A Metaphorical Chart of Our Mathematical Ontology.

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A **new paradigm** is emerging in fundamental physics. We know this because there is so much controversy about what is valid science, but what is the true nature of the new thinking? It may be too early to give a full answer but I think that an important part of it is in the way that we understand the vacuum and how it relates to particle physics. Thirty years ago all physicists would have assumed that the cold flat vacuum is a unique solution of the fundamental laws. The standard model of particle physics is now known to be successful beyond the wildest dreams of physicists back then and it has a single lowest energy state of broken symmetry from which all known laws other than gravity are known to follow. It was thought that a full unified theory was within our grasp. A fuller unification would be found at the GUT scale from a bigger but simpler gauge symmetry. Beyond that supersymmetry would provide the final unification with gravity. All the laws of the standard model including its 21 constants would be derived from some unique theory with fewer free parameters, or perhaps even none.

Today the more progressive physicists take a different view. Space and time are seen as emergent from a yet unknown new way of looking at the universe that must go beyond the bounds of standard quantum field theory, but that much is widely accepted and therefore is not the defining feature of the new paradigm. What is harder to accept is the **multiplicity of the vacuum** – the idea that there may be more than one stable solution for cold empty space and that the one we know is nothing special or unique. This concept bruises the egos of particle physicists who thought that the laws of physics they were unveiling were special in a very fundamental sense. They felt that their science as superior in a way that is different from other fields such as biology or geology. These are understood to be merely studies of one particular solution to the consequences of the laws of physics while many different solutions may be realized elsewhere. Surely the laws of particle physics could not turn out to be just as parochial.

Problems with the univacuum assumption have been around for a long time. It has been observed for years that the nature of physical laws appears fine-tuned for the convenience of life. With different values for the masses of particles and physical constants, chemistry and astronomy and therefore biology would not be present in the universe. Almost every natural occurring element of the periodic table plays some essential role in the making of multicellular life forms. How can this be explained if there is no alternative possibility for the values of those parameters? This question of anthropic principle was popularised in books by John Barrow and Frank Tipler in the 1980s [1] but physicists had grown used to the form of nature and took it for granted until they learnt from astronomy that the cosmological constant has a small positive value. More recently it was the failure to find supersymmetry in the first run of the LHC that rung alarm bells. SUSY is a natural consequence of string theory and would account for **fine-tuning** of the Higgs mechanism, but when the Higgs boson was found in its absence it was another feature of string theory that seemed to better provide an explanation, namely the multiplicity of the vacuum. When gravity and quantum theory come together we learn that there can be higher dimensions curled up with some undetermined topology and stabilised with fluxes. This provides a **landscape** of different solutions from which the vacuum we know may be selected almost at random. This might not be just a consequence of string theory but may also appear in the so-called alternatives such as spin-foams and Loop Quantum Gravity (I say "so-called" not because I do not accept their viability but rather because I still think they will turn out in the end to be aspects of the same theory from which strings emerge)

At this stage the multiplicity of vacua remains only a hypothesis, but eventually we will surely understand the theory of quantum gravity well-enough to know if it is correct. In the meantime it is normal and healthy that theorists will build more layers of speculation upon the idea to try and understand the range of possible consequences. It is equally predictable and healthy that such ideas will be criticised. It is all part of the entrance exam that a new paradigm must go through. If there is indeed a class of many possible solutions for the vacuum, is only one of these real? I think it is more parsimonious to accept that **all solutions exist** in some higher sense, whether inside or outside our universe. Some physicists have speculated that there is an eternal process of inflation with vacua decaying to different solutions so that our own universe is just one bubble inside a larger arena. Others have looked at evolving universes where the laws of physics evolve in leaps where new universes are born from old. We can learn a lot from thinking about such possibilities whether they are eventually testable or not but we should not get carried away by thinking they are less speculative or more testable than they really are.

Some physicists protest that this new way of thinking is "giving up" on fundamental physics. Nothing could be further from the truth. The only thing that is given up is the dream of a natural unified model of particle physics that explains it all from simple equations. There is still plenty of work to be done to understand dark matter and how inflation works. Experiments will probe higher energies, the stability of the proton, gravitational waves and we will learn more. Theorists still do not understand quantum field theory properly and there are plenty of unexplained patterns in the standard model. It is just going to take longer than expected at the end of the millennium to learn our place in the universe.

