2e^{j^2a+b} + e^{j^2a+jb} + je^{j^2a+j^2b} \\ 9e''_3(a)e''_3(b) &= e^{a+b} + j^2e^{a+jb} + je^{a+j^2b} + j^2e^{ja+b} + je^{ja+jb} + e^{ja+j^2b} + je^{j^2a+b} + e^{j^2a+jb} + j^2e^{j^2a+j^2b} \end{aligned} \quad (2.18)$$

and

$$\begin{aligned} 9e_3(a)e'_3(b) &= e^{a+b} + je^{a+jb} + j^2e^{a+j^2b} + e^{ja+b} + je^{ja+jb} + j^2e^{ja+j^2b} + e^{j^2a+b} + je^{j^2a+jb} + j^2e^{j^2a+j^2b} \\ 9e_3(a)e''_3(b) &= e^{a+b} + j^2e^{a+jb} + je^{a+j^2b} + e^{ja+b} + j^2e^{ja+jb} + je^{ja+j^2b} + e^{j^2a+b} + j^2e^{j^2a+jb} + je^{j^2a+j^2b} \\ 9e'_3(a)e''_3(b) &= e^{a+b} + j^2e^{a+jb} + je^{a+j^2b} + je^{ja+b} + e^{ja+jb} + j^2e^{ja+j^2b} + j^2e^{j^2a+b} + je^{j^2a+jb} + e^{j^2a+j^2b} \end{aligned} \quad (2.19)$$

and

$$\begin{aligned} 9e'_3(a)e_3(b) &= e^{a+b} + e^{a+jb} + je^{a+j^2b} + je^{ja+b} + je^{ja+jb} + je^{ja+j^2b} + j^2e^{j^2a+b} + j^2e^{j^2a+jb} + j^2e^{j^2a+j^2b} \\ 9e''_3(a)e_3(b) &= e^{a+b} + e^{a+jb} + e^{a+j^2b} + j^2e^{ja+b} + j^2e^{ja+jb} + j^2e^{ja+j^2b} + je^{j^2a+b} + je^{j^2a+jb} + je^{j^2a+j^2b} \\ 9e''_3(a)e'_3(b) &= e^{a+b} + je^{a+jb} + j^2e^{a+j^2b} + j^2e^{ja+b} + e^{ja+jb} + je^{ja+j^2b} + je^{j^2a+b} + j^2e^{j^2a+jb} + e^{j^2a+j^2b} \end{aligned} \quad (2.20)$$

Using (2.19) and (2.20), we have

$$\begin{aligned} 9(e'_3(a)e''_3(b) + e''_3(a)e'_3(b)) &= 2e^{a+b} - e^{a+jb} - e^{a+j^2b} - e^{ja+b} \\ &\quad + 2e^{ja+jb} - e^{ja+j^2b} - e^{j^2a+b} - e^{j^2a+jb} + 2e^{j^2a+j^2b} \\ &= 3(e^{a+b} + e^{ja+jb} + e^{j^2a+j^2b}) - (e^{a+b} + e^{a+jb} + e^{a+j^2b} \\ &\quad + e^{ja+b} + e^{ja+jb} + e^{ja+j^2b} + e^{j^2a+b} + e^{j^2a+jb} + e^{j^2a+j^2b}) \\ 9(e'_3(a)e''_3(b) + e''_3(a)e'_3(b)) &= 9e_3(a+b) - 9e_3(a)e_3(b) \end{aligned}$$

which gives

$$e_3(a+b) = e_3(a)e_3(b) + e'_3(a)e''_3(b) + e''_3(a)e'_3(b) \quad (2.21)$$

which the first relation of (2.17).