At the same time, theorists will desire to look beyond the laws of particle physics that depend on a particular vacuum state and ask what are the **meta-laws** to which they are a solution. If the challenges that face particle physics are already hard then this deeper problem may sound like something beyond the possibility of resolution but that seems not to be the case. It is known that the combination of quantum theory and general relativity imposes tough constraints on the possible range of consistent space-time models. Taking perturbations around a flat vacuum we find that the requirements of consistency in quantum field theory limits us to a range of particles with half integer spin up to the spin-two graviton. We can extend to higher spin states using string theory and it is highly likely (but not quite proven) that there is no other way to describe a consistent theory of quantum gravity in perturbation theory. Other approaches such spin foams and Loop Quantum Gravity approach the quantisation of general relativity from a different direction and tell us more about how to understand space-time in terms that transcend any particular space-time background or particle spectrum.

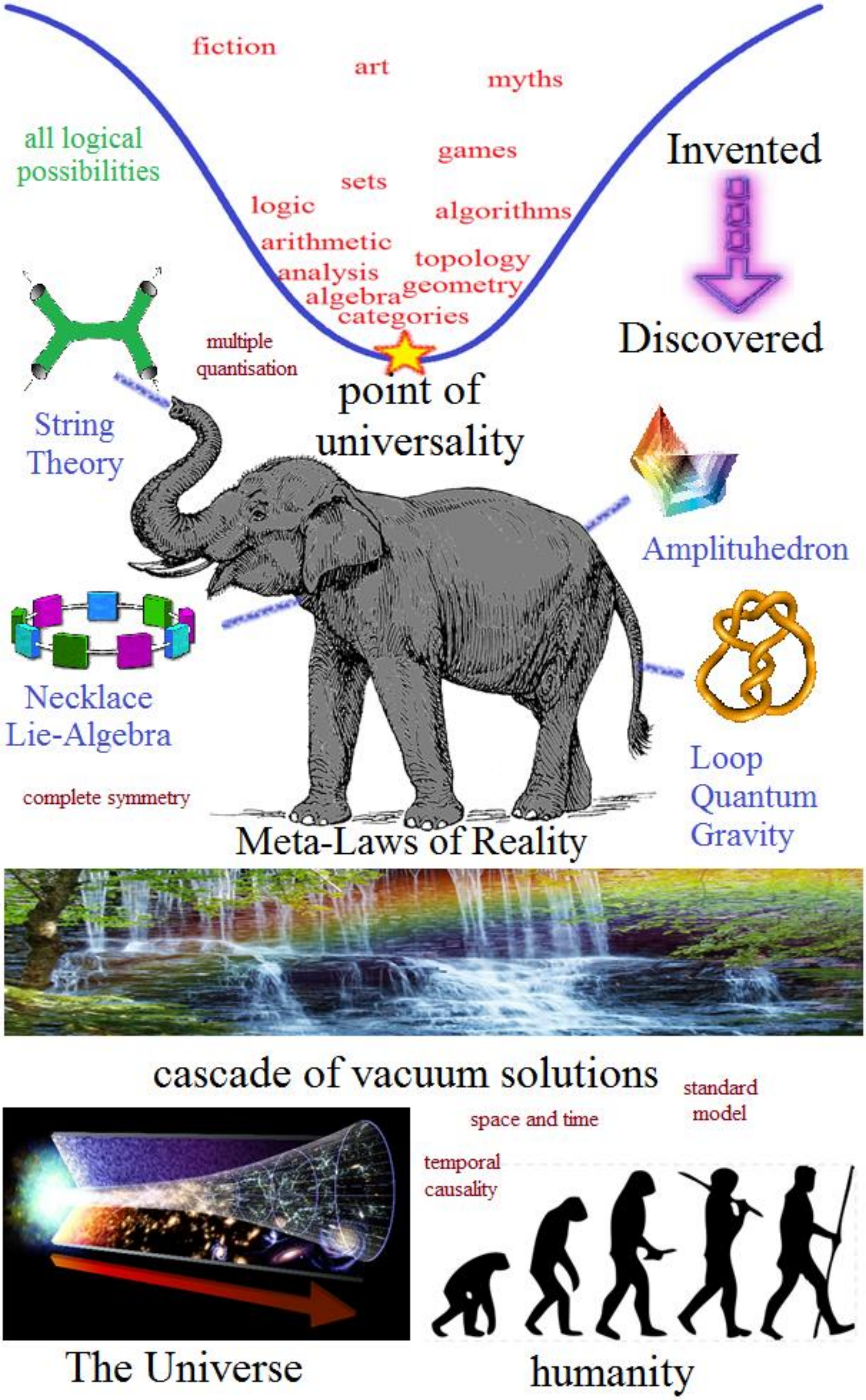
The ability of theorists to see so far beyond what experiment reveals directly is due to what Wigner called "**the unreasonable effectiveness of mathematics in the natural sciences**" [2] It is not hard to accept that mathematics can describe measurements we make and tell us what results we will get when we make different but similar measurements, but why is it that mathematics can take us from what we have observed to completely new phenomena not previously looked at? Wigner asked these questions in 1960 citing the use of complex numbers and matrices in quantum mechanics and differential geometry in relativity as examples of the way that ideas from pure mathematics have proved useful in physics. Fifty years later this mysterious power of mathematics is even clearer. Even new ideas from number theory and algebraic geometry have applications in advanced physics.

Critics will say that physicists have been carried away by the beauty of mathematical ideas and used them in ways that are not justified by experiment. This is not the case. Applications of advanced and abstract mathematical ideas appear to arise unexpectedly in physics research and are used as they are needed. Sometimes the physics even moves ahead of what has been previously known in mathematics and advances concepts that mathematicians had been struggling with. Today we might equally well ask why physical science is so effective in mathematics. A striking example is the proof by Richard Borcherds of the **Monstrous Moonshine Conjectures** for which he won the Fields Medal. Mathematicians had noticed mysterious and unexpected relationships between numbers and series that arose in unrelated areas of pure mathematics which they called Monstrous Moonshine. Borcherds eventually proved the connection using a clever construction based in string theory from physics. This remarkable connection leaves us to wonder how mathematicians would have proceeded if string theory had not been known. Would they simply have been stuck or would they have invented some purely mathematical form of string theory just to solve this problem? Whatever the answer it is clear that there are deep relations between ideas from physics and mathematics. It is this **unity** that lends hope to the idea that we really can come to understand the meta-laws that govern physics despite the limitations of our technical ability to measure phenomena at the relevant physical scales.

According to Tegmark's **Mathematical Universe Hypothesis** [3] our universe is just one mathematical structure of many whose existence is equally valid even if they are outside our own universe. I think it is unnecessary to concern ourselves with whether the words "exist" and "outside" have any meaning here. Such ideas are about concepts beyond our ordinary experience for which we do not have predefined words. To think about them we can only use **metaphors** with meaning that we understand within our own limits. In this sense we can build a picture of this mathematical universe and try to use it to comprehend the nature of reality.

In this spirit I offer a **metaphorical chart of the mathematical ontology**. It is a map of all things that are **logically possible** including us and our own universe. It is timeless and spaceless because these things are **emergent** features of our particular universe. In the mathematical ontology there are just relationships between mathematical objects. An ontology is a view of being, of how and why anything exists. I take it as self-evident that logical possibilities exist even if only in some metaphorical sense that we don't understand. It is just a way of saying that some things are possible.

Philosophers sometimes debate whether mathematical structures are **invented or discovered**. This is a key question and the answer is *both on a sliding scale*. A work of fiction has a logical structure both in the language it uses which is represented by a sequence of symbols in a book, and in the relationships and characteristics of the characters and objects in the plot. Mathematicians don't normally regard a story as a work of mathematics because it has a very arbitrary structure that is clearly invented. However, there is no sharp line between such an invention and the more mathematically interesting structures that most mathematicians would describe as discovered. An intermediate structure would be something like the game of chess. It has a very clear mathematical definition and can be analysed mathematically, yet its rules were invented. If we came across an alien civilisation and they played chess we would suspect that there had been some cultural communication that had passed the rules from us to them or vice versa. On the other hand, if the same aliens proved theorems about prime numbers we would assume they probably discovered them independently just as mathematicians often do. What about games with simpler rules like tic-tac-toe or Nim? Somewhere the distinction between invention and discovery blurs and depends on how much time people have had to think of new things to study.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Loop Quantum Gravity
complete symmetry
Meta-Laws of Reality
cascade of vacuum solutions
standard model
space and time
temporal causality
The Universe
humanity