In the same way, using (2.19) and (2.20), we have

$$\begin{aligned}
9(e_3(a)e_3''(b) + e_3''(a)e_3(b)) &= 2e^{a+b} - j^2e^{a+jb} - je^{a+j^2b} - j^2e^{ja+b} \\
&\quad + 2je^{ja+jb} - e^{ja+j^2b} - je^{j^2a+b} - e^{j^2a+jb} + 2j^2e^{j^2a+j^2b} \\
&= 3(e^{a+b} + je^{ja+jb} + j^2e^{j^2a+j^2b}) - (e^{a+b} + j^2e^{a+jb} + \\
&\quad je^{a+j^2b} + j^2e^{ja+b} + je^{ja+jb} + e^{ja+j^2b} + je^{j^2a+b} + e^{j^2a+jb} + j^2e^{j^2a+j^2b}) \\
9(e_3(a)e_3'(b) + e_3'(a)e_3(b)) &= 9e_3'(a+b) - 9e_3''(a)e_3'(b)
\end{aligned}$$

which gives

$$e_3'(a+b) = e_3''(a)e_3'(b) + e_3'(a)e_3(b) + e_3(a)e_3'(b) \quad (2.22)$$

which the second relation of (2.17).

In the same way, using (2.19) and (2.20), we have

$$\begin{aligned}
9(e_3(a)e_3''(b) + e_3''(a)e_3(b)) &= 2e^{a+b} - je^{a+jb} - j^2e^{a+j^2b} - je^{ja+b} \\
&\quad + 2j^2e^{ja+jb} - e^{ja+j^2b} - j^2e^{j^2a+b} - e^{j^2a+jb} + 2je^{j^2a+j^2b} \\
&= 3(e^{a+b} + j^2e^{ja+jb} + je^{j^2a+j^2b}) - (e^{a+b} + je^{a+jb} + j^2e^{a+j^2b} + je^{ja+b} \\
&\quad + j^2e^{ja+jb} + e^{ja+j^2b} + j^2e^{j^2a+b} + e^{j^2a+jb} + je^{j^2a+j^2b}) \\
9(e_3(a)e_3''(b) + e_3''(a)e_3(b)) &= 9e_3''(a+b) - 9e_3'(a)e_3''(b)
\end{aligned}$$

which gives

$$e_3''(a+b) = e_3'(a)e_3''(b) + e_3(a)e_3''(b) + e_3''(a)e_3(b) \quad (2.23)$$

which the third relation of (2.17).

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3 Orbifold emerging of the relation between the p-exponential and its derivatives

We give here the theorem without good proof but to understand it, read the case p=3.

Theorem 2. *There is a unique function g between the p-exponential and its derivatives such that*

$$g\left(e_p, e_p^{(1)}, e_p^{(2)}, \dots, e_p^{(p-1)}\right) = 1 \quad (3.24)$$

This relation defines a orbifold \mathcal{O} :

$$g(z_1, \dots, z_p) = 1 \quad (3.25)$$

Proof : Admitted

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Example :

For $p=3$, we have

Proposition 3. *The vector (e_3, e'_3, e''_3) is the unique solution of the equation*

$$g(x, y, z) = 1 \quad (3.26)$$

such that

$$g(x, y, z) = x^3 + y^3 + z^3 - 3xyz \quad (3.27)$$

The equation (3.26) defines an orbifold.

Proof :

From the addition relation, we have

$$\begin{aligned} e_3(a + b + c) &= e_3(a)e_3(b + c) + e'_3(a)e''_3(b + c) + e''_3(a)e'_3(b + c) \\ &= e_3(a) \left(e_3(b)e_3(c) + e'_3(b)e''_3(c) + e''_3(b)e'_3(c) \right) \\ &\quad + e'_3(a) \left(e'_3(b)e'_3(c) + e_3(b)e''_3(c) + e''_3(b)e_3(c) \right) \\ &\quad + e''_3(a) \left(e''_3(b)e''_3(c) + e'_3(b)e_3(c) + e_3(b)e'_3(c) \right) \end{aligned} \quad (3.28)$$

If we impose $b = ja$ and $c = j^2a$ and using (2.13), we find

$$\begin{aligned} e_3(a + ja + j^2a) &= e_3(a) \left(e_3(a)e_3(a) + je'_3(a)e''_3(a) + j^2e''_3(a)e'_3(a) \right) \\ &\quad + e'_3(a) \left(e'_3(a)e'_3(a) + j^2e_3(a)e''_3(a) + je''_3(b)e_3(c) \right) \\ &\quad + e''_3(a) \left(e''_3(a)e''_3(a) + j^2e'_3(a)e_3(a) + je_3(b)e'_3(c) \right) \\ 1 &= \left(e_3(a) \right)^3 + \left(e'_3(a) \right)^3 + \left(e''_3(a) \right)^3 + (j + j^2 + j + j^2 + j + j^2)e_3(a)e'_3(a)e'_3(a) \end{aligned}$$