What is it that distinguishes the good mathematical concepts that are discovered from the less interesting ones that seem invented? You might be excused for thinking that the answer lies in simplicity or mathematical beauty and elegance. Those qualities play their part in discovery but they are not the answer. The real answer is something more mysterious. It is referred to as **universality**. If you were a newcomer to mathematics you might expect that the field would consist of a mixture of methods and formula tailored to solve specific problems. If that were the case mathematics would be useful, but it would not be very interesting. What delights mathematicians is when they find connections between problems that had previously seemed unrelated. A mathematical concept like complex numbers or matrices originally formulated to solve one problem in analysis or algebra can turn out to be useful in unexpected areas such as number theory, or of course in physics. We don't really know why this happens so frequently but it seems to be a feature of universality, something that appears in the study of complexity rather than the study of simplicity.

Our understanding of universality is incomplete, but there are examples of it that give a good general idea. The most revealing is universality found in statistical physics which describes the behaviour of systems consisting of many particles in terms of temperature and entropy which do not depend on the specific microscopic properties of the system. Similar types of universality can be found in quantum field theory where scaling behaviour near a **critical point** in the phase diagram washes out the small scale description of the fields, a feature used in lattice gauge theories to approach a continuum limit.

What then would happen if we treat the whole of mathematics as a statistical physics system or as a **path integral** over the moduli space of all possible theories [4]? Would some universal behaviour emerge that could describe the meta-laws of physics?

That is essentially what I suspect happens on our ontological chart. The realm of all logical possibilities that includes all consistent things whether invented or discovered has a critical point around which universal behaviour can be found. This explains the unity of mathematics and why some concepts seem more discovered than invented, making them of interest to mathematicians. The precise theory of this universality would be algebraic rather than something that can be calculated using statistical physics, but thermodynamic metaphors help to understand it. As well as explaining the unity of mathematics this universality principle would also explain why mathematics is so effective in natural science. It also explains conversely why physics is so important in mathematicians, even to those who are not interested in practical applications. In my metaphorical chart I show how mathematical concepts funnel towards this critical point of where universality and the meta-laws of physics emerge.

The **meta-laws** themselves are depicted on the chart as an elephant in honour of the ancient metaphor of the elephant and the blind men which is popular in Asian philosophy, especially in India. The moral is that physicists are like blind men (and women) who feel something important as they touch different aspects of the fundamental laws of physics. In reality they are sensing a grand whole which is represented by the elephant. They see string theory when they look at how quantum perturbations of spacetime should propagate as gravitons and they see loop quantum gravity when they apply background independent methods of quantisation to gravity. Other features of the elephant appear as the amplituhedron, non-commutative geometry, spin-foams or twistors. Some of these things are known to be connected but the whole picture still eludes discovery.

Below the elephant we see the **cascade of vacua** which depicts the landscape of solutions to the meta-laws of which our universe is one example. The elephant is not to be envisaged as something that existed before the big bang. Geometry and time are part of specific vacua which can have different dimensions. The emergence of different vacua is ontological rather than temporal [5]. It is possible that different vacua exist in the same universe but a simpler picture is to see them as disconnected possible universes.

Uncovering the meta-laws is now the most important goal in our quest to understand the universe. New empirical data would help but theorists have to work from what they have and the need to combine quantum theory and general relativity under one roof appears to already be a tight constraint. To get answers they must pull together what they know from all the different approaches to quantum gravity and perhaps also from the nature of universality in mathematics.

From string theory there is the idea of **M-theory** from which consistent string theories may be derived as different vacuum solutions. M-theory is thought to be a theory of two and five dimensional membranes in eleven dimensions. However, there is also F-theory with an extra time dimension and also the bosonic string theories in 26 dimensions. M-theory may therefore be too restrictive to describe the full meta-laws. Instead we should regard M-theory as one aspect of a split in the cascade of vacua, albeit one that is quite high up in the hierarchy.

From work on **Matrix-theory**, the **amplituhedron** and even **Loop Quantum Gravity** we get the idea that space and time are **emergent** so the meta-laws should not be given directly in geometric terms. It has been said that geometry is an angel and algebra is a demon [6], but if so then the signs are that the devil rules at the deepest levels of existence. Perhaps the chart should be turned upside-down. How then can we understand the unholy alliance between these demons and angels that brings into being the graces of space and time from the devilish rules of algebra?

In the 17^{th} century Descartes provided the first components of an answer by defining **coordinates** so that geometric curves could be analysed using algebraic equations. This was followed by vector algebra, matrices, complex numbers, quaternions, tensor analysis and group theory bringing us to the beginning of the twentieth century. Physicists have made use of all those algebraic concepts to understand the geometry of space-time, but during the twentieth century mathematicians such as Grothendiecke moved on to new concepts of **algebraic geometry** that were more abstract and seemed further removed from physics. In the present century physicists have reconnected with the latest mathematical innovations after finding that some of the constructs from algebraic geometry arise naturally in quantum field theories. One of these ideas that I find particularly promising is the application of **iterated integrals** [7] to map algebraic structures like **necklace algebras** [8] to the geometry of particle worldliness, Feynman diagrams, string states and even path integrals. These things are new for both mathematicians and physicists so they will take time to assimilate.

There are people who argue that physicists are misguided and use sophisticated concepts just because of their mathematical beauty which has nothing to do with physical concepts and is not linked to experiment. I think those detractors are wrong and seem to be driven by a desire that the laws of physics should be simple so that more people can understand them. In fact all physicists would be more comfortable if simpler mathematics was all that was needed to find deeper laws of physics because it requires a huge effort to keep learning new things, but what really happens is that they study quantum field theory and relativity together and find that the hard mathematics just turns up naturally in the analysis. The reason for this confluence of mathematics and physics is mysterious but a possible explanation is that mathematicians and physicists are attracted towards the same critical point of universality. On the mathematicians side it is because the universal constructs are useful across different types of problems which makes them seem more discovered than invented. On the physicists side it is because the point of universality defines the meta-laws of which our own universe is one solution. Of course new experimental input will be needed to confirm that theories built in this way are right but theorists will have to explore the mathematical aspects of quantum field theory combined with relativity in all its forms to know what possibilities can work and which are inconsistent.

What is the role of **symmetry** in the meta-laws? It used to be conventional wisdom that symmetry is the key principle that determines the laws of physics. Group invariance is central to both general relativity and gauge fields so 30 years ago it made sense to look for larger symmetries that unified the forces. Now there is a growing movement among physicists that thinks symmetry is not so fundamental. They say that gauge fields can be regarded as redundant variables that just make the theory look simpler [9]. Different groups control different quantum field theories that are known to be dual to each other so how can the symmetry be what really counts? Many present day physicists prefer to think that symmetry is emergent just as space and time are. To some extent I agree. The fundamental principle that determines the laws of physics is universality, not symmetry, but I contend that symmetry is emergent at the critical point of universality and that it emerges as a **huge symmetry** present in the meta-laws of physics. As we descend the cascade of solutions this universal symmetry is broken by vacuum states and the symmetries we observe at the low energies are residual symmetries, not emergent symmetries. Different dual theories have different symmetries because the universal symmetry is broken in different ways according to the limit taken.

The big clue that this huge hidden symmetry exists is the **holographic principle** that is required to resolve the black hole information loss puzzle. The principle says that the laws of physics can be defined by variables on the surface of a region of space rather than the bulk volume. This must mean that the when we define the laws of physics in terms of field variable over space-time, those variables must in fact be redundant so that they can be replaced by variables on the boundary. The only mechanism that can realise this is a **complete symmetry** where there is one degree of symmetry for every field variable. In other words, the physical state is given by the adjoint representation of a symmetry Lie-algebra. Although the universal symmetry is hidden by the selection of the vacuum state the equations of motion for the fields still respect its gauge invariance and can be replaced by an infinite number of gauge charges on the boundary.

Does this mean that the algebraic form of the meta-laws is just a Lie-algebra? The largest and most general Lie-algebra is the **free Lie-algebra** whose universal enveloping algebra is the free associative tensor algebra. It is easy to see that this algebra may play a key role because it takes the form of a necklace algebra (a structure I first encountered in my work on **event-symmetry** [10]) which can be mapped through iterated integrations to both open and closed geometric string states. All other Lie-algebras can be derived from the free Lie-algebra over an infinite dimensional vector space through homomorphic mappings which are the analogue of solution selection in geometry. Could the principles of universal algebra be what governs the meta-laws?