So we find

$$\left(e_3(a) \right)^3 + \left(e'_3(a) \right)^3 + \left(e''_3(a) \right)^3 - 3e_3(a)e'_3(a)e'_3(a) = 1 \quad (3.29)$$

or

$$g(e_3(z), e'_3(z), e''_3(z)) = 1 \quad (3.30)$$

So $y(z) = e_3(z)$ is the only solution of the partial equation

$$\left(y(z) \right)^3 + \left(\frac{\partial y}{\partial z} \right)^3 + \left(\frac{\partial^2 y}{\partial z^2} \right)^3 - 3 \left(y(z) \right) \left(\frac{\partial y}{\partial z} \right) \left(\frac{\partial^2 y}{\partial z^2} \right) = 1 \quad (3.31)$$

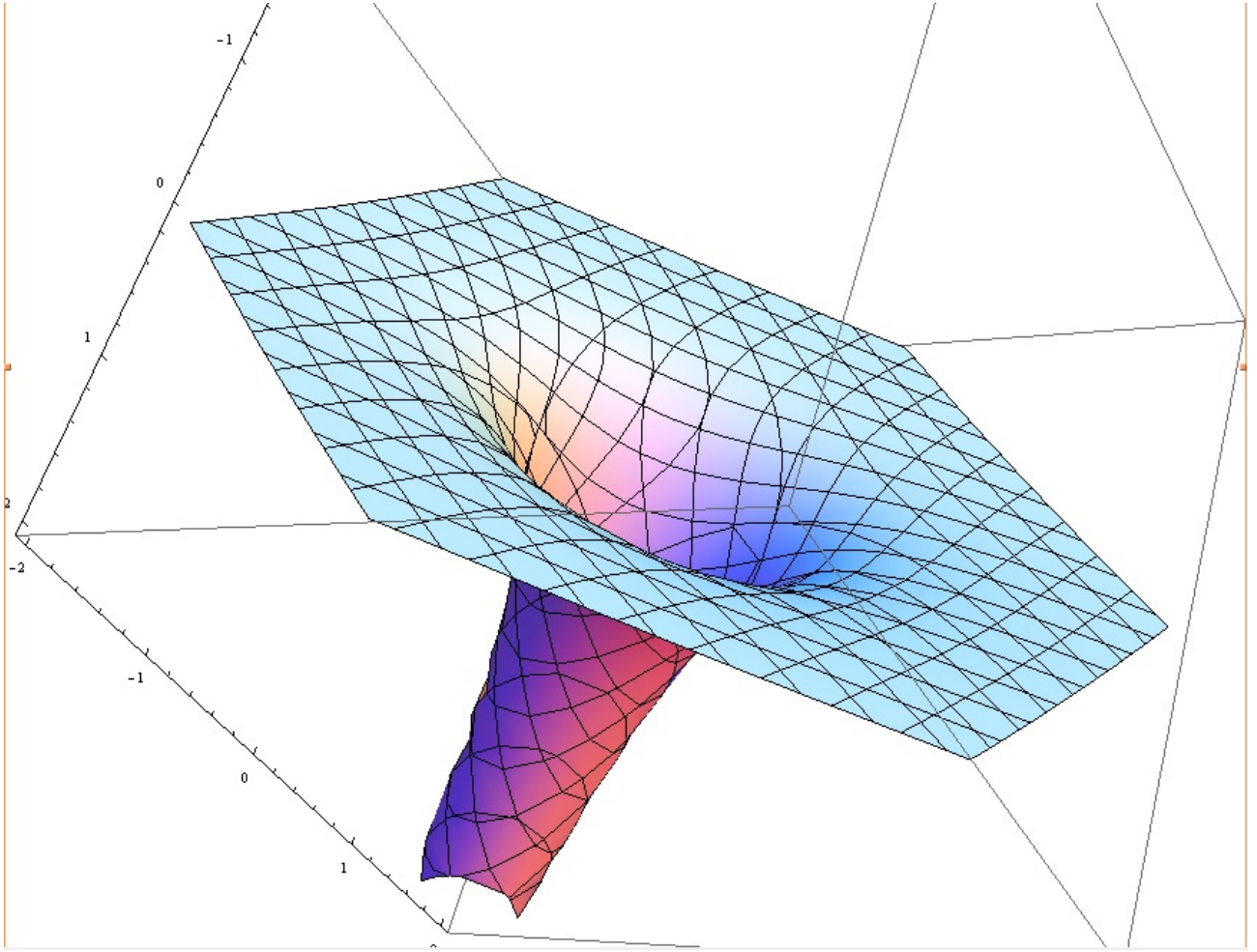


FIGURE 2 – Implicit plot of $x^3 + y^3 + z^3 - 3xyz = 1$

4 Hypercomplex numbers

From the definition of the orbifold emerging of the third exponential

$$x^3 + y^3 + z^3 - 3xyz = 1 \quad (4.32)$$

and because of the identity

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2} (x + y + z) \left[(x - y)^2 + (y - z)^2 + (z - x)^2 \right] \quad (4.33)$$

we obtain

$$\frac{1}{2} (x + y + z) \left[(x - y)^2 + (y - z)^2 + (z - x)^2 \right] = 1 \quad (4.34)$$

Now we expand the second factor which gives

$$\begin{aligned}
(x+y+z) \left[x^2 + y^2 + z^2 - xy - yz - xz \right] &= 1 \\
(x+y+z) \left[x^2 + y^2 + z^2 + (j+j^2)xy + (j+j^2)yz + (j+j^2)xz \right] &= 1 \\
(x+y+z) \left[x^2 + j^2 xy + j xz + y^2 + j yx + j^2 yz + z^2 + j^2 zx + j zy \right] &= 1 \\
(x+y+z) \left[(x+jy+j^2z)(x+j^2y+jz) \right] &= 1 \tag{4.35}
\end{aligned}$$

So if we parametrize

$$\begin{cases} x &= e_3(\theta) \\ y &= e'_3(\theta) \\ z &= e''_3(\theta) \end{cases} \tag{4.36}$$

We can write (4.35) as

$$e^\theta e^{j\theta} e^{j^2\theta} = 1 \tag{4.37}$$

since

$$e^\theta = \frac{e_3(\theta) + e'_3(\theta) + e''_3(\theta)}{3} \tag{4.38}$$

$$e^{j\theta} = \frac{e_3(\theta) + j^2 e'_3(\theta) + j e''_3(\theta)}{3} \tag{4.39}$$

$$e^{j^2\theta} = \frac{e_3(\theta) + j e'_3(\theta) + j^2 e''_3(\theta)}{3} \tag{4.40}$$

$$\tag{4.41}$$

Then we can define the hypercomplex numbers

$$\mathbb{S} = \left\{ x + j y + j^2 z \in \mathbb{C} \text{ with } x, y, z \in \mathbb{R} \right\} \tag{4.42}$$

5 Resolution of the 4th exponential

For $p=4$, the set of 4-exponential are

$$e_4(z) = \frac{e^z + e^{iz} + e^{-z} + e^{-iz}}{4} \quad (5.43)$$

$$e'_4(z) = \frac{e^z + ie^{iz} - e^{-z} - ie^{-iz}}{4} \quad (5.44)$$

$$e''_4(z) = \frac{e^z + i^2 e^{iz} + e^{-z} - e^{-iz}}{4} \quad (5.45)$$

or

$$e_4(z) = \frac{1}{2} (\cos(z) + \cosh(z)) \quad (5.46)$$

$$e'_4(z) = \frac{1}{2} (-\sin(z) + \sinh(z)) \quad (5.47)$$

$$e''_4(z) = \frac{1}{2} (-\cos(z) + \cosh(z)) \quad (5.48)$$

$$e'''_4(z) = \frac{1}{2} (\sin(z) + \sinh(z)) \quad (5.49)$$

From de usual duplication formula, we find

$$1/2 \cos(a) \cos(b) + 1/2 \cosh(a) \cosh(b) - 1/2 \sin(a) \sin(b) + 1/2 \sinh(a) \sinh(b) \quad (5.50)$$