No, it is not quite that simple. There are fermionic fields as well as bosonic fields, so complete symmetry implies a **super-Lie algebra**. Then there are other algebraic generalisation of symmetry that are likely to be relevant. The knotted nature of Loop Quantum Gravity suggests that **quantum deformations** of symmetry are needed and from **higher category theory** we know that there are higher dimensional *n*-groups that are also relevant to mathematical physics. In fact it is the language of these *n*-categories and operads that has taken over from universal algebra as tools to understand the universal structures of mathematics and there are good indications towards why they also feature in physics.

Universality brings together all the logical possibilities of mathematics under one metaphorical **path integral** that quantised the ensemble, but the structure that emerges from universality is also a mathematical structure in its own right. It should also therefore be one of the logical possibilities under the path integral. This gives a **self-referential** and **recursive** nature to universality so we should expect the meta-laws of physics to be not just quantised at one level, but to have the features of multiple layers of quantisation. The relationship between classical and quantum in physics is indeed more complex than it first appears, not only do we have first and second quantisation followed by hints of third and fourth quantisations [11], in addition we find that a given quantum theory can have more than one classical limit. For example theories with S-duality have two classical limits with different gauge symmetries related by **geometric Langlands duality**. These are features you would expect if physics is the convergent limit of **iterated quantisations**. When you use Newton-Raphson iteration for calculating square roots you converge to the same final answer for any positive choice of starting value. At each step of the iteration two values map to one value at the next step. We can expect the same to happen with iterated quantisation leading to the same result whether you start from a system of one **qubit** of information or anything else. There are less quantum theories than classical theories and the theory you get by recursively iterating quantisation should be unique.

The second quantisation process we know from quantum field theory is just a pale shadow of the full algebraic structure present in the meta-laws above the cascade of vacuum solutions. To understand it we need to see quantisation for what it is in its purest form. From one point of view quantisation is the process of taking all mappings from one structure to another that respect their operations, i.e. homomorphisms or *n*-functors in the more general language of higher category theory [12]. We see this in path-integrals and when we replace classical variables with operators. *N*-category theory is built in layers of abstraction where *n*-functors are replaced by (*n*+1) functors. At the first level of set theory it is set **exponentiation** in which we construct all functions from one set to another. This is why quantisation appears like a process of exponentiation with the exponential function appearing in the path-integral and the number of variables increased as an exponential. Taking this line of reason to its logical conclusion we might find that the meta-laws are described by the **free weak omega-category** for some suitably definition of that object. To understand the laws of physics we need to understand the natural rules of *n*-category theory and the mappings from algebraic geometry that map the algebraic structures to the geometric ones we are familiar with. It is an ambitious project that will no doubt occupy the minds of physicists and mathematicians for many decades.

The deepest questions we can ask about existence are “How do we exist?” and “Why are things as they are?” The Mathematical Universe Hypotheses tells us that all logical possibilities are equal. It does not require a magic spell to bring one chosen system of equations into reality. Our world is quantum because all those things can happen but it is the principle of universality that makes sense of what we experience. While all is possible in the quantum realm there is a hierarchy of classical limits determined by reversing the self-referential logic of universality. These limits define worlds in which mathematical rules are played out according to the law of quantum averages. They form a landscape of possible solutions of which our universe is just one. Apart from universality the only other constraint is that the solution we experience needs to be such that intelligent life can evolve through **lazy processes** that minimise fine-tuning giving preference to natural structures. Our understanding has been able to progress because universality also determines the elements of mathematics that are most useful and pleasing to mathematicians. The development of both the theory and the experiment is going to be tremendously hard, but during this century we will discover more about the relationships between algebra and geometry that determine the emergence of space and time in a universe governed by the laws of energy and entropy that are needed for life to evolve. At the same time technological progress will enable new empirical observations to help us understand inflation, dark matter, proton decay and other subtle phenomena that help to chart our course through the ontological realm to where we stand in it. They will enable us to pick out the universe's particular solution to the algebraic meta-laws. Thus we learn finally that there is no mysterious force that defines our consciousness. We have no existence beyond our journey in this material world. This universe of beauty is simply our Heaven or Hell according to the rules of universality and chance and what we make of them in the brevity of our life.

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