Putting $b = -a$ we obtain :

$$\frac{\cos^2(a) + \sin^2(a) + \cosh^2(a) - \sinh^2(a)}{2} = 1 \quad (5.51)$$

which leads to

$$(e_4(z))^2 + (e_4''(z))^2 - 2e_4'''(z)e_4'(z) = 1 \quad (5.52)$$

The other comes from

$$\begin{aligned} e_4(a+b+c+d) &= 1/2 \cos(a) \cos(b) \cos(c) \cos(d) + 1/2 \cosh(a) \cosh(b) \cosh(c) \cosh(d) \\ &\quad - 1/2 \cos(c) \cos(d) \sin(a) \sin(b) - 1/2 \cos(b) \cos(d) \sin(a) \sin(c) \\ &\quad - 1/2 \cos(a) \cos(d) \sin(b) \sin(c) - 1/2 \cos(b) \cos(c) \sin(a) \sin(d) \\ &\quad - 1/2 \cos(a) \cos(c) \sin(b) \sin(d) - 1/2 \cos(a) \cos(b) \sin(c) \sin(d) \\ &\quad + 1/2 \sin(a) \sin(b) \sin(c) \sin(d) + 1/2 \cosh(c) \cosh(d) \sinh(a) \sinh(b) \\ &\quad + 1/2 \cosh(b) \cosh(d) \sinh(a) \sinh(c) \\ &\quad + 1/2 \cosh(a) \cosh(d) \sinh(b) \sinh(c) + 1/2 \cosh(b) \cosh(c) \sinh(a) \sinh(d) \\ &\quad + 1/2 \cosh(a) \cosh(c) \sinh(b) \sinh(d) + 1/2 \cosh(a) \cosh(b) \sinh(c) \sinh(d) \\ &\quad + 1/2 \sinh(a) \sinh(b) \sinh(c) \sinh(d) \end{aligned}$$

Putting $b = -a, c = ia, d = -ia$, we obtain

$$(e_4^2(z) - e_4''^2(z))^2 - (e_4'''^2(z) - e_4'^2(z))^2 + 4(e_4(z)e_4'''^2(z) - e_4''^2(z)e_4'^2(z))(e_4''(z)e_4'''^2(z) - e_4^2(z)e_4'^2(z)) = 1 \quad (5.53)$$

So we obtain the 2 dimensional orbifold defined by

$$\begin{cases} (x^2 - y^2)^2 - (y'^2 - t^2)^2 + 4(xt - yz)(yt - xz) &= 1 \\ x^2 + y^2 - 2zt &= 1 \end{cases} \quad (5.54)$$

Conclusion

In fact there is a problem with the parametrization of $x^3 + y^3 + z^3 - 3xyz = 1$ with the curve $(e_3(x), e'_3(x), e_3(x))$. It is because why I have written this paper using $z \in \mathbb{C}$ because $e_3(z)$ is two dimensional like $x^3 + y^3 + z^3 - 3xyz = 1$. At the time I am writing those lines, I do not have solved this problem.

The dimension of the emerging orbifold seems to depend on the independence of the p -th root of unity back the previous roots of unity. You can see it in the fourth exponential case where

$$\omega_4^2 = i^2 = -1 = \omega_2 \tag{5.55}$$

Hence we have two defining equations for the orbifold emerging of the fourth exponential whereas we find only one implicit equation for the orbifold emerging of the third exponential case. This gives us the good dimensional number since the generalized exponential are defined on \mathbb{C} which is two dimensional.

Furthermore, I choose to give the generalized theorem for each $p \in \mathbb{N}$ to say that it is true for each p but I only formally prove it only for the third and the fourth exponential.

By the way, if somebody want to study the topology or the metric of the orbifold $x^3 + y^3 + z^3 - 3xyz = 1$, please contact me, I would be happy to work on it.

Références

- [1] Arm B. N., The p-Arm Theory